Calculus of Variation (2010, Spring) HW #4

變分學(2010,春季)第四次作業

繳交時間:六月11日(星期五)上課前

(Time Due: June 11 (Fri.) before class)

1. A system with one degree of freedom has a Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + A(q)p + B(q)$$

where A and B are certain functions of the coordinate q, and p is the momentum conjugate to q.

   (a) Find the velocity $\dot{q}$.

   (b) Find the Lagrangian $L(q,\dot{q})$

2. The equations of motion for a particle of mass m and charge e moving in a uniform magnetic field B which points in the z-direction can be obtained from a Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{eB}{2c}(x\dot{y} - y\dot{x})$$

   (a) Write down the equations of motion.

   (b) Find the momenta (p_x, p_y, p_z) conjugate to (x, y, z).

   (c) Find the Hamiltonian, expressing your answer first in terms of $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ and then in terms of (x, y, z, p_x, p_y, p_z).

   (d) Evaluate the Poisson bracket

$$[m\dot{x}, m\dot{y}], \quad [m\dot{y}, m\dot{z}], \quad [m\dot{z}, m\dot{x}], \quad [m\dot{x}, H], \quad [m\dot{y}, H], \quad [m\dot{z}, H]$$

   (e) Rewrite the Hamiltonian system in terms of Poisson bracket.

3. The Hamiltonian for a simple harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Introduce the complex quantities

$$a = \sqrt{\frac{m\omega}{2}}\left(x + \frac{ip}{m\omega}\right), \qquad a^* = \sqrt{\frac{m\omega}{2}}\left(x - \frac{ip}{m\omega}\right).$$

   (a) Express H in terms of a and a^*.

(b) Evaluate the Poisson bracket $[a, a^*]$, $[a, H]$, and $[a^*, H]$.

(c) Write down and solve the equations of motion for a and a^*.

4. The motion of a particle of mass m which moves vertically in the uniform gravitational field g near the surface of the earth can be described by an action principle with Lagrangian

$$L = \frac{1}{2}m\dot{z}^2 - mgz$$

(a) Show that the action principle is invariant under the transformation $z^* = z + \epsilon$ where ϵ is any constant, and find the associated constant of the motion by Noether's theorem.

(b) Show that the action principle is invariant under the transformation $z^* = z + \epsilon t$ where ϵ is any constant, and find the associated constant of the motion by.