Mathmatical Statistics

Chen, L.-A.

Notes:

(a) The number of success in one Bernoulli experiment has Bernoulli distribution Bernoulli(p).

(b) The number of success in n independent Bernoulli experiments has Binomial distribution b(n,p).

Normal Distribution
We say that a r.v. X has a normal distribution if it has p.d.f

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

for some fixed $\mu \in R$ and $\sigma > 0$. We denote by $X \sim N(\mu, \sigma^2)$.
If X has a normal distribution with $\mu = 0$ and $\sigma = 1$, we say that X has a standard normal distribution.

Note:
$\int_{-\infty}^{\infty} x f(x) dx = P(X \in (-\infty, \infty)) = P(X^{-1}(R)) = P(S) = 1$
$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$, for $\mu \in R$, $\sigma > 0$.

Thm. *If we let $\lambda = np$, then p.d.f of $b(n,p)$*

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \to \frac{\lambda^x e^{-\lambda}}{x!}$$

Proof.

$$
\begin{aligned}
f(x) &= \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x}, \lambda = np \\
&= \frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x} \\
&= \frac{\lambda^x}{x!}\frac{n(n-1)\cdots(n-(x-1))}{n^x}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x} \\
&= \frac{\lambda^x}{x!}\cdot 1\cdot(1-\frac{1}{n})\cdots(1-\frac{x-1}{n})(1+\frac{-\lambda}{n})^n(1-\frac{\lambda}{n})^{-x} \\
&= \frac{\lambda^x}{x!}(e^{-\lambda})
\end{aligned}
$$

□

Def. *We say that r.v. X has a Poisson distribution if it has p.d.f*

$$f(x) = \frac{\lambda^x}{x!}e^{-\lambda}, x = 0, 1, 2, \cdots$$

We denote by $X \sim Poisson(\lambda)$.

Notes:

(a) Binomial r.v. = number of success in n Bernoulli experiments.

(b) X = number of success in infinite Bernoulli experiments
$X \sim Poisson(\lambda)$

Gamma Distribution
Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$
Properties:
(1) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1),\ if\ \alpha > 1$
(2) $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$
(3) $\Gamma(n) = (n-1)\Gamma(n-1) = \cdots = (n-1)(n-2)\cdots 1\cdot\Gamma(1) = (n-1)!$
(4) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Def. *We say that X has a Gamma distribution if it has p.d.f*

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}},\ x > 0,\ for\ some\ \alpha > 0,\ \beta > 0$$

We denote by $X \sim Gamma(\alpha, \beta)$.
Note: $\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}dx = 1,\ \forall\ \alpha > 0,\ \beta > 0$

If X has Gamma distribution with $\beta = 2$ and $\alpha = \frac{r}{2}$, we say that X has a chi-square distribution with degrees of freedom r. The p.d.f is

$$f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, \ x > 0$$

We denote by $X \sim \chi^2(r)$.

Expectation:
Let g be a real valued function on R ($g : R \to R$). The expectation of $g(X)$ is

$$E[g(X)] = \begin{cases} \Sigma_{allx} g(x)f(x) & \text{, discrete r.v.} \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{, continuous r.v.} \end{cases}$$

Properties:
(a) $E[aX + b] = aE[X] + b$
(b) $E[c] = c$

The mean of r.v. X is $\mu = E[X]$.
The variance of r.v. X is $\sigma^2 = Var(X) = E[(X - \mu)^2]$.
Mean and variance can be divided through moment generating function.

Def. *The moment generating function of r.v. X is $M_X(t) = E[e^{tX}]$, a function of t. If there exists $\delta > 0$ such that $M_X(t)$ exists for $t \in (-\delta, \delta)$, then $D_t^k E[e^{tX}] = E[D_t^k e^{tX}]$, for all k.*

Thm. $M_X^{(k)}(0) = E[X^k], \ k = 1, 2, \cdots$

Proof.

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$M_X'(t) = D_t \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} (D_t e^{tx}) f(x)dx = \int_{-\infty}^{\infty} x e^{tx} f(x)dx$$

$$M_X''(t) = D_t \int_{-\infty}^{\infty} x e^{tx} f(x)dx = \int_{-\infty}^{\infty} x(D_t e^{tx}) f(x)dx = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)dx$$

$$\vdots$$

$$M_X^k(t) = D_t \int_{-\infty}^{\infty} x^{k-1} e^{tx} f(x)dx = \int_{-\infty}^{\infty} x^{k-1}(D_t e^{tx}) f(x)dx = \int_{-\infty}^{\infty} x^k e^{tx} f(x)dx$$

$$\Rightarrow M_X^k(0) = \int_{-\infty}^{\infty} x^k f(x)dx = E[X^k]$$

□

Notes:
(1) $M_X'(0) = E[X] = \mu = Mean$
(2) $M_X''(0) = E[X^2]$
(3)

$$\begin{aligned} Variance\ \sigma^2 &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + E[\mu^2] \\ &= E[X^2] - \mu^2 \\ &= M_X''(0) - (M_X'(0))^2 \end{aligned}$$

