Mathmatical Statisticals

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Chapter 4. Distribution of Function of Random variables

Sample space S : set of possible outcome in an experiment.

Probability set function P:
$(1)P(A) \geq 0, \forall A \subset S.$
$(2)P(S) = 1.$
$(3)P(\bigcup_1^\infty A_i) = \sum_1^\infty P(A_i), if A_i \cap A_j = \varnothing, \forall i \neq j.$

Random variable X:
$X : S \rightarrow R$
Given $B \subset R, P(X \in B) = P(\{s \in S : X(s) \in B\}) = P(X^{-1}(B))$ where $X^{-1}(B) \subset S.$

X is a discrete random variable if its range

$$X(s) = \{x \in R : \exists s \in S, X(s) = x\}$$

is countable. The probability density/mass function (p.d.f) of X is defined as

$$f(x) = P(X = x), x \in R.$$

Distribution function F:

$$F(x) = P(X \leq x), x \in R.$$

A r.v. is called a continuous r.v. if there exists $f(x) \geq 0$ such that

$$F(x) = \int_{-\infty}^{x} f(t)dt, x \in R.$$

where f is the p.d.f of continuous r.v. X.

Let X be a r.v. with p.d.f $f(x)$. Let $g : R \to R$
Q: What is the p.d.f. of $g(x)$? and is $g(x)$ a r.v.?(Yes)
Answer:
(a) distribution method :
Suppose that X is a continuous r.v.. Let $Y = g(X)$
The d.f(distribution function) of Y is

$$G(y) = P(Y \leq y) = P(g(X) \leq y)$$

If G is differentiable then the p.d.f. of $Y = g(X)$ is $g(y) = G'(y)$.
(b) mgf method :(moment generating function)

$$E[e^{tx}] = \begin{cases} \sum e^{tx} f(x) & \text{(discrete)} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & \text{(continuous)} \end{cases}$$

Thm. *m.g.f.* $M_x(t)$ *and its distribution (p.d.f. or d.f.) forms a* $1-1$ *functions.*

ex:

$$M_Y(t) = e^{\frac{1}{2}t} = M_{N(0,1)}(t) \Rightarrow Y \sim N(0,1)$$

Let $X_1, \ldots, X_n$ be random variables.
If they are discrete, the joint p.d.f. of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n), \forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^n$$

If $X_1, \ldots, X_n$ are continuous r.v.'s, there exists f such that

$$F(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(t_1, \ldots, t_n) dt_1 \ldots dt_n, \text{for} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^n$$

We call f the joint p.d.f. of $X_1, \ldots, X_n$.

If X is continuous, then

$$F(x) = \int_{-\infty}^{x} f(t)dt \text{ and } P(X = x) = \int_{x}^{x} f(t)dt = 0, \forall x \in R.$$

Marginal p.d.f's:
Discrete:

$$f_{X_i}(x) = P(X_i = x) = \sum_{x_n} \dots \sum_{x_{i+1}} \sum_{x_{i-1}} \dots \sum_{x_1} f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

Continuous:

$$f_{X_i}(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n$$

Events A and B are independent if $P(A \cap B) = P(A)P(B)$.
Q: If $A \cap B = \varnothing$, are A and B independent?
A: In general, they are not.

Let X and Y be r.v.'s with joint p.d.f. $f(x, y)$ and marginal p.d.f. $f_X(x)$ and $f_Y(y)$. We say that X and Y are independent if

$$f(x, y) = f_X(x) f_Y(y), \forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$$

Random variables X and Y are identically distributed (i.d.) if marginal p.d.f.'s f and g satisfy $f = g$ or d.f.'s F and G satisfy $F = G$.

We say that X and Y are **iid** random variables if they are independent and identically distributed.

