Chapter 11 Spectral Representation

Po-Ning Chen, Professor

Institute of Communications Engineering

National Chiao Tung University

Hsin Chu, Taiwan 30010, R.O.C.

11-1 Factorization and Innovations

Concern (for continuous-time processes)

- How to represent a real WSS process $\boldsymbol{x}(t)$ as a response of a *minimum-phase* system $\mathtt{L}(\omega)$ with a white input $\boldsymbol{i}(t)$ of unit power?

Definition (Minimum-phase system) A system is called *minimum-phase* if both $\mathtt{L}(\omega)$ and $1/\mathtt{L}(\omega)$ are causal and stable.
(A system is stable if a bounded input (BI) always induces a bounded output (BO). As a result, a linear system is stable in the BIBO sense if all poles of the system are in the strict left half of the s-plane.)

Definition (Causal filter) An causal filter is one whose output depends only on past and present inputs.

- A process that can be represented as a response of a *minimum-phase* system $\mathtt{L}(\omega)$ with a white input $\boldsymbol{i}(t)$ of unit power is called *regular*.

Oxford Dictionary - **Regular**. adj. Recurring at uniform intervals. Done or happening frequently.

- A formal definition of regular processes is given below.

Definition (Regular processes) A process $\boldsymbol{x}(t)$ is regular if

$$S_{xx}(\omega) = |\mathtt{L}(\omega)|^2$$

where $\mathtt{L}(s)$ $(s = j\omega)$ is analytic in the right-hand plane $\mathrm{Re}\{s\} > 0$.

- Roughly speaking, a function is analytic if its function values are determinate and finite (never indeterminate or infinity).

11-1 Factorization and Innovations

Filter with minimum group delay

- For all causal and stable systems that have the same magnitude response, the minimum phase system has the *minimum group delay*.
- Hence, a more appropriate name for *minimum-phase system* should be the "*minimum group delay*" system.
- We will come back to (provide a proof for) this later.

Some observations about regular process $\boldsymbol{x}(t)$

- $R_{ii}(\tau) = \delta(\tau) \Rightarrow S_{ii}(\omega) = 1$.
- $S_{xx}(\omega) = |\mathtt{L}(\omega)|^2 S_{ii}(\omega) = |\mathtt{L}(\omega)|^2$.
- So,

$$\boldsymbol{x}(t) = \int_{-\infty}^{\infty} \mathtt{l}(\tau)\boldsymbol{i}(t-\tau)d\tau,$$

where $\mathtt{L}(\omega)$ is *minimum-phase*, which is determined in terms of the desired *real, positive, even, finite-area* $S_{xx}(\omega)$, and $\mathtt{l}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathtt{L}(\omega)e^{j\omega\tau}d\omega$.

Terminologies and Paley-Wiener Condition

Innovation: $\boldsymbol{i}(t)$ is called the *innovation* of $\boldsymbol{x}(t)$.

Innovation Filter: $\mathtt{L}(\omega)$ is called the *innovation filter* of $\boldsymbol{x}(t)$.

Whitening Filter: $1/\mathtt{L}(\omega)$ is called the *whitening filter* of $\boldsymbol{x}(t)$.

Lemma (Paley-Wiener condition) A process $\boldsymbol{x}(t)$ is regular if the Paley-Wiener condition holds, i.e.,

$$\int_{-\infty}^{\infty} \frac{|\log S_{xx}(\omega)|}{1+\omega^2} d\omega < \infty.$$

- Hence, a BL process violates the Paley-Wiener condition.
- Paley-Wiener condition is only **sufficient**.
- We thus cannot prove that a BL process is not regular by showing it violates the Paley-Wiener condition.

Definition (Bandlimited processes) A process $\boldsymbol{x}(t)$ is called *bandlimited* (BL) if $\bar{S}_{xx}(\omega) = 0$ for $|\omega| > \sigma$, and $\bar{R}_{xx}(0) < \infty$.

Rational Spectra

How to find $\mathtt{L}(\omega)$ such that $|\mathtt{L}(\omega)|^2 = S(\omega)$ for a given real, positive, even, finite-area $S(\omega)$.

- Observation 1: $S(\omega) = S(-\omega)$ implies that $S(\omega)$ is a function of ω^2.
- Observation 2: $\mathtt{L}(\omega)$ can be easily determined if $S(\omega)$ is a rational spectrum.

Definition (Rational spectra) A rational spectrum is the ratio of two polynomials in ω^2:

$$S(\omega) = \frac{A(\omega^2)}{B(\omega^2)},$$

where $A(x)$ and $B(x)$ are both polynomials of x.

- Let $s = j\omega$. Then, $S(s) = A(-s^2)/B(-s^2)$.
- Observe that if s_i is a root (either zero or pole) of $S(s)$, $-s_i$ is also a root of $S(s)$. Also, the roots of $S(s)$ are either real or complex conjugate.
- Then, roots of $S(s)$ are symmetric with respect to the **imaginary axis**. So we can separate them into two groups: **Left** group that consists of all roots with $\mathrm{Re}\{s_i\} < 0$, and the **right** group that consists of all roots with $\mathrm{Re}\{s_i\} > 0$. (How to take care of those roots with $\mathrm{Re}\{s_i\} = 0$?)
- We can accordingly form $\mathtt{L}(s)$ by the ratio of two polynomials with the left roots of $S(s)$.

Example 11-1 $S(\omega) = \dfrac{N}{\alpha^2 + \omega^2}$.

Solution: $S(s) = \dfrac{N}{\alpha^2 - s^2} = \dfrac{N}{(|\alpha| + s)(|\alpha| - s)} \Rightarrow \mathtt{L}(s) = \dfrac{\sqrt{N}}{|\alpha| + s}$

$$\Rightarrow \mathtt{L}(\omega) = \frac{\sqrt{N}}{|\alpha| + j\omega} \qquad \left(\Rightarrow |\mathtt{L}(\omega)|^2 = \left| \frac{\sqrt{N}}{|\alpha| + j\omega} \right|^2 = \frac{N}{\alpha^2 + \omega^2} = S(\omega) \right)$$

$\square$

Example 11-2 $S(\omega) = \dfrac{49 + 25\omega^2}{9 + 10\omega^2 + \omega^4}$.

Solution: $S(s) = \dfrac{49 - 25s^2}{9 - 10s^2 + s^4} = \dfrac{(7 + 5s)(7 - 5s)}{(1 + s)(3 + s)(1 - s)(3 - s)}$

$$\Rightarrow \mathtt{L}(s) = \frac{7 + 5s}{(1 + s)(3 + s)} \qquad \Big(S(s) = \mathtt{L}(s)\mathtt{L}(-s) \Big)$$

$$\Rightarrow \mathtt{L}(\omega) = \frac{7 + 5j\omega}{(1 + j\omega)(3 + j\omega)}$$

$$\left(\Rightarrow |\mathtt{L}(\omega)|^2 = \left| \frac{7 + 5j\omega}{(1 + j\omega)(3 + j\omega)} \right|^2 = \frac{49 + 25\omega^2}{9 + 10\omega^2 + \omega^4} = S(\omega) \right)$$

$\square$

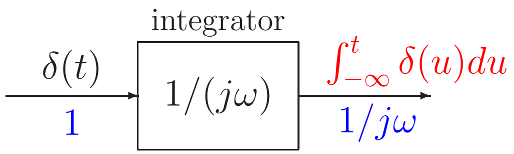
Minimum Phase System Revisited

Example 11-1

$$
\begin{aligned}
\mathtt{l}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathtt{L}(\omega) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sqrt{N}}{|\alpha| + j\omega} e^{j\omega\tau} d\omega \\
&= \sqrt{N} e^{-|\alpha|\tau} \int_{-\infty}^{\tau} \delta(u) du \\
&= \begin{cases} \sqrt{N} e^{-|\alpha|\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}
\end{aligned}
$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sqrt{N}}{|\alpha| + j\omega} e^{j\omega\tau} d\omega = \sqrt{N} e^{-|\alpha|\tau} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{j\omega'} e^{j\omega'\tau} d\omega' \right)$$

for $j\omega' = |\alpha| + j\omega$.

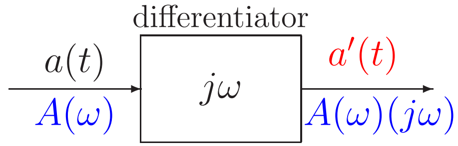


Example 11-2

$$
\begin{aligned}
\mathtt{l}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathtt{L}(\omega) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{7 + 5j\omega}{(1 + j\omega)(3 + j\omega)} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{1}{1 + j\omega} + \frac{4}{3 + j\omega} \right) e^{j\omega\tau} d\omega \\
&= \begin{cases} e^{-\tau} + 4e^{-3\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}
\end{aligned}
$$

Minimum Phase System Revisited

Example 11-1



$$
\begin{aligned}
\mathtt{l}_{\text{whitening}}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\mathtt{L}(\omega)} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\alpha| + j\omega}{\sqrt{N}} e^{j\omega\tau} d\omega \\
&= \frac{|\alpha|}{\sqrt{N}}\delta(\tau) + \frac{1}{2\pi\sqrt{N}}\int_{-\infty}^{\infty} (j\omega) e^{j\omega\tau} d\omega
\end{aligned}
$$

Example 11-2

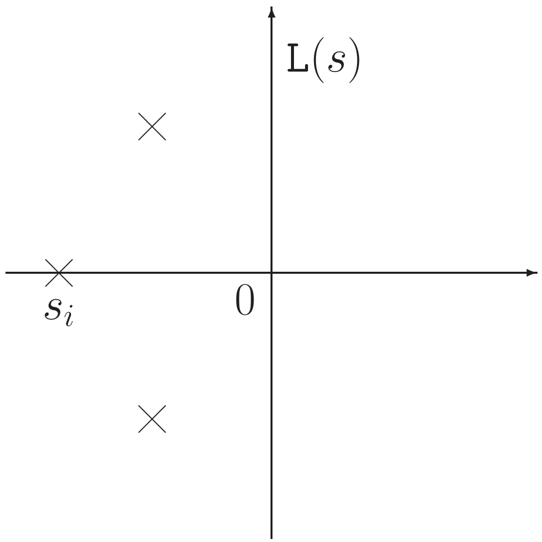
$$
\begin{aligned}
\mathtt{l}_{\text{whitening}}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\mathtt{L}(\omega)} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1+j\omega)(3+j\omega)}{7+5j\omega} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{5}\left(\frac{3+4(j\omega)+(j\omega)^2}{1.4+j\omega}\right) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{5}\left(-\frac{0.64}{1.4+j\omega} + 2.6 + j\omega\right) e^{j\omega\tau} d\omega \\
&= -0.128 e^{-1.4\tau}\mathbf{1}\{\tau > 0\} + 0.52\,\delta(\tau) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{5}(j\omega)\, e^{j\omega\tau} d\omega
\end{aligned}
$$

Minimum Phase System Revisited

Definition (Minimum-phase system) A system is called *minimum-phase* if both $\mathtt{L}(\omega)$ and $1/\mathtt{L}(\omega)$ are causal and stable.

Definition (Causal filter) A causal filter is one whose output depends only on past and present inputs.

Observation A system is minimum-phase if functions $\mathtt{L}(s)$ and $1/\mathtt{L}(s)$ are analytic in the right-hand plane $\mathrm{Re}\{s\} > 0$.
(I.e., no poles and zeros satisfy $\mathrm{Re}\{s\} > 0$.)



Implicitly, the above figure implies that $\mathtt{l}(\tau) \to 0$ as $\tau \to \infty$. See the two examples in the previous slide.

Minimum Phase System Revisited

Example 11-3 $S(\omega) = \dfrac{25}{\omega^4 + 1}$.

Solution: $S(s) = \dfrac{25}{s^4+1} = \dfrac{25}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)}$

$$\Rightarrow\ \mathtt{L}(s) = \frac{5}{s^2+\sqrt{2}s+1} \qquad \Big(S(s) = \mathtt{L}(s)\mathtt{L}(-s)\Big)$$

$$\Rightarrow\ \mathtt{L}(\omega) = \frac{5}{-\omega^2 + j\sqrt{2}\omega + 1}$$

$$\left(\ \Rightarrow\ |\mathtt{L}(\omega)|^2 = \left|\frac{5}{-\omega^2 + j\sqrt{2}\omega + 1}\right|^2 = \frac{25}{(1-\omega^2)^2 + 2\omega^2} = S(\omega)\right)$$

$$\begin{aligned}\Rightarrow\ \mathtt{l}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathtt{L}(\omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{5}{1-\omega^2+j\sqrt{2}\omega} e^{j\omega\tau} d\omega \\ &= \begin{cases} 5\sqrt{2}\sin(\tau/\sqrt{2})e^{-\tau/\sqrt{2}}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}\end{aligned}$$

$\square$

Minimum-Phase Discrete-Time Processes

Concern (for discrete-time processes)

- How to represent a real **discrete** WSS process $\boldsymbol{x}[t]$ as a response of a **discrete** *minimum-phase* system $\mathtt{L}[e^{j\omega}]$ with a **discrete** white input $\boldsymbol{i}[t]$ of unit power?

Definition (Minimum-phase system) A discrete system is called *minimum-phase* if both $\mathtt{L}[e^{j\omega}]$ and $1/\mathtt{L}[e^{j\omega}]$ are causal and stable.
(A system is stable if a bounded input (BI) always induces a bounded output (BO). As a result, a linear system is stable in the BIBO sense if all poles of the system are inside the unit circle in the z-plane.)

Definition (Causal filter) A causal filter is one whose output depends only on past and present inputs.

- A (discrete) process that can be represented as a response of a *minimum-phase* system $\mathtt{L}[e^{j\omega}]$ with a white input $\boldsymbol{i}[t]$ of unit power is called *regular*.
- A formal definition of (discrete) regular processes is given below.

