Chapter 5. Confidence Interval

Let Z be the r.v. with standard normal distribution $N(0,1)$
We can find z_α and $z_{\frac{\alpha}{2}}$ that satisfy

$$\alpha = P(Z \leq -z_\alpha) = P(Z \geq z_\alpha) \text{ and } 1-\alpha = P(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}).$$

A table of $z_{\frac{\alpha}{2}}$ is the following :

$1-\alpha$	$z_{\frac{\alpha}{2}}$
0.8	1.28 ($z_{0.1}$)
0.9	1.645 ($z_{0.05}$)
0.95	1.96 ($z_{0.025}$)
0.99	2.58 ($z_{0.005}$)
0.9973	3 ($z_{0.00135}$)

Def. *Suppose that we have a random sample from $f(x,\theta)$. For $0<\alpha<1$, if there exists two statistics $T_1 = t_1(X_1,\ldots,X_n)$ and $T_2 = t_2(X_1,\ldots,X_n)$ satisfying*

$$1-\alpha = P(T_1 \leq \theta \leq T_2)$$

We call the random interval (T_1,T_2) a $100(1-\alpha)\%$ confidence interval of parameter θ. If $X_1 = x_1,\ldots,X_n = x_n$ is observed, we also call $(t_1(X_1,\ldots,X_n), t_2(X_1,\ldots,X_n))$ a $100(1-\alpha)\%$ confidence interval(C.I.) for θ

Constructing C.I. by pivotal quantity:

Def. *A function of random sample and parameter, $Q = q(X_1,\ldots,X_n,\theta)$, is called a pivotal quantity if its distribution is independent of θ*

With a pivotal quantity $q(X_1,\ldots,X_n,\theta)$, there exists a,b such that

$$1-\alpha = P(a \leq q(X_1,\ldots,X_n,\theta) \leq b), \forall \theta \in \Theta.$$

The interest of pivotal quantity is that there exists statistics $T_1 = t_1(X_1,\ldots,X_n)$ and $T_2 = t_2(X_1,\ldots,X_n)$ with the following 1-1 transformation

$$a \leq q(X_1,\ldots,X_n,\theta) \leq b \text{ iff } T_1 \leq \theta \leq T_2$$

Then we have $1-\alpha = P(T_1 \leq \theta \leq T_2)$ and (T_1,T_2) is a $100(1-\alpha)\%$ C.I. for θ
Confidence Interval for Normal mean:
Let $X_1,\ldots,X_n$ be a random sample from $N(\mu,\sigma^2)$. We consider the C.I. of

parameter μ.
(I) $\sigma = \sigma_0$ is known

$$\overline{X} \sim N(\mu, \frac{\sigma_0^2}{n}) \Rightarrow \frac{\overline{X} - \mu}{\sigma_0/\sqrt{n}} \sim N(0,1)$$

$$\begin{aligned} 1-\alpha &= P(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}), Z \sim N(0,1) \\ &= P(-z_{\frac{\alpha}{2}} \leq \frac{\overline{X} - \mu}{\sigma_0/\sqrt{n}} \leq z_{\frac{\alpha}{2}}) \\ &= P(-z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \leq \overline{X} - \mu \leq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}) \\ &= P(\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}) \end{aligned}$$

$\Rightarrow (\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}})$ is a $100(1-\alpha)\%$ C.I. for μ.
ex: $n = 40, \sigma_0 = \sqrt{10}, \overline{x} = 7.164$ ($X_1, \ldots, X_{40} \overset{iid}{\sim} N(\mu, 10)$.)
Want a 80% C.I. for μ.
sol: A 80% C.I. for μ. is

$$\begin{aligned} (\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}) &= (7.164 - 1.28 \frac{\sqrt{10}}{\sqrt{40}}, 7.164 + 1.28 \frac{\sqrt{10}}{\sqrt{40}}) \\ &= (6.523, 7.805) \end{aligned}$$

$$P(\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}) = 1 - \alpha = 0.8$$
$$P(6.523 \leq \mu \leq 7.805) = 1 \text{ or } 0$$

(II)σ is unknown.

Def. *If $Z \sim N(0,1)$ and $\chi^2(r)$ are independent, we call the distribution of the r.v.*

$$T = \frac{Z}{\sqrt{\frac{\chi^2(r)}{r}}}$$

a t-distribution with r degrees of freedom.

The p.d.f of t-distribution is

$$f_T(t) = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})} \frac{1}{\sqrt{r\pi}(1 + \frac{t^2}{r})^{\frac{r+1}{2}}}, -\infty < t < \infty$$

$\because f_T(-t) = f_T(t)$
$\therefore$ t-distribution is symmetric at 0.
Now $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. We have

$$\begin{cases} \overline{X} \sim N(\mu, \frac{\sigma^2}{n}) \\ \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \end{cases} indep. \Rightarrow \begin{cases} \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \\ \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \end{cases} indep.$$

$$T = \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{\sigma^2(n-1)}}} = \frac{\overline{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$$

Let $t_{\frac{\alpha}{2}}$ satisfies

$$\begin{aligned} 1-\alpha &= P(-t_{\frac{\alpha}{2}} \le \frac{\overline{X}-\mu}{s/\sqrt{n}} \le t_{\frac{\alpha}{2}}) \\ &= P(-t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \le \overline{X} - \mu \le t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}) \\ &= P(\overline{X} - t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \le \mu \le \overline{X} + t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}) \end{aligned}$$

$\Rightarrow (\overline{X} - t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \overline{X} + t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}})$ is a $100(1-\alpha)\%$ C.I. for μ.

ex: Suppose that we have $n = 10, \overline{x} = 3.22$ and $s = 1.17$. We also have $t_{0.025} = 2.262$. Want a 95% C.I. for μ.
sol: A 95% C.I. for μ is

$$(3.22 - 2.262\frac{1.17}{\sqrt{10}}, 3.22 + 2.262\frac{1.17}{\sqrt{10}}) = (2.34, 4.10)$$