If $X \sim Bernoulli(p)$, m.g.f of X is

$$M_X(t) = E[e^{tX}] = \Sigma_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} = (1-p) + pe^t,\ t \in R$$
$$M_X'(t) = pe^t \Rightarrow Mean\ \mu = E[X] = M_X'(0) = p$$
$$M_X''(t) = pe^t \Rightarrow\ E[X^2] = M_X''(0) = p$$
$$\Rightarrow\ Variance\ \sigma^2 = M_X''(0) - (M_X'(0))^2 = p^2 - p = p(1-p)$$

If $X \sim b(n,p)$, m.g.f of X is

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \Sigma_{x=0}^{n} e^{tx} \begin{pmatrix} n \\ x \end{pmatrix} p^x (1-p)^{n-x} \\ &= \Sigma_{x=0}^{n} \begin{pmatrix} n \\ x \end{pmatrix} (pe^t)^x (1-p)^{n-x} \\ &= (1-p+pe^t)^n,\ t \in R \end{aligned}$$
$$M_X'(t) = n(1-p+pe^t)^{n-1} pe^t$$
$$\Rightarrow Mean\ \mu = E[X] = M_X'(0) = np$$
$$M_X''(t) = n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1} pe^t$$
$$\Rightarrow E[X^2] = M_X''(0) = n(n-1)p^2 + np$$
$$\Rightarrow\ Variance\ \sigma^2 = M_X''(0) - (M_X'(0))^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

Note:
Binomial expansion:
$(a+b)^n = \Sigma_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} b^k a^{n-k},\ \forall\ a,\ b \in R$

If $X \sim Poisson(\lambda)$, m.g.f of X is

$$M_X(t) = E[e^{tX}] = \Sigma_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \Sigma_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)},\ \forall\ t \in R$$

$$M_X^{'}(t) = \lambda e^t e^{\lambda(e^t-1)}$$

$$\Rightarrow Mean\ \mu = E[X] = M_X^{'}(0) = \lambda$$

$$M_X^{''}(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}$$

$$\Rightarrow E[X^2] = M_X^{''}(0) = \lambda + \lambda^2$$

$$\Rightarrow\ Variance\ \sigma^2 = M_X^{''}(0) - (M_X^{'}(0))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Note:
$e^a = \Sigma_{k=0}^{\infty} \frac{a^k}{k!},\ \forall\ a \in R$

If $X \sim N(\mu, \sigma^2)$, m.g.f of X is

$$\begin{aligned}
M_X(t) = E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2-2\sigma^2 tx}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2-2\mu x-2\sigma^2 tx+\mu^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2-2(\mu+\sigma^2 t)x}{2\sigma^2}-\frac{\mu^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}-\frac{\mu^2}{2\sigma^2}+\frac{(\mu+\sigma^2 t)^2}{2\sigma^2}} dx \\
&= e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{(\mu+\sigma^2 t)^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx \\
&= e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\mu^2+2\mu\sigma^2 t+\sigma^4 t^2}{2\sigma^2}} \\
&= e^{\mu t+\frac{\sigma^2}{2}t^2},\ t \in R \\
M_X^{'}(t) &= (\mu + \sigma^2 t) e^{\mu t+\frac{\sigma^2}{2}t^2} \\
&\Rightarrow Mean\ E[X] = M_X^{'}(0) = \mu \\
M_X^{''}(t) &= \sigma^2 e^{\mu t+\frac{\sigma^2}{2}t^2} + (\mu + \sigma^2 t)^2 e^{\mu t+\frac{\sigma^2}{2}t^2} \\
&\Rightarrow E[X^2] = M_X^{''}(0) = \sigma^2 + \mu^2 \\
&\Rightarrow\ Variance\ Var(X) = M_X^{''}(0) - (M_X^{'}(0))^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2
\end{aligned}$$

If $X \sim Gamma(\alpha, \beta)$, m.g.f of X is

$$\begin{aligned}
M_X(t) &= E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{(1-\beta t)x}{\beta}} dx \\
&= (1-\beta t)^{-\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)(\frac{\beta}{1-\beta t})^\alpha} x^{\alpha-1} e^{-\frac{x}{\frac{\beta}{1-\beta t}}} dx \\
&= (1-\beta t)^{-\alpha},\ t < \frac{1}{\beta} \\
&\because \frac{\beta}{1-\beta t} > 0 \Rightarrow 1 - \beta t > 0 \Rightarrow t < \frac{1}{\beta} \\
M_X^{'}(t) &= \alpha(1-\beta t)^{-\alpha-1}\beta \\
&\Rightarrow Mean\ \mu = E[X] = M_X^{'}(0) = \alpha\beta \\
M_X^{''}(t) &= \alpha(\alpha+1)(1-\beta t)^{-\alpha-2}\beta^2 \\
&\Rightarrow E[X^2] = M_X^{''}(0) = \alpha(\alpha+1)\beta^2 \\
&\Rightarrow\ Variance\ \sigma^2 = M_X^{''}(0) - (M_X^{'}(0))^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2
\end{aligned}$$