Definition (Discrete regular processes) A process $\boldsymbol{x}[t]$ is regular if

$$S_{xx}[\omega] = |\mathtt{L}[e^{j\omega}]|^2$$

where $\mathtt{L}[z]$ $(z = e^{j\omega})$ is analytic for $|z| > 1$.

- Roughly speaking, a function is analytic if its function values are determinate and finite (never indeterminate or infinity).

Minimum-Phase Discrete-Time Processes

Some observations about $\boldsymbol{x}[t]$ so defined

- $R_{ii}[\tau] = \delta[\tau] \Rightarrow S_{ii}[\omega] = 1$, where $\delta[\tau] = \begin{cases} 1, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$ is the Kronecker delta function.
- $S_{xx}[\omega] = |\mathtt{L}[e^{j\omega}]|^2 S_{ii}[\omega] = |\mathtt{L}[e^{j\omega}]|^2$.
- So,

$$\boldsymbol{x}[t] = \sum_{\tau=-\infty}^{\infty} \mathtt{l}[\tau]\boldsymbol{i}[t-\tau],$$

where $\mathtt{L}[e^{j\omega}]$ is minimum-phase, determined in terms of the desired *real, positive, even, finite-area* $S_{xx}[\omega]$, and $\mathtt{l}[\tau] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathtt{L}[e^{j\omega}]e^{j\omega\tau}d\omega$.

Conveniently, we will sometimes write $\mathtt{L}[e^{j\omega}]$ as $\mathtt{L}[\omega]$. These two expressions are actually equivalent.

Terminologies and Paley-Wiener Condition

Innovation: $\boldsymbol{i}[t]$ is called the *innovations* of $\boldsymbol{x}[t]$.

Innovation Filter: $\mathtt{L}[\omega]$ is called the *innovation filter* of $\boldsymbol{x}[t]$.

Whitening Filter: $1/\mathtt{L}[\omega]$ is called the *whitening filter* of $\boldsymbol{x}[t]$.

Lemma (Paley-Wiener condition) A process $\boldsymbol{x}[t]$ is regular if the Paley-Wiener condition holds, i.e.,

$$\int_{-\pi}^{\pi} \left|\log S_{xx}[\omega]\right| d\omega < \infty.$$

If $S_{xx}[\omega]$ is an integrable function, then the above condition reduces to

$$\int_{-\pi}^{\pi} \log(S_{xx}[\omega]) d\omega > -\infty.$$

- Obviously, $\displaystyle\int_{-\pi}^{\pi} \left|\log S_{xx}[\omega]\right| d\omega < \infty$ implies $\displaystyle\int_{-\pi}^{\pi} \log(S_{xx}[\omega]) d\omega > -\infty$.
  We need to prove the converse is also true if $S_{xx}[\omega]$ integrable.

Claim $\int_{-\pi}^{\pi} |S[\omega]| d\omega < \infty$ and $\int_{-\pi}^{\pi} \log(S[\omega]) d\omega > -\infty \Rightarrow \int_{-\pi}^{\pi} |\log S[\omega]|\, d\omega < \infty$.

Proof: By

$$\int_{\{\omega\in[-\pi,\pi):S[\omega]<1\}} \log(S[\omega])d\omega = \int_{-\pi}^{\pi} \log(S[\omega])d\omega - \int_{\{\omega\in[-\pi,\pi):S[\omega]\geq 1\}} \log(S[\omega])d\omega,$$

we derive:

$$\begin{aligned}
\int_{-\pi}^{\pi} |\log(S[\omega])|d\omega &= \int_{\{\omega\in[-\pi,\pi):S[\omega]<1\}} |\log(S[\omega])|d\omega + \int_{\{\omega\in[-\pi,\pi):S[\omega]\geq 1\}} |\log(S[\omega])|d\omega \\
&= -\int_{\{\omega\in[-\pi,\pi):S[\omega]<1\}} \log(S[\omega])d\omega + \int_{\{\omega\in[-\pi,\pi):S[\omega]\geq 1\}} \log(S[\omega])d\omega \\
&= -\int_{-\pi}^{\pi} \log(S[\omega])d\omega + 2\int_{\{\omega\in[-\pi,\pi):S[\omega]\geq 1\}} \log(S[\omega])d\omega \\
&\leq -\int_{-\pi}^{\pi} \log(S[\omega])d\omega + 2\int_{\{\omega\in[-\pi,\pi):S[\omega]\geq 1\}} (S[\omega]-1)d\omega \\
&\leq -\int_{-\pi}^{\pi} \log(S[\omega])d\omega + 2\int_{-\pi}^{\pi} (|S[\omega]|+1)d\omega < \infty.
\end{aligned}$$

$\square$

Terminologies and Paley-Wiener Condition

Theorem (Page 424 in textbook: Chapter 9) There exists a unique function

$$H[z] = \sum_{k=0}^{\infty} h[k]z^{-k} \text{ for } h[0] > 0 \text{ and } |z| > 1$$

that is analytic together with its inverse in $|z| > 1$ satisfying

$$\sum_{k=0}^{\infty} |h[k]|^2 < \infty \quad \text{and} \quad S[\omega] = |H[e^{-j\omega}]|^2 \; a.e.,$$

if, and only if, $S[\omega]$ as well as $\log(S[\omega])$ are integrable functions over $[-\pi, \pi)$, where

$$H[e^{-j\omega}] = \lim_{r \downarrow 1} H[re^{-j\omega}]$$

is defined as the exterior radial limit of $H[z]$ on the unit circle.

Rational Spectra for Discrete-Time Processes

How to find $\mathtt{L}[\omega]$ such that $|\mathtt{L}[\omega]|^2 = S[\omega]$ for a real, positive, even, finite-area $S[\omega]$.

- Observation 1: $S[\omega]\big(= S[e^{j\omega}] = S[e^{-j\omega}]\big) = S[-\omega]$ implies that $S[\omega]$ is a function of $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.
- Observation 2: $\mathtt{L}[\omega]$ can be easily determined if $S[\omega]$ is a rational spectrum.

Definition (Rational spectra) A rational spectrum is the ratio of two polynomials in $\cos(\omega)$:

$$S[\omega] = \frac{A(\cos(\omega))}{B(\cos(\omega))},$$

where $A(x)$ and $B(x)$ are both polynomials of x.

- Let $z = e^{j\omega}$. Then, $S[z] = A((z + z^{-1})/2)/B((z + z^{-1})/2)$.
- Observe that if z_i is a root (zero or pole) of $S[z]$, $1/z_i$ is also a root of $S[z]$. Also, the roots of $S[z]$ are either real or complex conjugate.
- Then, the roots of $S[z]$ are symmetric with respect to the **unit circle**. So we can separate them into two groups: **Inside** group that consists of all roots with $|z| < 1$, and the **outside** group that consists of all roots with $|z| > 1$.
- Form $\mathtt{L}[z]$ by the ratio of two polynomials with the **inside** roots of $S[z]$.

Example 11-4 $S[\omega] = \dfrac{5 - 4\cos(\omega)}{10 - 6\cos(\omega)}$.

Solution: $S[z] = \dfrac{5 - 2(z + z^{-1})}{10 - 3(z + z^{-1})} = \dfrac{2(1 - (1/2)z^{-1})}{3(1 - (1/3)z^{-1})} \cdot \dfrac{2(1 - (1/2)z)}{3(1 - (1/3)z)}$

$$\Rightarrow \mathtt{L}[z] = \frac{2(1 - (1/2)z^{-1})}{3(1 - (1/3)z^{-1})} \quad \Big(S[z] = \mathtt{L}[z]\mathtt{L}[1/z] \Big)$$

$$\Rightarrow \mathtt{L}[\omega] = \frac{2(1 - (1/2)e^{-j\omega})}{3(1 - (1/3)e^{-j\omega})}$$

$$\left(\Rightarrow |\mathtt{L}[\omega]|^2 = \left| \frac{2(1 - (1/2)e^{-j\omega})}{3(1 - (1/3)e^{-j\omega})} \right|^2 = S[\omega] \right)$$

$$
\begin{aligned}
\mathtt{l}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathtt{L}[\omega] e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2(1-(1/2)e^{-j\omega})}{3(1-(1/3)e^{-j\omega})} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - \frac{1/3}{1-(1/3)e^{-j\omega}}\right) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - 3^{-1}\left[1 + 3^{-1}e^{-j\omega} + 3^{-2}e^{-j2\omega} + \cdots\right]\right) e^{j\omega\tau} d\omega \\
&= \begin{cases} 0, & \tau < 0 \\ 1 - 3^{-1}, & \tau = 0 \\ -3^{-(1+\tau)}, & \tau > 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{l}_{\text{whitening}}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\mathtt{L}[\omega]} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{3(1-(1/3)e^{-j\omega})}{2(1-(1/2)e^{-j\omega})} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 + \frac{1/2}{1-(1/2)e^{-j\omega}}\right) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 + 2^{-1}\left[1 + 2^{-1}e^{-j\omega} + 2^{-2}e^{-j2\omega} + \cdots\right]\right) e^{j\omega\tau} d\omega \\
&= \begin{cases} 0, & \tau < 0 \\ 1 + 2^{-1}, & \tau = 0 \\ 2^{-(1+\tau)}, & \tau > 0 \end{cases}
\end{aligned}
$$

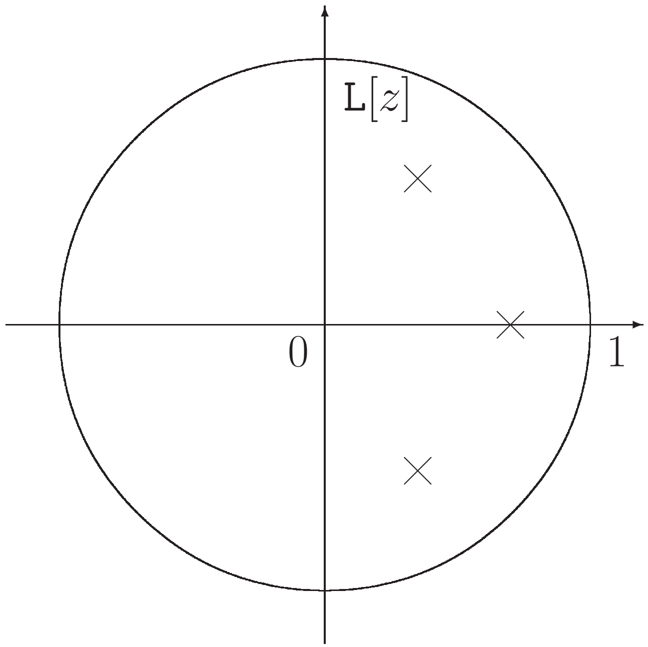
$\square$

Rational Spectra for Discrete-Time Processes

Definition (Minimum-phase system) A system is called *minimum-phase* if both $\mathtt{L}[\omega]$ and $1/\mathtt{L}[\omega]$ are causal and stable.

Definition (Causal filter) A causal filter is one whose output depends only on past and present inputs.

Observation A discrete system is minimum-phase if functions $\mathtt{L}[z]$ and $1/\mathtt{L}[z]$ are analytic in the exterior $|z| > 1$ of the unit circle.



Properties Regarding Minimum-Phase

Filter with minimum group delay

- For all causal and stable systems that have the same magnitude response, the minimum phase system has the *minimum group delay*.
- Hence, a more appropriate name for *minimum-phase system* is the "*minimum group delay*" system.

Delay of a filter: What is a proper definition for filter delay?

$$\boldsymbol{x}[t] \longrightarrow \boxed{\mathtt{L}[\omega] = e^{-j\omega n}} \longrightarrow \boldsymbol{y}[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\boldsymbol{X}[\omega]e^{-j\omega n}\right)e^{j\omega t}d\omega = \boldsymbol{x}[t-n]$$

Hence, the delay of a filter can be "defined" as:

$$-\frac{d}{d\omega}\left(\arg\{\mathtt{L}[\omega]\}\right) = -\frac{d}{d\omega}\left(-\omega n\right) = n.$$

The filter is named *minimum phase* due to that it minimizes the "**phase change**."

