Calculus of Variation (2010, Spring) HW #4

變分學(2010,春季)第四次作業之解答

繳交時間:六月11日(星期五)上課前

(Time Due: June 11 (Fri.) before class)

1. A system with one degree of freedom has a Hamiltonian

$$H(q,p)=\frac{p^2}{2m}+A(q)p+B(q)$$

where A and B are certain functions of the coordinate q, and p is the momentum conjugate to q.

(a) Find the velocity $\dot{q}$.

(b) Find the Lagrangian $L(q,\dot{q})$

Solution.

(a) The velocity $\dot{q}$ is given by the first of the Hamiltonian equations

$$\dot{q}=\frac{\partial H}{\partial p}=\frac{p}{m}+A(q)$$

(b) The Lagrangian L is given by the Legendre transform

$$L(q,\dot{q})=p\dot{q}-H=\frac{1}{2}m(\dot{q}-A)^2-B$$

2. The equations of motion for a particle of mass m and charge e moving in a uniform magnetic field B which points in the z-direction can be obtained from a Lagrangian

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{eB}{2c}(x\dot{y}-y\dot{x})$$

(a) Write down the equations of motion.

(b) Find the momenta (p_x,p_y,p_z) conjugate to (x,y,z).

(c) Find the Hamiltonian, expressing your answer first in terms of $(x,y,z,\dot{x},\dot{y},\dot{z})$ and then in terms of (x,y,z,p_x,p_y,p_z).

(d) Evaluate the Poisson bracket

$$[m\dot{x},m\dot{y}],\quad [m\dot{y},m\dot{z}],\quad [m\dot{z},m\dot{x}],\quad [m\dot{x},H],\quad [m\dot{y},H],\quad [m\dot{z},H]$$

(e) Rewrite the Hamiltonian system in terms of Poisson bracket.

Solution.

(a) We have

$$\frac{\partial L}{\partial \dot{x}} = -\frac{eB}{2c}y, \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y} - \frac{eB}{2c}x, \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial x} = \frac{eB}{2c}\dot{y}, \qquad \frac{\partial L}{\partial y} = -\frac{eB}{2c}\dot{x}, \qquad \frac{\partial L}{\partial z} = o$$

so Lagrange's equations are

$$m\ddot{x} = \frac{eB}{2c}\dot{y}, \qquad m\ddot{y} = -\frac{eB}{2c}\dot{x}, \qquad m\ddot{z} = 0$$

(b) The momenta (p_x, p_y, p_z) conjugate to (x, p, z) are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{eB}{2c}y, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{eB}{2c}x, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{x}$$

(c) The Hamiltonian is

$$\begin{aligned} H &= (p_x, p_y, p_z)\cdot(\dot{x}, \dot{y}, \dot{z}) - L \\ &= \left(m\dot{x} - \frac{eB}{2c}y\right)\dot{x} + \left(m\dot{y} + \frac{eB}{2c}x\right)\dot{y} + m\dot{z}^2 - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{eB}{2c}(x\dot{y} - y\dot{x}) \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \end{aligned}$$

and equals "kinetic energy". Writing H in terms of (x, y, z, p_x, p_y, p_z) we have

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{2c}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{eB}{2c}x\right)^2 + \frac{1}{2m}p_z^2$$

(d) The Poisson brackets are

$$\begin{aligned} [m\dot{x}, m\dot{y}] &= \left[p_x + \frac{eB}{2c}y, p_y - \frac{eB}{2c}x\right] \\ &= [p_x, p_y] - \frac{eB}{2c}[p_x, x] + \frac{eB}{2c}[y, p_y] - \left(\frac{eB}{2c}\right)^2[y, x] \\ &= \frac{eB}{c} \end{aligned}$$

together with

$$[m\dot{y}, m\dot{z}] = [m\dot{z}, m\dot{x}] = 0$$

The Poisson brackets of the components of the kinematic momentum with the Hamiltonian are then

$$\begin{aligned} &[m\dot{x}, H] = [m\dot{x}, \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)] = [m\dot{x}, m\dot{y}]\dot{y} = \frac{eB}{c}\dot{y} \\ &[m\dot{y}, H] = [m\dot{y}, \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)] = [m\dot{y}, m\dot{x}]\dot{x} = -\frac{eB}{c}\dot{x} \\ &[m\dot{z}, H] = 0 \end{aligned}$$

(e) The equations of motion in terms of the Poisson bracket are

$$\frac{d}{dt}(m\dot{x}) = [m\dot{x}, H] = \frac{eB}{c}\dot{y}$$
$$\frac{d}{dt}(m\dot{y}) = [m\dot{y}, H] = -\frac{eB}{c}\dot{x}$$
$$\frac{d}{dt}(m\dot{z}) = [m\dot{z}, H] = 0$$

3. The Hamiltonian for a simple harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Introduce the complex quantities

$$a = \sqrt{\frac{m\omega}{2}}\left(x + \frac{ip}{m\omega}\right), \qquad a^* = \sqrt{\frac{m\omega}{2}}\left(x - \frac{ip}{m\omega}\right).$$

(a) Express H in terms of a and a^*.

(b) Evaluate the Poisson bracket $[a, a^*]$, $[a, H]$, and $[a^*, H]$.

(c) Write down and solve the equations of motion for a and a^*.

Solution.

