Calculus of Variation (2010, Spring) HW #2

變分學(2010,春季)第二次作業

繳交時間:四月29日(星期五)上課前

(Time Due: April 29 (Fri.) before class)

1. Show that for $u'' + \lambda u = 0$ and the endpoint conditions

$$(a)\quad u(0) = u(\pi) = 0, \qquad\qquad (b)\quad u'(0) = u(\pi) = 0$$

$$(c)\quad u(0) = u'(\pi) = 0, \qquad\qquad (d)\quad u'(0) = u'(\pi) = 0$$

the eigenvalues are $(n+1)^2$, $(n+1/2)^2$, $(n+1/2)^2$ and n^2, respectively. What are the eigenfunctions?

2. Solve the eigenvalue problem

$$\begin{cases} u'' + \lambda u = 0, & 0 \le x \le 1 \\ u(0) = 0, \quad u'(1) + u(1) = 0 \end{cases}$$

3. Solve the eigenvalue problem

$$\begin{cases} \left[(1+x)^2 u'\right]' + \lambda u = 0, & 0 \le x \le 1 \\ u(0) = u(1) = 0 \end{cases}$$

4. Prove the Riemann metric in three different coordinate systems

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\theta^2 + dz^2 \qquad \text{(cylindrical coordinate)} \\ &= dr^2 + r^2 d\varphi^2 + r^2 \sin^2\varphi\, d\theta^2 \qquad \text{(spherical coordinate)} \end{aligned}$$

5. The motion of a particle of mass m is given by Lagrange's equations with Lagrangian

$$L = e^{\frac{\alpha t}{m}}(T - V)$$

where α is a constant, $T = \dfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ is the kinetic energy, and $V = V(x, y, z)$ is the potential energy. Write down the equations of motion and interpret.

6. A bead of mass m slides without friction along a wire bent in the shape of a cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi)\,.$$

Gravity g acts vertically down, parallel to the y-axis.

   (a) Find the displacement s along the cycloid, measured from the bottom, in terms of the parameter ϕ.

   (b) Write down the Lagrangian using s as generalized coordinates, and show that the motion is simple harmonic in s with period independent of amplitude. Thus the time required for the bead, starting from rest, to slide from any point on the cycloid to the bottom is independent of the starting point. What is the time?