Properties Regarding Minimum-Phase

Example: **Delay of a filter with a single zero**

$$\boldsymbol{x}[t] \longrightarrow \boxed{\mathtt{L}[z] = 1 - z_i z^{-1}} \longrightarrow \boldsymbol{y}[t] = \boldsymbol{x}[t] - z_i \boldsymbol{x}[t-1]$$

$$\begin{aligned}
-\frac{d}{d\omega}\left(\arg\{1 - z_i e^{-j\omega}\}\right) &= -\frac{d}{d\omega}\left(\arg\left\{|z_i|^{-1} - e^{-j(\omega - \arg\{z_i\})}\right\}\right)\\
&= -\frac{d}{d\omega}\left(\arg\left\{|z_i|^{-1} - \cos(\omega - \arg\{z_i\}) + j\sin(\omega - \arg\{z_i\})\right\}\right)\\
&= -\frac{d}{d\omega}\left(\tan^{-1}\left[\frac{\sin(\omega - \arg\{z_i\})}{|z_i|^{-1} - \cos(\omega - \arg\{z_i\})}\right]\right) \qquad \boxed{\text{See the next Slide.}}\\
&= \frac{|z_i| - \cos(\omega - \arg\{z_i\})}{|z_i| + |z_i|^{-1} - 2\cos(\omega - \arg\{z_i\})}.
\end{aligned}$$

Apparently, the choice between $z_i = |z_i| e^{j\arg\{z_i\}}$ and $1/z_i^* = |z_i|^{-1} e^{j\arg\{z_i\}}$ (or z_i^* and $1/z_i$), which minimizes the *filter delay*, is the one lying in the interior of unit circle since

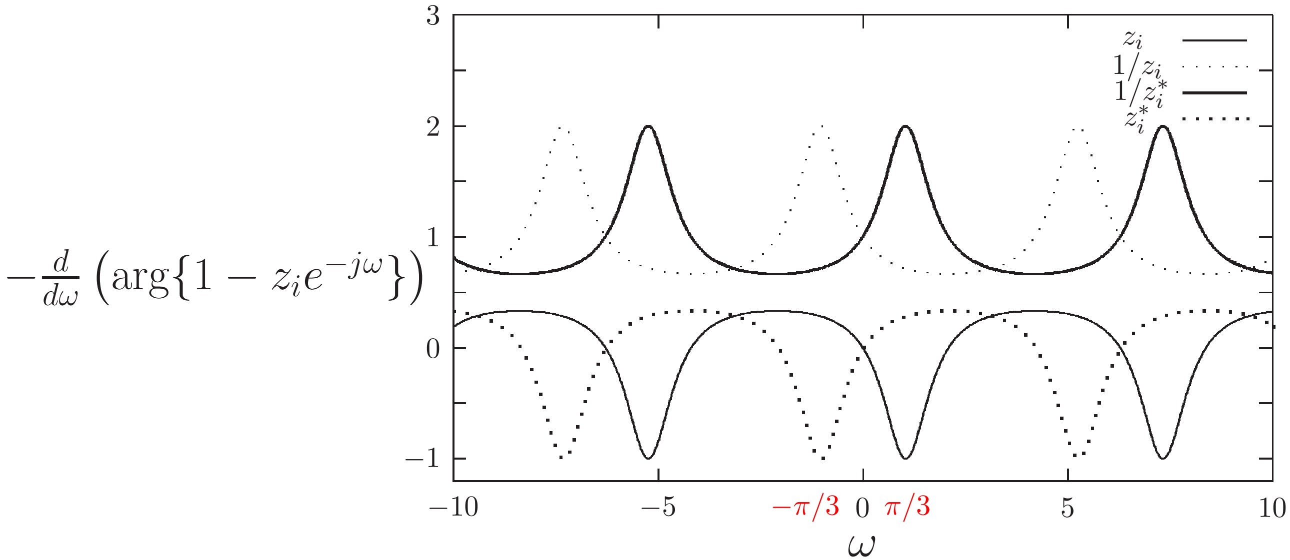
$$-\frac{d}{d\omega}\left(\arg\{1 - z_i e^{-j\omega}\}\right) = \frac{|z_i| + \text{fixed}}{\text{fixed}}.$$

Properties Regarding Minimum-Phase

$$\frac{d}{d\omega}\tan^{-1}\left(\frac{a}{b}\right)=\frac{b\,a'-a\,b'}{a^2+b^2}$$

$$\begin{aligned}
&\frac{d}{d\omega}\left(\tan^{-1}\left[\frac{\sin(\omega-\arg\{z_i\})}{|z_i|^{-1}-\cos(\omega-\arg\{z_i\})}\right]\right)\\
&=\frac{[|z_i|^{-1}-\cos(\omega-\arg\{z_i\})]\left(\frac{d}{d\omega}[\sin(\omega-\arg\{z_i\})]\right)}{\sin^2(\omega-\arg\{z_i\})+[|z_i|^{-1}-\cos(\omega-\arg\{z_i\})]^2}\\
&\qquad-\frac{\sin(\omega-\arg\{z_i\})\left(\frac{d}{d\omega}[|z_i|^{-1}-\cos(\omega-\arg\{z_i\})]\right)}{\sin^2(\omega-\arg\{z_i\})+[|z_i|^{-1}-\cos(\omega-\arg\{z_i\})]^2}\\
&=\frac{[|z_i|^{-1}-\cos(\omega-\arg\{z_i\})]\cos(\omega-\arg\{z_i\})-\sin^2(\omega-\arg\{z_i\})}{1-2|z_i|^{-1}\cos(\omega-\arg\{z_i\})+|z_i|^{-2}}\\
&=\frac{|z_i|^{-1}\cos(\omega-\arg\{z_i\})-1}{1-2|z_i|^{-1}\cos(\omega-\arg\{z_i\})+|z_i|^{-2}}\\
&=\frac{\cos(\omega-\arg\{z_i\})-|z_i|}{|z_i|-2\cos(\omega-\arg\{z_i\})+|z_i|^{-1}}
\end{aligned}$$

Example. Take $|z_i| = \frac{1}{2}$ and $\arg\{z_i\} = \frac{\pi}{3}$.



- The figure shows that

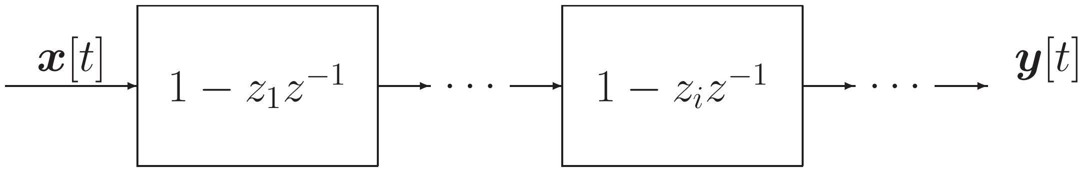
$$\frac{|z_i|-\cos(\omega-\arg\{z_i\})}{|z_i|+|z_i|^{-1}-2\cos(\omega-\arg\{z_i\})} \quad \text{and} \quad \frac{|z_i^*|-\cos(\omega-\arg\{z_i^*\})}{|z_i^*|+|z_i^*|^{-1}-2\cos(\omega-\arg\{z_i^*\})}$$

  are identical except with some shift.

- The figure shows that the phase change corresponding to z_i is always smaller than that corresponding to $1/z_i$ (and of course, to $1/z_i^*$). So, the choice between $z_i = |z_i|e^{j\arg\{z_i\}}$ and $1/z_i = |z_i|^{-1}e^{-j\arg\{z_i\}}$, which minimizes the *filter delay*, is the one lying in the interior of unit circle.

Properties Regarding Minimum-Phase

Delay of a filter with multiple zeros $\mathtt{L}[z] = \prod_i (1 - z_i z^{-1})$



Choose half of the zeros, among all the zero-pairs (one in inside group and one in outside group) of the target $S[z] = \mathtt{L}[z]\mathtt{L}[1/z]$, such that the group delay is minimized.

Apparently, the choice of all zeros in the inside group will satisfy the need.

Properties Regarding Minimum-Phase

Delay of a filter with a single pole

$$\boldsymbol{x}[t] \longrightarrow \boxed{\mathtt{L}[z] = 1/(1-p_i z^{-1})} \longrightarrow \boldsymbol{y}[t] = \boldsymbol{x}[t] + p_i \boldsymbol{y}[t-1]$$

$$\mathtt{L}[e^{j\omega}] = \frac{1 - p_i^* e^{j\omega}}{|1 - p_i e^{-j\omega}|^2} \Rightarrow \arg\left\{\mathtt{L}[e^{j\omega}]\right\} = \arg\left\{1 - p_i^* e^{j\omega}\right\}$$

and

$$-\frac{d}{d\omega}\left(\arg\{1 - p_i^* e^{j\omega}\}\right) \;=\; \frac{|p_i^*| - \cos(\omega + \arg\{p_i^*\})}{|p_i^*| + |p_i^*|^{-1} - 2\cos(\omega + \arg\{p_i^*\})}.$$

Again, the choice between $p_i^* = |p_i^*|e^{j\arg\{p_i^*\}}$ and $1/p_i = |p_i^*|^{-1}e^{j\arg\{p_i^*\}}$ (or p_i and $1/p_i^*$), which minimizes the *filter delay*, is the one lying in the interior of unit circle since

$$-\frac{d}{d\omega}\left(\arg\{1 - p_i^* e^{j\omega}\}\right) \;=\; \frac{|p_i^*| + \text{fixed}}{\text{fixed}}.$$

All the conclusions for zeros can be applied to poles.

Properties Regarding Minimum-Phase

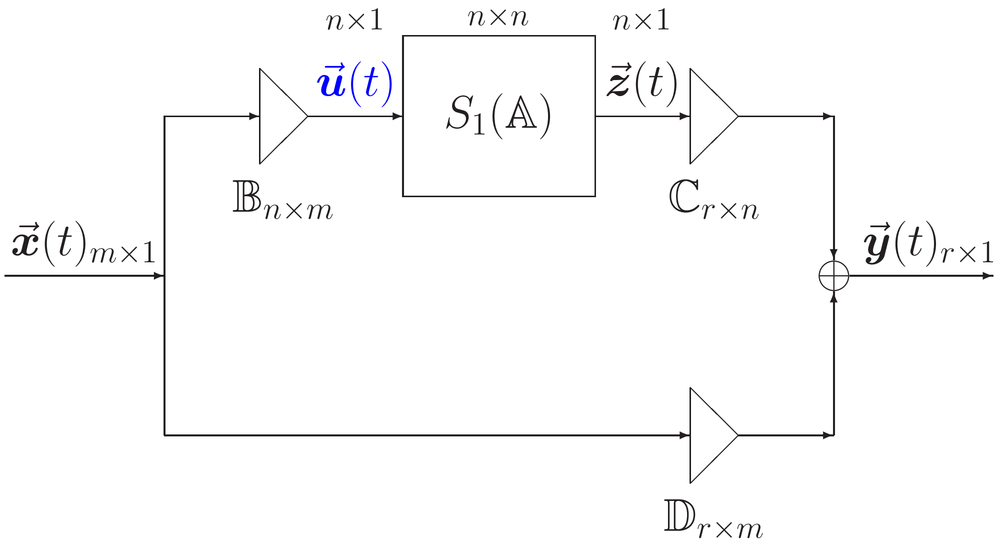
Delay of a filter with multiple zeros and multiple poles

$$\mathtt{L}[z] = \frac{\prod_i (1 - z_i z^{-1})}{\prod_k (1 - p_k z^{-1})}$$

Choose half of the zeros and poles, among all the zero-pairs and pole-pairs (one in inside group and one in outside group) of the desired $S[z] = \mathtt{L}[z]\mathtt{L}[1/z]$ such that the group delay is minimized.

Apparently, the choice of all zeros and poles in the inside group will satisfy the need.

$\boxed{\textbf{The end of Section 11-1 Factorization and Innovations}}$



A system with state variables

- Consider a system with input $\vec{\boldsymbol{x}}(t)$ and output $\vec{\boldsymbol{y}}(t)$, in which their relationship is defined through an internal state variable $\vec{\boldsymbol{z}}(t)$ as:

$$\begin{cases} \frac{d}{dt}\vec{\boldsymbol{z}}(t) &= \mathbb{A}\vec{\boldsymbol{z}}(t) + \vec{\boldsymbol{u}}(t) = \mathbb{A}\vec{\boldsymbol{z}}(t) + \mathbb{B}\vec{\boldsymbol{x}}(t) \\ \vec{\boldsymbol{y}}(t) &= \mathbb{C}\vec{\boldsymbol{z}}(t) + \mathbb{D}\vec{\boldsymbol{x}(t)} \end{cases} \qquad (*)$$

The relationship between input $\vec{\boldsymbol{u}}(t)$ and output $\vec{\boldsymbol{z}}(t)$ of the subsystem S_1 is given by $(*)$.

Terminology

- The order of the system is defined as the dimension of the state variable $\vec{\boldsymbol{z}}(t)$, which is n in our case.

Derivation of the impulse response

- The impulse response of the subsystem S_1 can be derived from relationship

  $\boxed{\vec{\boldsymbol{z}}(t)_{n\times 1} = \int_{-\infty}^{\infty} \phi(\alpha)_{n\times n}\vec{\boldsymbol{u}}(t-\alpha)_{n\times 1}d\alpha}$ equivalently $\boxed{\vec{\boldsymbol{z}}(s)_{n\times 1} = \phi(s)_{n\times n}\vec{\boldsymbol{u}}(s)_{n\times 1}}$.

  Taking the Laplace transform of both sides of Eq. $(*)$ yields:

$$
\begin{aligned}
& s\vec{\boldsymbol{z}}(s)_{n\times 1} = \mathbb{A}_{n\times n}\vec{\boldsymbol{z}}(s)_{n\times 1} + \vec{\boldsymbol{u}}(s)_{n\times 1} \\
& \Rightarrow\ s\phi(s)_{n\times n}\vec{\boldsymbol{u}}(s)_{n\times 1} = \mathbb{A}_{n\times n}\phi(s)_{n\times n}\vec{\boldsymbol{u}}(s)_{n\times 1} + \vec{\boldsymbol{u}}(s)_{n\times 1} \\
& \Rightarrow\ s\Phi(s)_{n\times n} - \mathbb{A}_{n\times n}\Phi(s)_{n\times n} = \mathbb{I}_{n\times n} \\
& \Rightarrow\ \Phi(s)_{n\times n} = \left(s\mathbb{I}_{n\times n} - \mathbb{A}_{n\times n}\right)^{-1} \\
& \Rightarrow\ \phi(t)_{n\times n} = \exp\left\{\mathbb{A}_{n\times n}t\right\} \quad t > 0.
\end{aligned}
$$

  where $\mathbb{I}$ is the identity matrix.

$$e^{\mathbb{A}t} \triangleq \begin{cases} \mathbb{S}e^{\Lambda t}\mathbb{S}^{-1} = \mathbb{S}\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}\mathbb{S}^{-1}, & \text{if } \mathbb{S}^{-1} \text{ exists} \\ \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}(\mathbb{A}t)^k, & \text{holds no matter whether } \mathbb{S}^{-1} \text{ exists or not} \end{cases}$$

where λ_1, λ_2, ..., λ_n are the eigenvalues of $\mathbb{A}$, and $\mathbb{S}$ is the matrix with its columns being the linearly independent eigenvectors of $\mathbb{A}$.