Transformation of r.v.'s (discrete case)
Univariate: $Y = g(X)$, p.d.f. of Y is

$$g(y) = P(Y = y) = P(g(x) = y) = P(\{x \in \text{Range of } X : g(x) = y\}) = \sum_{\{x: g(x) = y\}} f(x)$$

For random variables $X_1, \dots, X_n$ with joint p.d.f. $f(x_1, \dots, x_n)$, define transformations

$$Y_1 = g_1(X_1, \dots, X_n), \dots, Y_m = g_m(X_1, \dots, X_n).$$

The joint p.d.f. of $Y_1, \dots, Y_m$ is

$$
\begin{aligned}
g(y_1, \dots, y_m) &= P(Y_1 = y_1, \dots, Y_m = y_m) \\
&= P(\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : g_1(x_1, \dots, x_n) = y_1, \dots, g_m(x_1, \dots, x_n) = y_m \}) \\
&= \sum_{\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : g_1(x_1,\dots,x_n)=y_1,\dots,g_m(x_1,\dots,x_n)=y_m \}} f(x_1, \dots, x_n)
\end{aligned}
$$

Example: joint p.d.f. of X_1, X_2, X_3 is

(x_1, x_2, x_3)	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
$f(x_1, x_2, x_3)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$$Y_1 = X_1 + X_2 + X_3, Y_2 = |X_3 - X_2|$$

Space of (Y_1, Y_2) is $\{(0,0), (1,1), (2,0), (2,1), (3,0)\}$.
Joint p.d.f. of Y_1 and Y_2 is

(y_1, y_2)	$(0,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$(3,0)$
$g(y_1, y_2)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

Continuous one-to-one transformations:
Let X be a continuous r.v. with joint p.d.f. $f(x)$ and range $A = X(s)$.

Consider $Y = g(x)$, a differentiable function. We want p.d.f. of Y.

Thm. *If g is 1-1 transformation, then the p.d.f. of Y is*

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{dg^{-1}(y)}{dy}| & y \in g(A) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The d.f. of Y is

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

(a) If g is $\nearrow$, g^{-1} is also $\nearrow$.$(\frac{dg^{-1}}{dy} > 0)$

$$F_Y(y) = P(X \le g^{-1}(y)) = \int_{-\infty}^{g^{-1}(y)} f_X(x)dx$$

$\Rightarrow$ p.d.f. of Y is

$$
\begin{aligned}
f_Y(y) &= D_y \int_{-\infty}^{g^{-1}(y)} f_X(x)dx \\
&= f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} \\
&= f_X(g^{-1}(y)) |\frac{dg^{-1}(y)}{dy}|
\end{aligned}
$$

(b) If g is $\searrow$, g^{-1} is also $\searrow$. ($\frac{dg^{-1}}{dy} < 0$)

$$
F_Y(y) = P(X \geq g^{-1}(y)) = \int_{g^{-1}(y)}^{\infty} f_X(x)dx = 1 - \int_{-\infty}^{g^{-1}(y)} f_X(x)dx
$$

$\Rightarrow$ p.d.f. of Y is

$$
\begin{aligned}
f_Y(y) &= D_y(1 - \int_{-\infty}^{g^{-1}(y)} f_X(x)dx) \\
&= -f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} \\
&= f_X(g^{-1}(y)) |\frac{dg^{-1}(y)}{dy}|
\end{aligned}
$$

$\square$

Example : $X \sim U(0,1), Y = -2\ \ln(x) = g(x)$
sol: p.d.f. of X is

$$
f_X(x) = \begin{cases} 1 & \text{, if } 0 < x < 1 \\ 0 & \text{, elsewhere.} \end{cases}
$$

$A = (0,1), g(A) = (0, \infty)$,

$$
x = e^{-\frac{y}{2}} = g^{-1}(y), \frac{dx}{dy} = -\frac{1}{2} e^{-\frac{y}{2}}
$$

p.d.f. of Y is

$$
f_Y(y) = f_X(g^{-1}(y)) |\frac{dy}{dx}| = \frac{1}{2} e^{-\frac{y}{2}}, y > 0
$$

$$
(X \sim U(a,b) \text{ if } f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{elsewhere.} \end{cases})
$$