(a) We have

$$a^*a = \frac{m\omega}{2}\left(x^2 + \frac{p^2}{m^2\omega^2}\right).$$

so the Hamiltonian for a simple harmonic oscillator can be written

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \omega a^* a$$

(b) The required Poisson brackets are

$$[a, a^*] = \frac{m\omega}{2}\left[x + \frac{ip}{m\omega}, x - \frac{ip}{m\omega}\right] = -i$$
$$[a, H] = \omega[a, a^*a] = \omega[a, a^*]a = -i\omega a$$
$$[a^*, H] = \omega[a^*, a^*a] = \omega a^*[a^*, a] = i\omega a^*$$

(c) The equations of motion are

$$\frac{da}{dt} = [a, H] = -i\omega a, \qquad \frac{da^*}{dt} = [a^*, H] = i\omega a^*.$$

These can be integrated to give

$$a = a_0 e^{-i\omega t}, \qquad a^* = a_0^* e^{i\omega t}$$

where a_0 and a_0^* are the initial values of a and a^*. This provides yet another way to obtain the general solution to the harmonic oscillator problem,

$$x = \sqrt{\frac{2}{m\omega}}(a + a^*) = \sqrt{\frac{2}{m\omega}}(a_0 e^{-i\omega t} + a_0^* e^{i\omega t})$$
$$p = -i\sqrt{2m\omega}(a - a^*) = -i\sqrt{2m\omega}(a_0 e^{-i\omega t} - a_0^* e^{i\omega t})$$

4. The motion of a particle of mass m which moves vertically in the uniform gravitational field g near the surface of the earth can be described by an action principle with Lagrangian

$$L = \frac{1}{2} m \dot{z}^2 - mgz$$

(a) Show that the action principle is invariant under the transformation $z^* = z + \epsilon$ where ϵ is any constant, and find the associated constant of the motion by Noether's theorem.

(b) Show that the action principle is invariant under the transformation $z^* = z + \epsilon t$ where ϵ is any constant, and find the associated constant of the motion by.

Solution.

(a) Setting

$$z = z^* - \epsilon, \qquad \dot{z} = \dot{z}^*$$

we have

$$\begin{aligned} L^*(z^*, \dot{z}^*) &= L(z, \dot{z}) \\ &= \frac{1}{2} m \dot{z}^2 - mgz \\ &= \frac{1}{2} m (\dot{z}^*)^2 - mg(z^* - \epsilon) \\ &= \frac{1}{2} m (\dot{z}^*)^2 - mgz^* + mg\epsilon \\ &= L(z, \dot{z}) + \frac{d\Lambda}{dt} \end{aligned}$$

where $\Lambda = mgt\epsilon$. Thus the action principle and system is invariant under this transformation. The corresponding infinitesimal invariance transformation is obtained by replacing ϵ by $\delta\epsilon$ and setting

$$\delta z = z^* - z = t\delta\epsilon, \qquad \delta\Lambda = -m(z - \frac{1}{2}gt^2)\delta\epsilon.$$

The associated constant of the motion is

$$\frac{\partial L}{\partial \dot{z}} \delta z + \delta\Lambda = m\dot{z}\delta\epsilon + mgt\delta\epsilon = m(\dot{z} + gt)\delta\epsilon$$

The constant $m(\dot{z} + gt)$ equals mv_0 where v_0 is the initial velocity of the particle.

(b) Setting

$$z = z^* - \epsilon t, \qquad \dot{z} = \dot{z}^* - \epsilon$$

we have

$$\begin{aligned} L^*(z^*, \dot{z}^*) &= L(z, \dot{z}) \\ &= \frac{1}{2}m\dot{z}^2 - mgz \\ &= \frac{1}{2}m(\dot{z}^* - \epsilon)^2 - mg(z^* - \epsilon t) \\ &= \frac{1}{2}(\dot{z}^*)^2 - mgz^* - m\dot{z}^*\epsilon + \frac{1}{2}m\epsilon^2 + mgt\epsilon \\ &= L(z, \dot{z}) + \frac{d\Lambda}{dt} \end{aligned}$$

where $\Lambda = -m(z^* - \frac{1}{2}gt^2)\epsilon + \frac{1}{2}m\epsilon^2 t$. Thus the action principle and system are invariant under this transformsation. The corresponding infinitesimal invariance transformation is obtained by replacing ϵ by $\delta\epsilon$ and setting

$$\delta z = z^* - z = t\delta\epsilon, \qquad \delta\Lambda = -m(z - \frac{1}{2}gt^2)\delta\epsilon.$$

The associated constant of the motion is

$$\begin{aligned} \frac{\partial L}{\partial \dot{z}}\delta z + \delta\Lambda &= m\dot{z}t\delta\epsilon - m(z - \frac{1}{2}gt^2)\delta\epsilon \\ &= m(\dot{z}t - z + \frac{1}{2}gt^2)\delta\epsilon \end{aligned}$$

The constant $m(\dot{z}t - z + \frac{1}{2}gt^2)$ equals $-mz_0$ where z_0 is the initial position of the particle. The expressions

$$\dot{z} + gt = v_0, \qquad \dot{z} - z + \frac{1}{2}gt^2 = -z_0$$

for the two constants of the motion can be inverted to obtain the velocity and position of the particle as functions of time,

$$\dot{z} = v_0 - gt, \qquad z = z_0 + v_0 t - \frac{1}{2}gt^2$$

thus solving the equations of motion.