Derivation of the impulse response (continued)

- For the overall system,

$$\begin{aligned}
\vec{\boldsymbol{y}}(t)_{r\times 1} &= \mathbb{C}_{r\times n}\vec{\boldsymbol{z}}(t)_{n\times 1} + \mathbb{D}_{r\times m}\vec{\boldsymbol{x}(t)}_{m\times 1} \\
&= \int_{-\infty}^{\infty} \mathbb{C}_{r\times n}\phi(\alpha)_{n\times n}\vec{\boldsymbol{u}}(t-\alpha)_{n\times 1}d\alpha + \int_{-\infty}^{\infty} \delta(\alpha)\mathbb{D}_{r\times m}\vec{\boldsymbol{x}}(t-\alpha)_{m\times 1}d\alpha \\
&= \int_{-\infty}^{\infty} \mathbb{C}_{r\times n}\phi(\alpha)_{n\times n}\mathbb{B}_{n\times m}\vec{\boldsymbol{x}}(t-\alpha)_{m\times 1}d\alpha + \int_{-\infty}^{\infty} \delta(\alpha)\mathbb{D}_{r\times m}\vec{\boldsymbol{x}}(t-\alpha)_{m\times 1}d\alpha \\
&= \int_{-\infty}^{\infty} \left(\mathbb{C}_{r\times n}\phi(\alpha)_{n\times n}\mathbb{B}_{n\times m} + \delta(\alpha)\mathbb{D}_{r\times m}\right)\vec{\boldsymbol{x}}(t-\alpha)_{m\times 1}d\alpha.
\end{aligned}$$

Hence,

$$h(t)_{r\times m} = \mathbb{C}_{r\times n}\phi(t)_{n\times n}\mathbb{B}_{n\times m} + \delta(t)\mathbb{D}_{r\times m}$$

and

$$H(s)_{r\times m} = \mathbb{C}_{r\times n}\Phi(s)_{n\times n}\mathbb{B}_{n\times m}+\mathbb{D}_{r\times m} = \boxed{\mathbb{C}_{r\times n}\left(s\mathbb{I}_{n\times n} - \mathbb{A}_{n\times n}\right)^{-1}\mathbb{B}_{n\times m} + \mathbb{D}_{r\times m}}.$$

By Theorem 9-4 that tells:

$$\begin{cases} S_{xy}(\omega)_{m\times r} & = & S_{xx}(\omega)_{m\times m} H^{\dagger}(\omega)_{r\times m} \\ S_{yy}(\omega)_{r\times r} & = & H(\omega)_{r\times m} S_{xy}(\omega)_{m\times r} \\ & = & H(\omega)_{r\times m} S_{xx}(\omega)_{m\times m} H^{\dagger}(\omega)_{r\times m} \end{cases}$$

we can infer that,

$$\begin{cases} S_{xy}(s)_{m\times r} & = & S_{xx}(s)_{m\times m} H^{\dagger}(-s)_{r\times m} \\ S_{yy}(s)_{r\times r} & = & H(s)_{r\times m} S_{xy}(s)_{m\times r} \\ & = & H(s)_{r\times m} S_{xx}(s)_{m\times m} H^{\dagger}(-s)_{r\times m} \end{cases}$$

Example 1

- Suppose $r = n = m$ and $\mathbb{B}_{n\times n} = \mathbb{C}_{n\times n} = \mathbb{I}_{n\times n}$ and $\mathbb{D}_{n\times n} = \mathbf{0}_{n\times n}$. Then,

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\vec{\boldsymbol{z}}(t) & = & \mathbb{A}\vec{\boldsymbol{z}}(t) + \vec{\boldsymbol{u}}(t) = \mathbb{A}\vec{\boldsymbol{z}}(t) + \vec{\boldsymbol{x}}(t) \\ \vec{\boldsymbol{y}}(t) & = & \vec{\boldsymbol{z}}(t) \end{array} \right.$$

implies

$$\frac{d}{dt}\vec{\boldsymbol{y}}(t) = \mathbb{A}\vec{\boldsymbol{y}}(t) + \vec{\boldsymbol{x}}(t).$$

- Then,

$$H(s)_{n\times n} = \mathbb{C}_{r\times n}\left(s\mathbb{I}_{n\times n} - \mathbb{A}_{n\times n}\right)^{-1}\mathbb{B}_{n\times m} + \mathbb{D}_{r\times m} = \boxed{\left(s\mathbb{I}_{n\times n} - \mathbb{A}_{n\times n}\right)^{-1}}\,.$$

Example 2

- Suppose
$$\boldsymbol{y}^{(n)}(t) + a_1\boldsymbol{y}^{(n-1)}(t) + \cdots + a_n\boldsymbol{y}(t) = \boldsymbol{x}(t).$$

- By assuming that $\vec{\boldsymbol{z}}(t) = [\boldsymbol{y}(t), \boldsymbol{y}^{(1)}(t), \cdots, \boldsymbol{y}^{(n-1)}(t)]^{\mathsf{T}}$, the system can be equivalently transformed to:

$$\begin{cases} \dfrac{d}{dt}\vec{\boldsymbol{z}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}}_{\mathbb{A}} \vec{\boldsymbol{z}}(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbb{B}} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{\mathbb{C}} \vec{\boldsymbol{z}}(t) + \underbrace{0}_{\mathbb{D}} \end{cases}$$

- Hence,

$$
\begin{aligned}
H(s)_{r\times m} &= \boxed{\mathbb{C}_{r\times n}\left(s\mathbb{I}_{n\times n}-\mathbb{A}_{n\times n}\right)^{-1}\mathbb{B}_{n\times m}+\mathbb{D}_{r\times m}} \\
&= \mathbb{C}_{1\times n}\left(s\mathbb{I}_{n\times n}-\mathbb{A}_{n\times n}\right)^{-1}\mathbb{B}_{n\times 1} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}
\begin{bmatrix}
s & -1 & 0 & \cdots & 0 \\
0 & s & -1 & \cdots & 0 \\
0 & 0 & s & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & a_{n-1} & a_{n-2} & \cdots & s+a_1
\end{bmatrix}^{-1}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
&= \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}.
\end{aligned}
$$

□

Finite-Order Processes

Definition (Finite-order processes) A (WSS) process $\boldsymbol{x}(t)$ is of finite order if its innovation filter is a rational function of s, i.e.,

$$\mathtt{L}(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)},$$

satisfying that $N(s)$ and $D(s)$ are Hurwitz polynomials.

A Hurwitz polynomial is a polynomial whose zeros are located in the left half-plane of the complex plane, namely, the real part of every zero is negative.

Autocorrelation function of finite-order process $\boldsymbol{x}(t)$

- Let $\{s_i\}_{i=1}^n$ be the roots of $D(s)$, and assume $m < n$.
- Then, $\mathtt{L}(s)$ can be expanded into partial fractions as:

$$\mathtt{L}(s) = \sum_{i=1}^n \frac{\gamma_i}{s - s_i} \quad \text{and} \quad \mathtt{l}(\tau) = \sum_{i=1}^n \gamma_i e^{s_i \tau} \int_{-\infty}^{\tau} \delta(u)du.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\gamma_i}{j\omega - s_i} e^{j\omega\tau} d\omega = \gamma_i e^{s_i\tau}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega'} e^{j\omega'\tau} d\omega'\right) \text{ for } j\omega' = j\omega - s_i.$$

See Slide 11-6.

Finite-Order Processes

- We can then derive:

$$
\begin{aligned}
S_{xx}(s) &= \mathtt{L}(s)\mathtt{L}(-s) \\
&= \left(\sum_{i=1}^{n} \frac{\gamma_i}{s - s_i}\right)\left(\sum_{k=1}^{n} \frac{\gamma_k}{-s - s_k}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\gamma_i \gamma_k}{(s - s_i)(-s - s_k)} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{-\gamma_i\gamma_k/(s_i + s_k)}{s - s_i} + \frac{-\gamma_i\gamma_k/(s_i + s_k)}{-s - s_k}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{-\gamma_i\gamma_k/(s_i + s_k)}{s - s_i} + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{-\gamma_i\gamma_k/(s_i + s_k)}{-s - s_k} \\
&= \sum_{i=1}^{n} \frac{\alpha_i}{s - s_i} + \sum_{k=1}^{n} \frac{\alpha_k}{-s - s_k} \triangleq S_{xx}^{+}(s) + S_{xx}^{+}(-s),
\end{aligned}
$$

where

$$
\alpha_k = \gamma_k \sum_{i=1}^{n} \frac{\gamma_i}{-s_k - s_i} = \gamma_k\, \mathtt{L}(-s_k).
$$

Finite-Order Processes

- This gives that:

$$
\begin{aligned}
R_{xx}^{+}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}^{+}(\omega)e^{j\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n}\frac{\alpha_i}{j\omega - s_i}\right)e^{j\omega\tau}d\omega \\
&= \begin{cases} \displaystyle\sum_{i=1}^{n}\alpha_i e^{s_i\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
R_{xx}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}(S_{xx}^{+}(\omega) + S_{xx}^{+}(-\omega))e^{j\omega\tau}d\omega \\
&= R_{xx}^{+}(\tau) + R_{xx}^{+}(-\tau) \\
&= R_{xx}^{+}(|\tau|) \quad (\text{for } \tau \neq 0)
\end{aligned}
$$

Example 11-5 $\mathtt{L}(s) = 1/(s+\alpha)$

Solution:

$$S_{xx}(s) = \frac{1}{(s+\alpha)(-s+\alpha)} = \frac{1/(2\alpha)}{s+\alpha} + \frac{1/(2\alpha)}{-s+\alpha}.$$

Then,

$$R_{xx}(\tau) = \frac{1}{2\alpha} e^{-\alpha|\tau|}.$$

□

Example 11-6 $\boldsymbol{x}''(t) + 3\boldsymbol{x}'(t) + 2\boldsymbol{x}(t) = \boldsymbol{i}(t)$.

Solution

$$\mathtt{L}(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} + \frac{-1}{s+2}$$

$$\Rightarrow S_{xx}(s) = \mathtt{L}(s)\mathtt{L}(-s) = \frac{1}{(s^2+3s+2)(s^2-3s+2)} = \frac{s/12+1/4}{s^2+3s+2} + \frac{-s/12+1/4}{s^2-3s+2}.$$

Hence,

$$S_{xx}^{+}(s) = \frac{1/6}{s+1} + \frac{(-1/12)}{s+2} \Rightarrow R_{xx}(\tau) = R_{xx}^{+}(|\tau|) = \frac{1}{6}e^{-|\tau|} - \frac{1}{12}e^{-2|\tau|}.$$

□

Finite-Order Processes

Given

$$\boldsymbol{x}(t) = \int_{-\infty}^{\infty} \mathtt{l}(\tau)\boldsymbol{i}(t-\tau)d\tau,$$

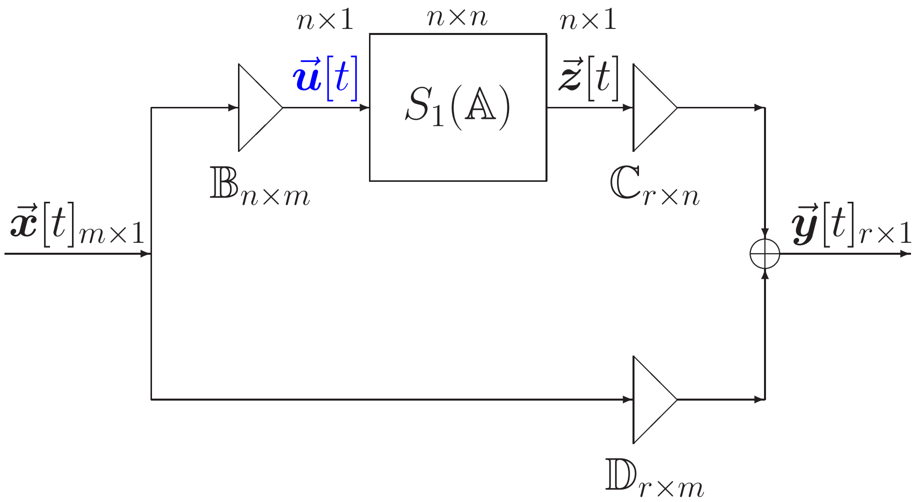
we derive

$$\begin{aligned}
R_{xx}(\tau) &= E[\boldsymbol{x}(t+\tau)\boldsymbol{x}(t)] \\
&= E\left[\left(\int_{-\infty}^{\infty} \mathtt{l}(u)\boldsymbol{i}(t+\tau-u)du\right)\left(\int_{-\infty}^{\infty} \mathtt{l}(v)\boldsymbol{i}(t-v)dv\right)\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathtt{l}(u)\mathtt{l}(v)E\left[\boldsymbol{i}(t+\tau-u)\boldsymbol{i}(t-v)\right]dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathtt{l}(u)\mathtt{l}(v)\delta(\tau-u+v)dudv \\
&= \int_{-\infty}^{\infty} \mathtt{l}(v)\mathtt{l}(\tau+v)dv \left(= \int_{-\infty}^{\infty} \mathtt{l}(-v)\mathtt{l}(\tau-v)dv = \mathtt{l}(-\tau) * \mathtt{l}(\tau)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{xx}(-\tau) &= \int_{-\infty}^{\infty} \mathtt{l}(v)\mathtt{l}(-\tau+v)dv \\
&= \int_{-\infty}^{\infty} \mathtt{l}(u+\tau)\mathtt{l}(u)du \quad (u = -\tau+v) = R_{xx}(\tau)
\end{aligned}$$

Discrete Finite-Order System



A system with state variables

- Consider a system with input $\vec{\boldsymbol{x}}[t]$ and output $\vec{\boldsymbol{y}}[t]$, in which their relationship is defined through an internal state variable $\vec{\boldsymbol{z}}[t]$ as:

$$\begin{cases} \vec{\boldsymbol{z}}[t+1] = \mathbb{A}\vec{\boldsymbol{z}}[t] + \vec{\boldsymbol{u}}[t] = \mathbb{A}\vec{\boldsymbol{z}}[t] + \mathbb{B}\vec{\boldsymbol{x}}[t] & (*) \\ \vec{\boldsymbol{y}}[t] = \mathbb{C}\vec{\boldsymbol{z}}[t] + \mathbb{D}\boldsymbol{x}\vec{[t]} \end{cases}$$

  The relationship between input $\vec{\boldsymbol{u}}[t]$ and output $\vec{\boldsymbol{z}}[t]$ of the subsystem S_1 is given by $(*)$.