$\Rightarrow Y \sim \chi^2(2)$

$$(X \sim \chi^2(r) \text{ if } f_X(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, x > 0)$$

Continuous n-r.v.-to-m-r.v., $n > m$, case :

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{cases} Y_1 = g_1(X_1, \ldots, X_n) \\ \vdots \\ Y_m = g_m(X_1, \ldots, X_n) \end{cases} \quad R^n \overset{\begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}}{\longrightarrow} R^m$$

Q : What are the marginal p.d.f. of $Y_1, \cdots, Y_m$
A : We need to define $Y_{m+1} = g_{m+1}(X_1, \ldots, X_n), \cdots, Y_n = g_n(X_1, \ldots, X_n)$ such that $\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$ is 1-1 from R^n to R^n.

Theory for change variables :

$$P(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in A) = \int \cdots \int f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

Let $y_1 = g_1(x_1, \ldots, x_n), \cdots, y_n = g_n(x_1, \ldots, x_n)$ be a $1-1$ function with inverse $x_1 = w_1(y_1, \ldots, y_n), \cdots, x_n = w_n(y_1, \ldots, y_n)$ and Jacobian

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

Then

$$\begin{aligned} &\int \cdots \int f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_1 \cdots dx_n \\ &= \int \cdots \int f_{X_1,\ldots,X_n}(w_1(y_1, \ldots, y_n), \ldots, w_n(y_1, \ldots, y_n))|J| dy_1 \cdots dy_n \end{aligned}$$

Hence, joint p.d.f. of $Y_1, \cdots, Y_n$ is

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{X_1,\ldots,X_n}(w_1, \ldots, w_n)|J|$$

Thm. *Suppose that* X_1 *and* X_2 *are two r.v.'s with continuous joint p.d.f.* f_{X_1,X_2} *and sample space* A.
If $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$ *forms a* $1-1$ *transformation inverse function*

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} w_1(Y_1,Y_2) \\ w_2(Y_1,Y_2) \end{pmatrix} \text{ and Jacobian } J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

the joint p.d.f. of Y_1, Y_2 *is*

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(w_1(y_1,y_2), w_2(y_1,y_2))|J|, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(A).$$

Steps :
(a) joint p.d.f. of X_1, X_2, space A.
(b) check if it is $1-1$ transformation.
Inverse function $X_1 = w_1(Y_1, Y_2), X_2 = w_2(Y_1, Y_2)$
(c) Range of $\binom{Y_1}{Y_2} = \binom{g_1}{g_2}(A)$

Example : For $X_1, X_2 \overset{iid}{\sim} U(0,1)$, let $Y_1 = X_1 + X_2, Y_2 = X_1 - X_2$.
Want marginal p.d.f. of Y_1, Y_2
Sol : joint p.d.f. of X_1, X_2 is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1 & \text{if } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$A = \{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} : 0 < x_1 < 1, 0 < x_2 < 1\}$$

Given y_1, y_2, solve $y_1 = x_1 + x_2, y_2 = x_1 - x_2$.

$$\Rightarrow x_1 = \frac{y_1 + y_2}{2} = w_1(y_1, y_2), x_2 = \frac{y_1 - y_2}{2} = w_2(y_1, y_2)$$

$$(1-1 \text{ transformation})$$

Jacobian is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

The joint p.d.f. of Y_1, Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(w_1, w_2)|J|, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in B$$

Marginal p.d.f. of Y_1, Y_2 are

$$f_{Y_1}(y_1) = \begin{cases} \int_{-y_1}^{y_1} \frac{1}{2} dy_2 = y_1 & , 0 < y_1 < 1 \\ \int_{y_1-2}^{2-y_1} \frac{1}{2} dy_2 = 1 - y_1 & , 1 < y_1 < 2 \\ 0 & , \text{elsewhere.} \end{cases}$$

$$f_{Y_2}(y_2) = \begin{cases} \int_{-y_2}^{2+y_2} \frac{1}{2} dy_1 = y_2 + 1 & , -1 < y_2 < 0 \\ \int_{y_2}^{2-y_2} \frac{1}{2} dy_1 = 1 - y_2 & , 0 < y_2 < 1 \\ 0 & , \text{elsewhere.} \end{cases}$$