Discrete Finite-Order Systems

Terminology

- The order of the system is defined as the dimension of the state variable $\vec{\boldsymbol{z}}[t]$, which is n in our case.

Derivation of the impulse response

- The impulse response of the subsystem S_1 can be derived from relationship

  $\boxed{\vec{\boldsymbol{z}}[t]_{n\times 1} = \sum_{\alpha=-\infty}^{\infty} \phi[\alpha]_{n\times n}\vec{\boldsymbol{u}}[t-\alpha]_{n\times 1}}$ equivalently $\boxed{\vec{\boldsymbol{z}}[z]_{n\times 1} = \phi[z]_{n\times n}\vec{\boldsymbol{u}}[z]_{n\times 1}}$

  Taking the z-transform of both sides of $(*)$ yields:

$$
\begin{aligned}
& z\vec{\boldsymbol{z}}[z]_{n\times 1} = \mathbb{A}_{n\times n}\vec{\boldsymbol{z}}[z]_{n\times 1} + \vec{\boldsymbol{u}}[z]_{n\times 1} \\
& \Rightarrow \;\; z\phi[z]_{n\times n}\vec{\boldsymbol{u}}[z]_{n\times 1} = \mathbb{A}_{n\times n}\phi[z]_{n\times n}\vec{\boldsymbol{u}}[z]_{n\times 1} + \vec{\boldsymbol{u}}[z]_{n\times 1} \\
& \Rightarrow \;\; z\Phi[z]_{n\times n} - \mathbb{A}_{n\times n}\Phi[z]_{n\times n} = \mathbb{I}_{n\times n} \\
& \Rightarrow \;\; \Phi[z]_{n\times n} = \left(z\mathbb{I}_{n\times n} - \mathbb{A}_{n\times n}\right)^{-1} \\
& \Rightarrow \;\; \phi[t]_{n\times n} = \exp\left\{\mathbb{A}_{n\times n}t\right\}.
\end{aligned}
$$

  where $\mathbb{I}$ is the identity matrix.

Derivation of the impulse response (continued)

- For the overall system,

$$\begin{aligned}
\vec{\boldsymbol{y}}[t]_{r\times 1} &= \mathbb{C}_{r\times n}\vec{\boldsymbol{z}}[t]_{n\times 1} + \mathbb{D}_{r\times m}\vec{\boldsymbol{x}[t]}_{m\times 1}\\
&= \sum_{\alpha=-\infty}^{\infty} \mathbb{C}_{r\times n}\phi[\alpha]_{n\times n}\vec{\boldsymbol{u}}[t-\alpha]_{n\times 1} + \sum_{\alpha=-\infty}^{\infty} \delta[\alpha]\mathbb{D}_{r\times m}\vec{\boldsymbol{x}}[t-\alpha]_{m\times 1}\\
&= \sum_{\alpha=-\infty}^{\infty} \mathbb{C}_{r\times n}\phi[\alpha]_{n\times n}\mathbb{B}_{n\times m}\vec{\boldsymbol{x}}[t-\alpha]_{m\times 1} + \sum_{\alpha=-\infty}^{\infty} \delta[\alpha]\mathbb{D}_{r\times m}\vec{\boldsymbol{x}}[t-\alpha]_{m\times 1}\\
&= \sum_{\alpha=-\infty}^{\infty} \left(\mathbb{C}_{r\times n}\phi[\alpha]_{n\times n}\mathbb{B}_{n\times m} + \delta[\alpha]\mathbb{D}_{r\times m}\right)\vec{\boldsymbol{x}}[t-\alpha]_{m\times 1}.
\end{aligned}$$

Hence,

$$h[t]_{r\times m} = \mathbb{C}_{r\times n}\phi[t]_{n\times n}\mathbb{B}_{n\times m} + \delta[t]\mathbb{D}_{r\times m}$$

and

$$H[z]_{r\times m} = \mathbb{C}_{r\times n}\Phi[z]_{n\times n}\mathbb{B}_{n\times m}+\mathbb{D}_{r\times m} = \boxed{\mathbb{C}_{r\times n}\left(z\mathbb{I}_{n\times n} - \mathbb{A}_{n\times n}\right)^{-1}\mathbb{B}_{n\times m} + \mathbb{D}_{r\times m}}.$$

Discrete Finite-Order Processes

Definition (Discrete finite-order processes) A (WSS) discrete process $\boldsymbol{x}[t]$ is of finite order if its innovation filter is a rational function of z, i.e.,

$$\mathtt{L}[z] = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{N[z]}{D[z]},$$

satisfying that the roots of $N[z]$ and $D[z]$ are within the unit circle.

Autocorrelation function of discrete finite-order process $\boldsymbol{x}[t]$

- Let $\{z_i\}_{i=1}^n$ be the roots of $D[z]$, and assume $m \leq n$.
  - We allow $m = n$ with $z_1 = 0$ in some practical case. In such case, $\frac{\gamma_1}{1 - z_1 z^{-1}}$ below is equal to $\gamma_1 = b_n / a_n$.
  - Here, we further assume that $z_i \neq 0$ for $i \geq 2$.
- Then, $\mathtt{L}[z]$ can be expanded into partial fractions as:

$$\mathtt{L}[z] = \sum_{i=1}^n \frac{\gamma_i}{1 - z_i z^{-1}} \text{ and } \mathtt{l}(\tau) = \gamma_1 \delta[\tau] \mathbf{1}\{z_1 = 0\} + \gamma_1 z_1^\tau \mathbf{1}\{\tau \geq 0\} \mathbf{1}\{z_1 \neq 0\} + \sum_{i=2}^n \gamma_i z_i^\tau \mathbf{1}\{\tau \geq 0\}.$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\gamma_i}{1 - z_i e^{-j\omega}} e^{j\omega\tau} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_i \left(1 + z_i e^{-j\omega} + z_i^2 e^{-j2\omega} + \cdots\right) e^{j\omega\tau} d\omega.$$

- We can then derive (and correct (11-37) in text) that:

$$
\begin{aligned}
S_{xx}[z] &= \mathtt{L}[z]\mathtt{L}[z^{-1}] \\
&= \left(\sum_{i=1}^{n} \frac{\gamma_i}{1 - z_i z^{-1}}\right)\left(\sum_{k=1}^{n} \frac{\gamma_k}{1 - z_k z}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\gamma_i\gamma_k}{(1 - z_i z^{-1})(1 - z_k z)} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{\gamma_i\gamma_k/(1 - z_i z_k)}{1 - z_i z^{-1}} + \frac{\gamma_i\gamma_k/(1 - z_i z_k)}{1 - z_k z} - \frac{\gamma_i\gamma_k}{1 - z_i z_k}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\gamma_i\gamma_k/(1 - z_i z_k)}{1 - z_i z^{-1}} + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\gamma_i\gamma_k/(1 - z_i z_k)}{1 - z_k z} - \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\gamma_i\gamma_k}{1 - z_i z_k} \\
&= \sum_{i=1}^{n} \frac{\alpha_i}{1 - z_i z^{-1}} + \sum_{k=1}^{n} \frac{\alpha_k}{1 - z_k z} - \sum_{i=1}^{n} \alpha_i = S_{xx}^{+}[z] + S_{xx}^{+}[1/z] - \sum_{i=1}^{n} \alpha_i,
\end{aligned}
$$

where

$$
\alpha_k = \gamma_k \sum_{i=1}^{n} \frac{\gamma_i}{1 - z_i z_k} = \gamma_k \mathtt{L}[z_k^{-1}].
$$

Discrete Finite-Order Processes

- This gives that:

$$
\begin{aligned}
R^{+}_{xx}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S^{+}_{xx}[e^{j\omega}]e^{j\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{i=1}^{n} \frac{\alpha_i}{1 - z_i e^{-j\omega}}\right) e^{j\omega\tau}d\omega \\
&= \begin{cases} \alpha_1 \delta[\tau]\mathbf{1}\{z_1 = 0\} + \alpha_1 z_1^{\tau}\mathbf{1}\{z_1 \neq 0\} + \displaystyle\sum_{i=2}^{n} \alpha_i z_i^{\tau}, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
R_{xx}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}[e^{j\omega}]e^{j\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(S^{+}_{xx}[e^{j\omega}] + S^{+}_{xx}[e^{-j\omega}] - \sum_{i=1}^{n}\alpha_i\right) e^{j\omega\tau}d\omega \\
&= R^{+}_{xx}[\tau] + R^{+}_{xx}[-\tau] - \delta[\tau]R^{+}_{xx}[0] \\
&= R^{+}_{xx}[|\tau|].
\end{aligned}
$$

Autoregressive Processes

Definition (AR processes) The discrete (finite-order) process $\boldsymbol{x}[t]$ is called autoregressive (AR) if its innovation filter is of the form:

$$\mathtt{L}[z] = \frac{b_0}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}.$$

Remarks

- For AR processes,

$$\boldsymbol{x}[t] + a_1 \boldsymbol{x}[t-1] + \cdots + a_n \boldsymbol{x}[t-n] = b_0 \boldsymbol{i}[t]. \tag{11.1}$$

- It is named AR because the output will continue indefinitely in an self-regressive fashion only with one excitation.

- Since $\boldsymbol{x}[t-m]$ can be completely determined by

$$\boldsymbol{x}[t-m-1] \text{ upto } \boldsymbol{x}[t-m-n] \text{ and } \boldsymbol{i}[t-m],$$

  it only depends on

$$\boldsymbol{i}[t-m], \boldsymbol{i}[t-m-1], \boldsymbol{i}[t-m-2], \ldots.$$

  Accordingly under the assumption that $\boldsymbol{x}[t]$ is WSS,

$$R_{xi}[-m] = E\{\boldsymbol{x}[t-m]\boldsymbol{i}[t]\} = E\{\boldsymbol{x}[t-m]\}E\{\boldsymbol{i}[t]\} = 0 \text{ for } m > 0.$$

- By multiplying $\boldsymbol{i}[t]$ followed by taking expectation of both sides of (11.1), we obtain:

$$R_{xi}[0] + a_1 \cancel{R_{xi}[-1]} + a_2 \cancel{R_{xi}[-2]} + \cdots + a_n \cancel{R_{xi}[-n]} = \boxed{R_{xi}[0] = b_0}.$$

- By multiplying $\boldsymbol{x}[t-m]$ for $0 \le m \le n$ followed by taking expectation of both sides of (11.1), we obtain:

$$\begin{array}{rcrcl}
\times\boldsymbol{x}[t] & : & R_{xx}[0] + a_1 R_{xx}[-1] + \cdots + a_n R_{xx}[-n] & = & b_0^2 \\
\times\boldsymbol{x}[t-1] & : & R_{xx}[1] + a_1 R_{xx}[0] + \cdots + a_n R_{xx}[-n+1] & = & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\times\boldsymbol{x}[t-n] & : & R_{xx}[n] + a_1 R_{xx}[n-1] + \cdots + a_n R_{xx}[0] & = & 0,
\end{array}$$

or equivalently,

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[-1] & R_{xx}[-2] & \cdots & R_{xx}[-n] \\ R_{xx}[1] & R_{xx}[0] & R_{xx}[-1] & \cdots & R_{xx}[-n+1] \\ R_{xx}[2] & R_{xx}[1] & R_{xx}[0] & \cdots & R_{xx}[-n+2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{xx}[n] & R_{xx}[n-1] & R_{xx}[n-2] & \cdots & R_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is named the *Yule-Walker* equations.

The Yule-Walker equations can be used to determine $a_1, \cdots, a_n$ and b_0 for known $R_{xx}[m]$, or to determine $R_{xx}[m]$ recursively for known $a_1, \cdots, a_n$ and b_0.

Autoregressive Processes

Example 11-7 $\boldsymbol{x}[t] - a\boldsymbol{x}[t-1] = b\boldsymbol{i}[t]$.

Solution:

- $\mathtt{L}[z] = \dfrac{b}{1 - az^{-1}} \Rightarrow z_1 = a$ and $\gamma_1 = b$ and $\alpha_1 = \gamma_1 \mathtt{L}[1/z_1] = b^2/(1-a^2)$.
- Then, $R_{xx}[\tau] = \alpha_1 z_1^{|\tau|} = \dfrac{b^2}{1-a^2}\, a^{|\tau|}$. $\square$

If $a > 1$, then $b\,(1 + az^{-1} + a^2 z^{-2} + \cdots)$ does not converge unless $|z| < 1/|a|$; hence,

$$\frac{b}{1 - az^{-1}} = b\,(1 + az^{-1} + a^2 z^{-2} + \cdots)$$

is not valid for $|z| = |e^{j\omega}| = 1$. In short, an AR process with roots outside the unit circle is not stationary!

Two cases that are not included in Slide 11-43:

1. Case of $m = 0$ and $b_0 = 0$, such as the autoregressive processes with line spectrum.
2. Case of $m > n$, such as the moving average processes.

These will be covered in next few slides.

Line Spectra

Definition (Line spectra) A line spectrum only consists of lines, i.e.,

$$S(\omega) = 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i).$$

- The autocorrelation function of a process with line spectrum is:

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \right) e^{j\omega\tau} d\omega = \sum_i \sigma_i^2 e^{j\omega_i \tau}.$$

- An exemplified process that results in a line spectrum is:

$$\boldsymbol{x}(t) = \sum_i \boldsymbol{c}_i e^{j\omega_i t},$$

where $\{\boldsymbol{c}_i\}$ are uncorrelated with zero mean, and $\sigma_i^2 = E\{|\boldsymbol{c}_i|^2\}$.

Discrete Line Spectra

Definition (Discrete line spectra) A line spectrum for discrete processes only consists of lines, i.e.,

$$S[\omega] = 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \text{ for } -\pi \le \omega < \pi,$$

where each $-\pi \le \omega_i < \pi$.

- The autocorrelation function of a discrete process with line spectrum is:

$$R[\tau] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \right) e^{j\omega\tau} d\omega = \sum_i \sigma_i^2 e^{j\omega_i \tau}.$$

- An exemplified process that results in a line spectrum is:

$$\boldsymbol{x}[t] = \sum_i \boldsymbol{c}_i e^{j\omega_i t},$$

where $\{\boldsymbol{c}_i\}$ are uncorrelated with zero mean, and $\sigma_i^2 = E\{|\boldsymbol{c}_i|^2\}$.

Example of AR processes with line spectra

- Suppose that

$$\boldsymbol{x}[t] = \sum_{i=1}^{n} \boldsymbol{c}_i e^{j\omega_i t},$$

  where $\{\boldsymbol{c}_i\}$ are real and uncorrelated with zero mean and variance $\sigma_i^2 = E\{\boldsymbol{c}_i^2\}$, and each $-\pi \leq \omega_i < \pi$.

- Let $z_i = e^{j\omega_i}$.

- Find $a_1, a_2, \ldots, a_n$ such that

$$\begin{bmatrix} 1 & z_1^{-1} & z_1^{-2} & \cdots & z_1^{-n} \\ 1 & z_2^{-1} & z_2^{-2} & \cdots & z_2^{-n} \\ 1 & z_3^{-1} & z_3^{-2} & \cdots & z_3^{-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n^{-1} & z_n^{-2} & \cdots & z_n^{-n} \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

  Then

$$\begin{aligned} &\boldsymbol{x}[t] + a_1\boldsymbol{x}[t-1] + a_2\boldsymbol{x}[t-2] + \cdots + a_n\boldsymbol{x}[t-n] \\ &= \sum_{i=1}^{n} \boldsymbol{c}_i z_i^t \left(1 + a_1 z_i^{-1} + \cdots + a_n z_i^{-n}\right) = 0. \end{aligned}$$

Specifically, if $n = 2$, we require

$$\begin{aligned} D(z_1) &= 1 + a_1 z_1^{-1} + a_2 z_1^{-2} = 0 \\ D(z_2) &= 1 + a_1 z_2^{-1} + a_2 z_2^{-2} = 0 \end{aligned}$$

Then, $a_1 = -(z_1 + z_2)$ and $a_2 = z_1 z_2$.
If $n = 3$, we require

$$\begin{aligned} D(z_1) &= 1 + a_1 z_1^{-1} + a_2 z_1^{-2} + a_3 z_1^{-3} = 0 \\ D(z_2) &= 1 + a_1 z_2^{-1} + a_2 z_2^{-2} + a_3 z_2^{-3} = 0 \\ D(z_3) &= 1 + a_1 z_3^{-1} + a_2 z_3^{-2} + a_3 z_3^{-3} = 0 \end{aligned}$$

Then, $a_1 = -(z_1 + z_2 + z_3)$, $a_2 = z_1 z_2 + z_1 z_3 + z_2 z_3$ and $a_3 = -z_1 z_2 z_3$.
In fact, $D(z) = \prod_{i=1}^{n}(1 - z_i z^{-1})$.

- This turns out to be a special case of the AR processes for which $b_0 = 0$ and $D(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}$. It is usually referred to as the predictable process.

Definition (Predictable process) A process is called *predictable* if its present value can be determined by its past.

Discrete Line Spectra

Autocorrelation and line power spectrum of $\boldsymbol{x}[t]$

•

$$\begin{aligned} R_{xx}[\tau] &= E\left[\left(\sum_{i=1}^{n} \boldsymbol{c}_i e^{j\omega_i(t+\tau)}\right)\left(\sum_{k=1}^{n} \boldsymbol{c}_k^* e^{-j\omega_k t}\right)\right] \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} E\left[\boldsymbol{c}_i \boldsymbol{c}_k^*\right] e^{j\omega_i(t+\tau)} e^{-j\omega_k t} \\ &= \sum_{i=1}^{n} \sigma_i^2 e^{j\omega_i \tau} \end{aligned}$$

and for $-\pi \leq \omega < \pi$,

$$\begin{aligned} S_{xx}[\omega] &= \sum_{\tau=-\infty}^{\infty} R_{xx}[\tau] e^{-j\omega\tau} = \sum_{\tau=-\infty}^{\infty}\sum_{i=1}^{n} \sigma_i^2 e^{j\omega_i\tau} e^{-j\omega\tau} \\ &= \sum_{i=1}^{n} \sigma_i^2 \sum_{\tau=-\infty}^{\infty} e^{-j(\omega-\omega_i)\tau} = 2\pi \sum_{i=1}^{n} \sigma_i^2 \delta(\omega-\omega_i). \quad \text{(Line spectra!)} \end{aligned}$$

where $\sum_{\tau=-\infty}^{\infty} e^{-j(\omega-\omega_i)\tau} = \sum_{\tau=-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_i + 2\pi\tau)$.

Moving Average Processes

Definition (MA processes) The discrete process $\boldsymbol{x}[t]$ is called moving average (MA) if its innovation filter is of the form:

$$\mathtt{L}[z] = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}.$$

Autocorrelation function of MA processes

- For an MA process,

$$\boldsymbol{x}[t] = b_0 \boldsymbol{i}[t] + b_1 \boldsymbol{i}[t-1] + \cdots + b_m \boldsymbol{i}[t-m].$$

- Hence, the symmetric autocorrelation function (i.e., $R_{xx}[\tau] = R_{xx}[-\tau]$) equals

$$\begin{aligned}
R_{xx}[\tau] &= E\{\boldsymbol{x}[t+\tau]\boldsymbol{x}[t]\} \\
&= E\left\{\left(\sum_{i=0}^{m} b_i \boldsymbol{i}[t+\tau-i]\right)\left(\sum_{k=0}^{m} b_k \boldsymbol{i}[t-k]\right)\right\} \\
&= \sum_{i=0}^{m}\sum_{k=0}^{m} b_i b_k E\left\{\boldsymbol{i}[t+\tau-i]\boldsymbol{i}[t-k]\right\} = \sum_{i=0}^{m}\sum_{k=0}^{m} b_i b_k \delta[\tau - i + k] \\
&= \begin{cases} \displaystyle\sum_{k=0}^{m-\tau} b_{k+\tau} b_k, & \text{for } 0 \le \tau \le m \\ 0, & \text{for } \tau > m \end{cases}
\end{aligned}$$

Autoregressive Moving Average Processes

Definition (ARMA processes) The discrete process $\boldsymbol{x}[t]$ is called autoregressive moving average (ARMA) if its innovation filter is of the form:

$$\mathtt{L}[z] = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{N[z]}{D[z]}.$$

- The analysis of the ARMA processes has been done; so we omit it. See the slides after Slide 11-43.

The end of Section 11-2 Finite-Order Systems and State Variables

Question: Given that $\omega_0 = 2\pi/T$,

$$\hat{\boldsymbol{x}}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n e^{jn\omega_0 t} \quad \text{and} \quad \boldsymbol{c}_n = \frac{1}{T}\int_0^T \boldsymbol{x}(t) e^{-jn\omega_0 t} dt,$$

whether does $\hat{\boldsymbol{x}}(t)$ well approximate the WSS $\boldsymbol{x}(t)$?

Theorem $\hat{\boldsymbol{x}}(t)$ equals $\boldsymbol{x}(t)$ for $0 < t < T$ in the MS sense, i.e.,

$$E[|\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)|^2] = 0$$

for $0 < t < T$.

Proof: Observe that for $0 < t < T$,

$$\begin{aligned}
E[|\hat{\boldsymbol{x}}(t)|^2] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} \boldsymbol{c}_m^* e^{-jm\omega_0 t}\right)\right] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[\boldsymbol{c}_n \boldsymbol{c}_m^*\right] e^{jn\omega_0 t} e^{-jm\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \left(\frac{1}{T^2}\int_0^T\int_0^T E\left[\boldsymbol{x}(u)\boldsymbol{x}^*(v)\right] e^{-jn\omega_0 u} e^{jm\omega_0 v} du dv\right) e^{jn\omega_0 t} e^{-jm\omega_0 t}
\end{aligned}$$

(continued)

$$
\begin{aligned}
\left(E[|\hat{\boldsymbol{x}}(t)|^2]\right) &= \int_0^T \int_0^T R_{xx}(u-v) \left(\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0(t-u)}\right)\left(\frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\omega_0(v-t)}\right) du dv \\
&= \int_0^T \int_0^T R_{xx}(u-v) \left(\sum_{n=-\infty}^{\infty} \delta(t-u+nT)\right)\left(\sum_{m=-\infty}^{\infty} \delta(v-t+mT)\right) du dv \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \int_0^T \int_0^T R_{xx}(u-v)\delta(t-u+nT)\delta(v-t+mT) du dv \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \int_0^T R_{xx}(t+nT-v)\mathbf{1}\{0<t+nT<T\}\delta(v-t+mT) dv \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_{xx}((n+m)T)\mathbf{1}\{0<t+nT<T\}\mathbf{1}\{0<t-mT<T\} \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_{xx}((n+m)T)\mathbf{1}\left\{-\frac{t}{T}<n<1-\frac{t}{T} \text{ and } \frac{t}{T}-1<m<\frac{t}{T}\right\} \\
&= R_{xx}(0) = E[|\boldsymbol{x}(t)|^2],
\end{aligned}
$$

and

$$\begin{aligned}
E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty}\boldsymbol{c}_n e^{jn\omega_0 t}\right)\boldsymbol{x}^*(t)\right] \\
&= \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n\boldsymbol{x}^*(t)]e^{jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} E\left[\left(\frac{1}{T}\int_0^T \boldsymbol{x}(s)e^{-jn\omega_0 s}ds\right)\boldsymbol{x}^*(t)\right]e^{jn\omega_0 t} \\
&= \int_0^T E[\boldsymbol{x}(s)\boldsymbol{x}^*(t)]\left(\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0(t-s)}\right)ds \\
&= \int_0^T R_{xx}(s-t)\left(\sum_{n=-\infty}^{\infty}\delta(t-s+nT)\right)ds \\
&= \sum_{n=-\infty}^{\infty} R_{xx}(nT)\mathbf{1}\{0<t+nT<T\} \quad (\text{i.e., } \frac{t}{T}-1<-n<\frac{t}{T}) \\
&= R_{xx}(0)
\end{aligned}$$

Fourier Series

Similarly,

$$\begin{aligned}
E[\hat{\boldsymbol{x}}^*(t)\boldsymbol{x}(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n^* e^{-jn\omega_0 t}\right)\boldsymbol{x}(t)\right] \\
&= \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n^* \boldsymbol{x}(t)] e^{-jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} E\left[\left(\frac{1}{T}\int_0^T \boldsymbol{x}^*(s) e^{jn\omega_0 s} ds\right)\boldsymbol{x}(t)\right] e^{-jn\omega_0 t} \\
&= \int_0^T E[\boldsymbol{x}(t)\boldsymbol{x}^*(s)] \left(\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 (s-t)}\right) ds \\
&= \int_0^T R_{xx}(t-s) \left(\sum_{n=-\infty}^{\infty} \delta(s-t+nT)\right) ds \\
&= \sum_{n=-\infty}^{\infty} R_{xx}(nT)\mathbf{1}\{0 < t-nT < T\} \quad (\text{i.e., } \frac{t}{T}-1 < n < \frac{t}{T}) \\
&= R_{xx}(0)
\end{aligned}$$

Hence,

$$\begin{aligned} E[|\hat{\boldsymbol{x}}(t)-\boldsymbol{x}(t)|^2] &= E[|\hat{\boldsymbol{x}}(t)|^2]-E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)]-E[\hat{\boldsymbol{x}}^*(t)\boldsymbol{x}(t)]+E[|\boldsymbol{x}(t)|^2] \\ &= R_{xx}(0)-R_{xx}(0)-R_{xx}(0)+R_{xx}(0) \\ &= 0. \end{aligned}$$

$\square$

Remarks

- It is tricky to say the theorem holds at $t = 0$ (respectively, $t = T$) since $\int_0^T \delta(s)ds$ or (respectively, $\int_0^T \delta(s-T)ds$) is actually indeterminate.
- It can be similarly proved that if $\boldsymbol{x}(t)$ is MS-periodic with period T,

$$E[|\hat{\boldsymbol{x}}(t)-\boldsymbol{x}(t)|^2] = 0 \quad \text{for a.e. } t \in \Re.$$

Definition A process $\boldsymbol{x}(t)$ is called *MS periodic* if

$$E[|\boldsymbol{x}(t+T)-\boldsymbol{x}(t)|^2] = 0$$

for every t.

Theorem 9-1 A process $\boldsymbol{x}(t)$ is *MS periodic* if, and only if, its autocorrelation function is *doubly periodic*, namely,

$$R_{xx}(t_1 + mT, t_2 + nT) = R_{xx}(t_1, t_2) \text{ for every integer } m \text{ and } n.$$

- In addition, for a MS-periodic WSS process $\boldsymbol{x}(t)$,

$$\left\{ \boldsymbol{c}_n = \frac{1}{T} \int_0^T \boldsymbol{x}(t) e^{-jn\omega_0 t} dt \right\}_{n=-\infty}^{\infty}$$

  are uncorrelated with zero-mean except possibly non-zero-mean at $n = 0$.

- These remarks are summarized into the next theorem.

Theorem 11-1 For a MS-periodic (with period T) WSS process $\boldsymbol{x}(t)$, $\hat{\boldsymbol{x}}(t)$ equals $\boldsymbol{x}(t)$ in the MS sense, i.e., $E[|\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)|^2] = 0$.