Def. *If a sequence of r.v.'s $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.),then they are called a **random sample**.*

If $X_1, \ldots, X_n$ is a random sample from a distribution with p.d.f. f_0, then the joint p.d.f. of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_0(x_i), \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^n$$

Def. *Any function $g(X_1, \ldots, X_n)$ of a random sample $X_1, \ldots, X_n$ which is not dependent on a parameter θ is called a **statistic**.*

<u>Note</u> : If X is a random sample with p.d.f. $f(x, \theta)$, where θ is an unknown constant, then θ is called **parameter**.

For example, $N(\mu, \sigma^2)$: μ, σ^2 are parameters.
Poisson(λ) : λ is a parameter.

Example of statistics :
$X_1, \ldots, X_n$ are iid r.v.'s $\Rightarrow \overline{X}$ and S^2 are statistics.

<u>Note</u> : If $X_1, \ldots, X_n$ are r.v.'s, the m.g.f of $X_1, \ldots, X_n$ is

$$M_{X_1, \ldots, X_n}(t_1, \ldots, t_n) = \mathrm{E}(e^{t_1 X_1 + \cdots + t_n X_n})$$

m.g.f

$$M_x(t) = \mathrm{E}(e^{tx}) = \int e^{tx} f(x) dx$$

$$\longrightarrow D_t M_x(t) = D_t \mathrm{E}(e^{tx}) = D_t \int e^{tx} f(x) dx = \int D_t e^{tx} f(x) dx$$

Lemma. *X_1 and X_2 are independent if and only if*

$$M_{X_1,X_2}(t_1,t_2) = M_{X_1}(t_1)M_{X_2}(t_2), \forall t_1, t_2.$$

Proof. $\Rightarrow$) If X_1, X_2 are independent,

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \mathrm{E}(e^{t_1X_1+t_2X_2}) \\
&= \int\int e^{t_1x_1+t_2x_2} f(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty} e^{t_1x_1} f_{X_1}(x_1)dx_1 \int_{-\infty}^{\infty} e^{t_2x_2} f_{X_2}(x_2)dx_2 \\
&= \mathrm{E}(e^{t_1X_1})\mathrm{E}(e^{t_2X_2}) \\
&= M_{X_1}(t_1)M_{X_2}(t_2)
\end{aligned}$$

$\Leftarrow$)

$$M_{X_1,X_2}(t_1,t_2) = \mathrm{E}(e^{t_1X_1+t_2X_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(x_1,x_2)dx_1dx_2$$

$$\begin{aligned}
M_{X_1}(t_1)M_{X_2}(t_2) &= \mathrm{E}(e^{t_1X_1})\mathrm{E}(e^{t_2X_2}) \\
&= \int_{-\infty}^{\infty} e^{t_1x_1} f_{X_1}(x_1)dx_1 \int_{-\infty}^{\infty} e^{t_2x_2} f_{X_2}(x_2)dx_2 \\
&= \int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(x_1,x_2)dx_1dx_2
\end{aligned}$$

With $1-1$ correspondence between m.g.f and p.d.f,
then $f(x_1,x_2) = f_1(x_1)f_2(x_2), \forall x_1, x_2$
$\Rightarrow X_1, X_2$ are independent. $\square$

X and Y are independent, denote by $X \coprod Y$.

$$\begin{cases}
X \sim N(\mu,\sigma^2) & , M_x(t) = e^{\mu t + \frac{\sigma^2}{2}t^2}, \forall t \in R \\
X \sim \mathrm{Gamma}(\alpha,\beta) & , M_x(t) = (1-\beta t)^{-\alpha}, t < \frac{1}{\beta} \\
X \sim b(n,p) & , M_x(t) = (1-p+pe^t)^n, \forall t \in R \\
X \sim \mathrm{Poisson}(\lambda) & , M_x(t) = e^{\lambda(e^t-1)}, \forall t \in R
\end{cases}$$

Note :

(a) If $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ are independent, then $g(X_1, \ldots, X_n)$ and $h(Y_1, \ldots, Y_m)$ are also independent.