In addition, $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ are uncorrelated with zero mean except possibly for $n = 0$.

Proof: It remains to prove that $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ are uncorrelated with zero mean except possibly for $n = 0$.

For an MS-periodic WSS $\boldsymbol{x}(t)$,

$$E[\boldsymbol{c}_n] = \frac{1}{T}\int_0^T E[\boldsymbol{x}(t)]e^{-jn\omega_0 t}dt = \mu_x \delta[n], \quad \text{(because } \omega_0 = T/(2\pi)\text{)}$$

and

$$
\begin{aligned}
E[\boldsymbol{c}_n \boldsymbol{c}_m^*] &= E\left[\left(\frac{1}{T}\int_0^T \boldsymbol{x}(t)e^{-jn\omega_0 t}dt\right)\left(\frac{1}{T}\int_0^T \boldsymbol{x}(s)e^{-jn\omega_0 s}ds\right)^*\right] \\
&= \frac{1}{T^2}\int_0^T\int_0^T E\left[\boldsymbol{x}(t)\boldsymbol{x}^*(s)\right]e^{-jn\omega_0 t}e^{jm\omega_0 s}dtds \\
&= \frac{1}{T^2}\int_0^T\int_0^T R_{xx}(t-s)e^{-jn\omega_0 t}e^{jm\omega_0 s}dtds, \quad u = t-s \\
&= \frac{1}{T^2}\int_0^T\int_{-s}^{T-s} R_{xx}(u)e^{-jn\omega_0 (u+s)}e^{jm\omega_0 s}duds \\
&= \left(\frac{1}{T}\int_0^T e^{-j(n-m)\omega_0 s}ds\right)\left(\frac{1}{T}\int_0^T R_{xx}(u)e^{-jn\omega_0 u}du\right) \\
&= \delta[n-m]\left(\frac{1}{T}\int_0^T R_{xx}(u)e^{-jn\omega_0 u}du\right) \quad \text{(since } \omega_0 = T/(2\pi)\text{)}
\end{aligned}
$$

□

Remarks

- $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ may not be uncorrelated if $\boldsymbol{x}(t)$ is not MS-periodic!
- Even if $\boldsymbol{x}(t)$ is MS-periodic, $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ may not be uncorrelated when the chosen T is not the MS-period for $\boldsymbol{x}(t)$.
- **Concern:** Can we find an alternative expression for $\boldsymbol{x}(t)$ for which the coefficients are guaranteed to be **uncorrelated**?

  Answer: Karhunen-Loève Expansions.

Karhunen-Loève Expansions

Question: Given a set of orthonormal functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ over $[0, T)$, define

$$\hat{\boldsymbol{x}}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \varphi_n(t) \quad \text{and} \quad \boldsymbol{c}_n = \int_0^T \boldsymbol{x}(t)\varphi_n^*(t)dt.$$

Whether does $\hat{\boldsymbol{x}}(t)$ well approximate $\boldsymbol{x}(t)$?

Theorem $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ are orthogonal, if

$$\int_0^T R_{xx}(t,s)\varphi_n(s)ds = \lambda_n \varphi_n(t)$$

for some λ_n for every n.

For MS-periodic WSS process $\boldsymbol{x}(t)$ with MS-period T,

$$\int_0^T R_{xx}(t-s)\left(\frac{1}{\sqrt{T}}e^{jn\omega_0 s}\right)ds = \int_{t-T}^{t} R_{xx}(u)\frac{1}{\sqrt{T}}e^{jn\omega_0(t-u)}du$$

$$= \frac{1}{\sqrt{T}}e^{jn\omega_0 t}\int_0^T R_{xx}(u)e^{-jn\omega_0 u}du = \lambda_n\left(\frac{1}{\sqrt{T}}e^{jn\omega_0 t}\right),$$

where $\omega_0 = 2\pi/T$ and $\lambda_n = \displaystyle\int_0^T R_{xx}(u)e^{-jn\omega_0 u}du.$

Proof:

$$\begin{aligned}
E[\boldsymbol{c}_n\boldsymbol{c}_m^*] &= E\left[\left(\int_0^T \boldsymbol{x}(t)\varphi_n^*(t)dt\right)\left(\int_0^T \boldsymbol{x}(s)\varphi_m^*(s)ds\right)^*\right]\\
&= \int_0^T\int_0^T E\left[\boldsymbol{x}(t)\boldsymbol{x}^*(s)\right]\varphi_n^*(t)\varphi_m(s)dtds\\
&= \int_0^T\left(\int_0^T R_{xx}(t,s)\varphi_m(s)ds\right)\varphi_n^*(t)dt\\
&= \int_0^T \lambda_m\varphi_m(t)\varphi_n^*(t)dt\\
&= \lambda_m\delta[m-n].
\end{aligned}$$

$\square$

Remarks

- $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ and $\{\lambda_n\}_{n=-\infty}^{\infty}$ are respectively called the eigenfunctions and eigenvalues of $R_{xx}(t,s)$.

Karhunen-Loève Expansions

- For a random process $\boldsymbol{x}(t)$, projection λ_n is real and non-negative for every n.

Proof:

$$\begin{aligned}
E\left[\left|\int_0^T \boldsymbol{x}(t)\varphi_n^*(t)dt\right|^2\right] &= E\left[\left(\int_0^T \boldsymbol{x}(t)\varphi_n^*(t)dt\right)\left(\int_0^T \boldsymbol{x}^*(s)\varphi_n(s)ds\right)\right] \\
&= \int_0^T \left(\int_0^T R_{xx}(t,s)\varphi_n(s)ds\right)\varphi_n^*(t)dt \\
&= \int_0^T \lambda_n \varphi_n(t)\varphi_n^*(t)dt \\
&= \lambda_n.
\end{aligned}$$

$\square$

- $R_{xx}(t,t) = \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2$ for $0 \le t < T$. (Property of the eigen-system)

Theorem $E[|\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)|^2] = 0$ for $0 < t < T$.

Proof: Observe that

$$
\begin{aligned}
E[|\hat{\boldsymbol{x}}(t)|^2] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \varphi_n(t)\right)\left(\sum_{m=-\infty}^{\infty} \boldsymbol{c}_m^* \varphi_m^*(t)\right)\right] \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left[\boldsymbol{c}_n \boldsymbol{c}_m^*\right] \varphi_n(t) \varphi_m^*(t) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \lambda_m \delta[m-n] \varphi_n(t) \varphi_m^*(t) \quad \text{(because } \{\boldsymbol{c}_n\} \text{ orthogonal)} \\
&= \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 \quad \big(= R_{xx}(t,t)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \varphi_n(t)\right)\boldsymbol{x}^*(t)\right] \\
&= \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n \boldsymbol{x}^*(t)]\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} E\left[\left(\int_0^T \boldsymbol{x}(s)\varphi_n^*(s)ds\right)\boldsymbol{x}^*(t)\right]\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} \left(\int_0^T R_{xx}(t,s)\varphi_n(s)ds\right)^* \varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} \lambda_n^* \varphi_n^*(t)\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} \lambda_n^* |\varphi_n(t)|^2 = \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 \quad (\lambda_n \text{ real and non-negative})
\end{aligned}
$$

Similarly,

$$
E[\hat{\boldsymbol{x}}^*(t)\boldsymbol{x}(t)] = \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2.
$$

Hence,

$$
\begin{aligned}
E[|\hat{\boldsymbol{x}}(t)-\boldsymbol{x}(t)|^2] &= E[|\hat{\boldsymbol{x}}(t)|^2]-E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)]-E[\hat{\boldsymbol{x}}^*(t)\boldsymbol{x}(t)]+E[|\boldsymbol{x}(t)|^2] \\
&= \sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2-\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2-\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2+R_{xx}(t,t) \\
&= R_{xx}(t,t)-\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2,
\end{aligned}
$$

which equals zero by property of the eigen-system. □

Mercer's Theorem tells that $R_{xx}(t,s)=\sum_{n=-\infty}^{\infty}\lambda_n\varphi_n(t)\varphi_n^*(s)$.

Example 11-10: Wiener process. Suppose

- $\boldsymbol{n}[0,0)=0$,
- $\boldsymbol{n}[t_1,t_2)$ is Gaussian distributed with mean zero and variance $\alpha(t_2-t_1)$,
- and $\boldsymbol{n}[t_1,t_2)$ and $\boldsymbol{n}[t_3,t_4)$ are independent if $[t_1,t_2)$ and $[t_3,t_4)$ are non-overlapping intervals.

Please determine the Karhunen-Loève expansion of real process $\boldsymbol{x}(t)\triangleq\boldsymbol{n}[0,t)$.

Answer:

- $$\begin{aligned}
R_{xx}(t_1, t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] \\
&= E[\boldsymbol{n}[0, t_1)\boldsymbol{n}[0, t_2)] \\
&= E\left[\left(\boldsymbol{n}[0, t_{\min}) + \boldsymbol{n}[t_{\min}, t_{\max})\right)\boldsymbol{n}[0, t_{\min})\right] \\
&= E\left[\boldsymbol{n}^2[0, t_{\min})\right] + E\left[\boldsymbol{n}[t_{\min}, t_{\max})\boldsymbol{n}[0, t_{\min})\right] \\
&= E\left[\boldsymbol{n}^2[0, t_{\min})\right] + \cancel{E\left[\boldsymbol{n}[t_{\min}, t_{\max})\right] E\left[\boldsymbol{n}[0, t_{\min})\right]} \\
&= \alpha \min\{t_1, t_2\},
\end{aligned}$$

  where $t_{\min} \triangleq \min\{t_1, t_2\}$ and $t_{\max} \triangleq \{t_1, t_2\}$.

- $$\int_0^T R_{xx}(t, s)\varphi(s)ds = \lambda\varphi(t) \Leftrightarrow \alpha \int_0^T \min\{t, s\}\varphi(s)ds = \lambda\varphi(t)$$

  $$\Leftrightarrow \; \alpha \int_0^t s\varphi(s)ds + \alpha t \int_t^T \varphi(s)ds = \lambda\varphi(t) \quad (\texttt{a1})$$

  $$\Leftrightarrow \; \begin{cases} \alpha \displaystyle\int_t^T \varphi(s)ds = \lambda\varphi'(t) \quad (\texttt{a2}) \\ \lambda\varphi''(t) + \alpha\varphi(t) = 0 \end{cases} \text{with initially} \begin{cases} (\texttt{a1}) \; \varphi(0) = 0 \\ (\texttt{a2}) \; \varphi'(T) = 0 \end{cases}$$

Theorem 8.6 [Tom M. Apostoal, *Calculus*, pp. 326, Volume 1, 2nd Edition, 1967] The solution of the equation $y''(x) + by(x) = 0$ is

$$y(x) = c_1 u_1(x) + c_2 u_2(x),$$

where c_1 and c_2 are constants determined by initial conditions, and

1. $u_1(x) = 1$ and $u_2(x) = x$ if $b = 0$;
2. $u_1(x) = e^{kx}$ and $u_2(x) = e^{-kx}$ if $b = -k^2 < 0$;
3. $u_1(x) = \cos(kx)$ and $u_2(x) = \sin(kx)$ if $b = k^2 > 0$.

- Consequently, $\varphi_n(t) = c_1 \cos(t\sqrt{\alpha/\lambda_n}) + c_2 \sin(t\sqrt{\alpha/\lambda_n})$, and the two initial conditions give that $c_1 = 0$ ($\varphi_n(0) = 0$) and $T\sqrt{\alpha/\lambda_n} = (2k_n + 1)\pi/2$ for integer k_n ($\varphi_n'(T) = 0$). Moreover,

$$\begin{aligned}\int_0^T \varphi_n^2(t)dt &= \int_0^T c_2^2 \sin^2\left(\frac{(2k_n+1)\pi}{2T}t\right)dt \\ &= \int_0^1 c_2^2 \sin^2\left(\frac{(2k_n+1)\pi}{2}u\right)T du = \frac{T}{2}c_2^2 = 1\end{aligned}$$

gives that $c_2 = \sqrt{2/T}$.