(b) If X , Y are independent, then

$$\mathrm{E}[g(X)h(Y)] = \mathrm{E}[g(X)]\mathrm{E}[h(Y)].$$

Thm. *If $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, then*

$$\begin{aligned}
&(a)\overline{X} \sim N(\mu, \frac{\sigma^2}{n})\\
&(b)\overline{X} \text{ and } S^2 \text{ are independent .}\\
&(c)\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)
\end{aligned}$$

Proof. (a) m.g.f. of $\overline{X}$ is

$$\begin{aligned}
M_{\overline{X}}(t) &= \mathrm{E}(e^{t\overline{X}}) = \mathrm{E}(e^{t\frac{1}{n}\sum_{i=1}^n X_i})\\
&= \mathrm{E}(e^{\frac{t}{n}X_1}e^{\frac{t}{n}X_2}\cdots e^{\frac{t}{n}X_n})\\
&= \mathrm{E}(e^{\frac{t}{n}X_1})\mathrm{E}(e^{\frac{t}{n}X_2})\mathrm{E}(e^{\frac{t}{n}X_n})\\
&= M_{X_1}(\frac{t}{n})M_{X_2}(\frac{t}{n})\cdots M_{X_n}(\frac{t}{n})\\
&= (e^{\mu\frac{t}{n}+\frac{\sigma^2}{2}(\frac{t}{n})^2})^n\\
&= e^{\mu t+\frac{\sigma^2/n}{2}t^2}
\end{aligned}$$

$\Rightarrow \overline{X} \sim (\mu, \frac{\sigma^2}{n})$
(b) First we want to show that $\overline{X}$ and $(X_1 - \overline{X}, X_2 - \overline{X}, \ldots, X_n - \overline{X})$ are

independent. Joint m.g.f. of $\overline{X}$ and $(X_1-\overline{X}, X_2-\overline{X}, \ldots, X_n-\overline{X})$ is

$$
\begin{aligned}
&M_{\overline{X}, X_1-\overline{X}, X_2-\overline{X}, \ldots, X_n-\overline{X}}(t, t_1, \ldots, t_n) \\
&= \mathrm{E}[e^{t\overline{X}+t_1(X_1-\overline{X})+\cdots+t_n(X_n-\overline{X})}] \\
&= \mathrm{E}[e^{\frac{t}{n}\sum_{i=1}^n X_i+\sum_{i=1}^n t_iX_i-\sum_{i=1}^n t_i\frac{\sum X_i}{n}}] \\
&= \mathrm{E}[e^{\sum_{i=1}^n(\frac{t}{n}+t_i-\bar{t})X_i}], \bar{t}=\frac{1}{n}\sum_{i=1}^n t_i \\
&= \mathrm{E}[e^{\sum_{i=1}^n \frac{n(t_i-\bar{t})+t}{n}X_i}] \\
&= \mathrm{E}[\prod_{i=1}^n e^{\frac{n(t_i-\bar{t})+t}{n}X_i}] \\
&= \prod_{i=1}^n e^{\mu\frac{n(t_i-\bar{t})+t}{n}+\frac{\sigma^2}{2}\frac{(n(t_i-\bar{t})+t)^2}{n^2}} \\
&= e^{\frac{\mu}{n}\sum\limits_{i=1}^n(n(t_i-\bar{t})+t)+\frac{\sigma^2}{2n^2}\sum\limits_{i=1}^n(n(t_i-\bar{t})+t)^2} \\
&= e^{\mu t+\frac{\sigma^2/n}{2}t^2+\mu\sum(t_i-\bar{t})+\frac{\sigma^2}{2}\sum(t_i-\bar{t})^2+\frac{\sigma^2}{n^2}nt\sum(t_i-\bar{t})} \\
&= e^{\mu t+\frac{\sigma^2/n}{2}t^2}e^{\frac{\sigma^2}{2}\sum(t_i-\bar{t})^2} \\
&= M_{\overline{X}}(t)M_{(X_1-\overline{X}, X_2-\overline{X}, \ldots, X_n-\overline{X})}(t_1, \ldots, t_n)
\end{aligned}
$$

$\Rightarrow \overline{X}$ and $(X_1-\overline{X}, X_2-\overline{X}, \ldots, X_n-\overline{X})$ are independent.
$\Rightarrow \overline{X}$ and $S^2=\frac{1}{n-1}\sum\limits_{i=1}^n(X_i-\overline{X})^2$ are independent.

(c)

(1) $Z \sim N(0,1), \Rightarrow Z^2 \sim \chi^2(1)$

(2)

$$X \sim \chi^2(r_1) \text{ and } Y \sim \chi^2(r_2) \text{ are independent. } \Rightarrow X+Y \sim \chi^2(r_1+r_2)$$

Proof. m.g.f. of $X+Y$ is

$$
\begin{aligned}
M_{X+Y}(t) &= \mathrm{E}(e^{t(X+Y)}) = \mathrm{E}(e^{tX+tY}) = \mathrm{E}(e^{tX})\mathrm{E}(e^{tY}) = M_X(t)M_Y(t) \\
&= (1-2t)^{-\frac{r_1}{2}}(1-2t)^{-\frac{r_2}{2}} = (1-2t)^{-\frac{r_1+r_2}{2}}
\end{aligned}
$$

$\Rightarrow X+Y \sim \chi^2(r_1+r_2)$

(3)

$$(X_1, \ldots, X_n) \overset{iid}{\sim} N(\mu, \sigma)$$

$$\frac{X_1-\mu}{\sigma}, \frac{X_2-\mu}{\sigma}, \ldots, \frac{X_n-\mu}{\sigma} \overset{iid}{\sim} N(0,1)$$

$$\frac{(X_1-\mu)^2}{\sigma^2}, \frac{(X_2-\mu)^2}{\sigma^2}, \ldots, \frac{(X_n-\mu)^2}{\sigma^2} \overset{iid}{\sim} \chi^2(1)$$

$$\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^2} = \frac{\sum\limits_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{(n-1)s^2}{\sigma^2} = \frac{\sum\limits_{i=1}^{n}(X_i-\overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\begin{aligned}
(1-2t)^{-\frac{n}{2}} &= M_{\frac{\sum(X_i-\mu)^2}{\sigma^2}}(t) = \mathrm{E}(e^{t\frac{\sum(X_i-\mu)^2}{\sigma^2}}) \\
&= \mathrm{E}(e^{t\frac{\sum(X_i-\overline{X}+\overline{X}-\mu)^2}{\sigma^2}}) = \mathrm{E}(e^{t\frac{\sum(X_i-\overline{X})^2+n(\overline{X}-\mu)^2}{\sigma^2}}) \\
&= \mathrm{E}(e^{t\frac{(n-1)s^2}{\sigma^2}} e^{t\frac{(\overline{X}-\mu)^2}{\sigma^2/n}}) \\
&= \mathrm{E}(e^{t\frac{(n-1)s^2}{\sigma^2}})\mathrm{E}(e^{t\frac{(\overline{X}-\mu)^2}{\sigma^2/n}}) \\
&= M_{\frac{(n-1)s^2}{\sigma^2}}(t) M_{\frac{(\overline{X}-\mu)^2}{\sigma^2/n}}(t) \\
&= M_{\frac{(n-1)s^2}{\sigma^2}}(t)(1-2t)^{-\frac{1}{2}}
\end{aligned}$$

$$\Rightarrow M_{\frac{(n-1)s^2}{\sigma^2}}(t) = (1-2t)^{-\frac{n-1}{2}} \Rightarrow \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

□