- To sum up,

$$\varphi_n(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{(2k_n+1)\pi}{2T}t\right), \quad \lambda_n = \frac{4\alpha T^2}{(2k_n+1)^2\pi^2},$$

and

$$\begin{aligned}
\int_0^T \varphi_n(t)\varphi_m^*(t)dt &= \int_0^T \frac{2}{T}\sin\left(\frac{(2k_n+1)\pi}{2T}t\right)\sin\left(\frac{(2k_m+1)\pi}{2T}t\right)dt \\
&= \int_0^1 2\sin\left(\frac{(2k_n+1)\pi}{2}u\right)\sin\left(\frac{(2k_m+1)\pi}{2}u\right)du \\
&= \int_0^1 \cos[(k_n-k_m)\pi u]du - \int_0^1 \cos[(k_n+k_m+1)\pi u]du \\
&= \delta[k_n-k_m] - \delta[k_n+k_m+1] \\
&= \begin{cases} 1, & k_n = k_m \text{ (equivalently } (2k_n+1) = (2k_m+1)) \\ -1, & (2k_n+1) = -(2k_m+1) \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

So, it only requires to take those k_n's that make $(2k_n + 1)$ strictly positive. This concludes that the Winner process $\boldsymbol{x}(t)$ for $t \in [0, T)$ can be written as a sum of sine waves:

$$\boldsymbol{x}(t) = \sqrt{\frac{2}{T}} \sum_{n=0}^{\infty} \boldsymbol{c}_n \sin\left(\frac{(2n+1)\pi}{2T} t\right)$$

and

$$\boldsymbol{c}_n = \sqrt{\frac{2}{T}} \int_0^T \boldsymbol{x}(t) \sin\left(\frac{(2n+1)\pi}{2T} t\right) dt.$$

$\square$

By assigning $\tilde{\boldsymbol{c}}_n = \boldsymbol{c}_n \sqrt{2/T}$, we can simplify the expression as:

$$\boldsymbol{x}(t) = \sum_{n=0}^{\infty} \tilde{\boldsymbol{c}}_n \sin\left(\frac{(2n+1)\pi}{2T} t\right) \quad \text{and} \quad \tilde{\boldsymbol{c}}_n = \frac{2}{T} \int_0^T \boldsymbol{x}(t) \sin\left(\frac{(2n+1)\pi}{2T} t\right) dt.$$

Example. Suppose $\boldsymbol{x}(t)$ is WSS. Then, from

$$\int_{-\infty}^{\infty} R_{xx}(t-s)\varphi_\lambda(s)ds = \lambda\,\varphi_\lambda(t),$$

we know that the Fourier transform $\Phi_\lambda(\omega)$ of eigenfunction $\varphi_\lambda(t)$ and eigenvalue λ should satisfy:

$$S_{xx}(\omega)\Phi_\lambda(\omega) = \lambda\,\Phi_\lambda(\omega).$$

This implies

$$(S_{xx}(\omega) - \lambda)\,\Phi_\lambda(\omega) = 0.$$

Suppose $S_{xx}(\omega) = \lambda$ only at $\omega = u$. (There could be other value of ω such as $\omega = v$ that also makes $S_{xx}(v) = \lambda$. We would treat this case as the eigenvalue λ has several eigenfunctions.)
Then, $\Phi_\lambda(\omega) = \sqrt{2\pi}\delta(\omega - u)$ is an eigenfunction corresponding to eigenvalue λ, which implies

$$\varphi_\lambda(t) = \frac{1}{\sqrt{2\pi}}e^{jut}.$$

Hence,

$$\boldsymbol{x}(t) = \int_{-\infty}^{\infty} \boldsymbol{c}_\lambda \varphi_\lambda(t) d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boldsymbol{c}_\lambda e^{j\,u(\lambda)\,t} d\lambda$$

and

$$\boldsymbol{c}_\lambda = \int_{-\infty}^{\infty} \boldsymbol{x}(t) \varphi_\lambda^*(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boldsymbol{x}(t) e^{-jut} dt.$$

We can redenote $\boldsymbol{c}_\lambda$ by $\frac{1}{\sqrt{2\pi}} \boldsymbol{X}(u)$ and yield:

$$\boldsymbol{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boldsymbol{X}(u) e^{jut} du \quad \text{and} \quad \boldsymbol{X}(u) = \int_{-\infty}^{\infty} \boldsymbol{x}(t) e^{-jut} dt.$$

$\square$

- This example justifies the viewpoint that the Fourier transform of a WSS process is simply the Karhunen-Loève expansion of this random process.

- We will show later that $E[\boldsymbol{X}(u)\boldsymbol{X}^*(v)] = 2\pi S_{xx}(u)\delta(u-v)$ (resp. $E[\boldsymbol{c}_{\lambda_1}\boldsymbol{c}^*_{\lambda_2}] = \frac{1}{2\pi}E[\boldsymbol{X}(u)\boldsymbol{X}^*(v)] = S_{xx}(u)\delta(u-v)$ for distinct eigenvalues λ_1 and λ_2).

- The eigenvalue corresponding to eigenvector $\dfrac{1}{\sqrt{2\pi}}e^{jut}$ is $\sqrt{2\pi}S_{xx}(u)$.

  (I.e., the eigenvalue corresponding to eigenvector $\dfrac{1}{\sqrt{2\pi}}e^{j\omega t}$ is $\sqrt{2\pi}S_{xx}(\omega)$.)

- The eigenvectors $\dfrac{1}{\sqrt{2\pi}}e^{j\omega_1 t}$ and $\dfrac{1}{\sqrt{2\pi}}e^{j\omega_2 t}$ are orthogonal to each other (namely,

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{j\omega_1 t}\frac{1}{\sqrt{2\pi}}e^{-j\omega_2 t}dt = \delta(\omega_1-\omega_2)\ ).$$

The end of Section 11-3 Fourier Series and Karhunen-Loève Expansions

11-4 Spectral Representation of Random Processes

- The Fourier transform of a random process $\boldsymbol{x}(t)$ is also a random process, defined as:

$$\boldsymbol{X}(u) \triangleq \int_{-\infty}^{\infty} \boldsymbol{x}(t)e^{-jut}dt.$$

Lemma

- Let $R_{XX}(u_1, u_2)$ and $S_{XX}(\lambda_1, \lambda_2)$ be the autocorrelation function and two-dimensional power spectrum of $\boldsymbol{X}(t)$, respectively.
- Let $R_{xx}(t_1, t_2)$ and $S_{xx}(f_1, f_2)$ be the autocorrelation function and two-dimensional power spectrum of $\boldsymbol{x}(t)$, respectively.

Then,

$$R_{XX}(u_1, u_2) = S_{xx}(u_1, -u_2) \quad \text{and} \quad S_{XX}(\lambda_1, \lambda_2) = 4\pi^2 R_{xx}(-\lambda_1, \lambda_2).$$

Proof:

$$\begin{aligned} R_{XX}(u_1,u_2) &= E[\boldsymbol{X}(u_1)\boldsymbol{X}^*(u_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)]e^{-j(u_1t_1-u_2t_2)}dt_1dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1,t_2)e^{-j[u_1t_1+(-u_2)t_2]}dt_1dt_2 \\ &= S_{xx}(u_1,-u_2) \end{aligned}$$

and

$$\begin{aligned} S_{XX}(\lambda_1,\lambda_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(u_1,u_2)e^{-j(\lambda_1u_1+\lambda_2u_2)}du_1du_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(u_1,-u_2)e^{-j(\lambda_1u_1+\lambda_2u_2)}du_1du_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(u_1,u_2')e^{j[(-\lambda_1)u_1+\lambda_2u_2']}du_1du_2' \\ &= 4\pi^2R_{xx}(-\lambda_1,\lambda_2). \end{aligned}$$

$\square$

Example (Theorem 11-2: Nonstationary white noise) If

$$R_{xx}(t_1,t_2) = q(t_1)\delta(t_1-t_2) \text{ with } q(t_1) > 0,$$

(which defines the so-called nonstationary white noise) then

$$\begin{aligned} S_{xx}(f_1,f_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1,t_2)e^{-j(f_1t_1+f_2t_2)}dt_1dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(t_1)\delta(t_1-t_2)e^{-j(f_1t_1+f_2t_2)}dt_1dt_2 \\ &= \int_{-\infty}^{\infty} q(t_2)e^{-j(f_1+f_2)t_2}dt_2 \\ &= Q(f_1+f_2) \end{aligned}$$

$$R_{XX}(u_1,u_2) = S_{xx}(u_1,-u_2) = Q(u_1-u_2),$$

and

$$S_{XX}(\lambda_1,\lambda_2) = 4\pi^2 R_{xx}(-\lambda_1,\lambda_2) = 4\pi^2 q(-\lambda_1)\delta(-\lambda_1-\lambda_2) = 4\pi^2 q(\lambda_2)\delta(\lambda_1+\lambda_2).$$

From the above derivation, it is apparent that if a nonstationary white noise $\boldsymbol{x}(t)$ has zero mean, then $\boldsymbol{X}(u)$ becomes WSS. $\square$

Example If $\boldsymbol{x}(t)$ is WSS, then

$$\begin{aligned} S_{xx}(f_1, f_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1 - t_2)e^{-j(f_1t_1+f_2t_2)}dt_1dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(s)e^{-j(f_1s+f_1t_2+f_2t_2)}dsdt_2 \\ &= S_{xx}(f_1)\int_{-\infty}^{\infty} e^{-j(f_1+f_2)t_2}dt_2 \\ &= 2\pi S_{xx}(f_1)\delta(f_1 + f_2). \end{aligned}$$

Hence,

$$R_{XX}(u, v) = S_{xx}(u, -v) = 2\pi S_{xx}(u)\delta(u - v) \quad \left(\text{where } S_{xx}(u) \geq 0\right).$$

$\square$

In summary:

- The Fourier transform of a zero-mean nonstationary white process becomes WSS.
- The Fourier transform of a WSS process becomes nonstationary white.

Example If $\boldsymbol{x}(t)$ is real and WSS, then

$$R_{XX}(u,v) = E[\boldsymbol{X}(u)\boldsymbol{X}^*(v)] = S_{xx}(u,-v) = 2\pi S_{xx}(u)\delta(u-v).$$

Taking $u=\omega$ and $v=-\omega$ for $\omega \neq 0$, together with the fact that $\boldsymbol{X}(\omega) = \boldsymbol{X}^*(-\omega)$, yields:

$$\begin{aligned}
R_{XX}(\omega,-\omega) &= E[\boldsymbol{X}(\omega)\boldsymbol{X}^*(-\omega)] \\
&= E[\boldsymbol{X}^2(\omega)] \\
&= E[\mathrm{Re}\{\boldsymbol{X}(\omega)\}^2] - E[\mathrm{Im}\{\boldsymbol{X}(\omega)\}^2] + 2jE[\mathrm{Re}\{\boldsymbol{X}(\omega)\}\cdot\mathrm{Im}\{\boldsymbol{X}(\omega)\}] \\
&\left(= 2\pi S_{xx}(\omega)\delta(2\omega)\right) = 0.
\end{aligned}$$

This concludes:

$$E[\mathrm{Re}\{\boldsymbol{X}(\omega)\}^2] = E[\mathrm{Im}\{\boldsymbol{X}(\omega)\}^2] \quad \text{and} \quad E[\mathrm{Re}\{\boldsymbol{X}(\omega)\}\cdot\mathrm{Im}\{\boldsymbol{X}(\omega)\}] = 0.$$

$\square$

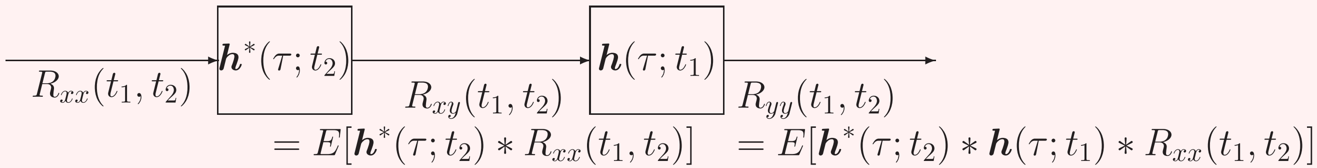
Windowing

A windowing filter is of the form $\boldsymbol{h}(\tau; t) = w(t)\delta(\tau)$ that induces

$$\boldsymbol{y}(t) = \boldsymbol{x}(t)w(t) = \int_{-\infty}^{\infty} \underbrace{w(t)\delta(\tau)}_{\boldsymbol{h}(\tau;t)} \boldsymbol{x}(t-\tau)d\tau$$

Example 11-11 $w(t) = \mathbf{1}\{|t| \le T\}$ for WSS $\boldsymbol{x}(t)$.

Fundamental Theorem and Theorem 9-2 For any linear system,



- For the windowing filter,

$$\begin{aligned} R_{xy}(t_1, t_2) &= E[\boldsymbol{h}^*(\tau; t_2) * R_{xx}(t_1, t_2)] \\ &= E\left[\int_{-\infty}^{\infty} \boldsymbol{h}^*(\tau; t_2) R_{xx}(t_1, t_2 - \tau) d\tau\right] \\ &= w^*(t_2) \int_{-\infty}^{\infty} \delta(\tau) R_{xx}(t_1, t_2 - \tau) d\tau \\ &= w^*(t_2) R_{xx}(t_1, t_2) \end{aligned}$$

and

$$
\begin{aligned}
R_{yy}(t_1, t_2) &= E[\boldsymbol{h}(\tau; t_1) * R_{xy}(t_1, t_2)] \\
&= E\left[\int_{-\infty}^{\infty} \boldsymbol{h}(\tau; t_1) R_{xy}(t_1 - \tau, t_2) d\tau\right] \\
&= w(t_1) \int_{-\infty}^{\infty} \delta(\tau) R_{xy}(t_1 - \tau, t_2) d\tau \\
&= w(t_1) R_{xy}(t_1, t_2) \\
&\left(= w(t_1) w^*(t_2) R_{xx}(t_1, t_2)\right)
\end{aligned}
$$

Windowing

- For the windowing filter,

$$
\begin{aligned}
&S_{xy}(u_1, u_2)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(t_1, t_2)e^{-j(t_1u_1+t_2u_2)}dt_1dt_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^*(t_2)R_{xx}(t_1, t_2)e^{-j(t_1u_1+t_2u_2)}dt_1dt_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^*(t_2)\left(\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(v_1, v_2)e^{j(t_1v_1+t_2v_2)}dv_1dv_2\right)e^{-j(t_1u_1+t_2u_2)}dt_1dt_2\\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-jt_1(u_1-v_1)}dt_1\left(\int_{-\infty}^{\infty} w(t_2)e^{-jt_2(v_2-u_2)}dt_2\right)^*\right)S_{xx}(v_1, v_2)dv_1dv_2\\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(2\pi\delta(u_1-v_1)W^*(v_2-u_2)\right)S_{xx}(v_1, v_2)dv_1dv_2\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} W^*(v_2-u_2)S_{xx}(u_1, v_2)dv_2,
\end{aligned}
$$

and

$$
\begin{aligned}
S_{yy}(u_1,u_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j(t_1u_1+t_2u_2)}dt_1dt_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(t_1)R_{xy}(t_1,t_2)e^{-j(t_1u_1+t_2u_2)}dt_1dt_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(t_1)\left(\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xy}(v_1,v_2)e^{j(t_1v_1+t_2v_2)}dv_1dv_2\right)e^{-j(t_1u_1+t_2u_2)}dt_1dt_2 \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} w(t_1)e^{-jt_1(u_1-v_1)}dt_1\int_{-\infty}^{\infty} e^{-jt_2(u_2-v_2)}dt_2\right)S_{xy}(v_1,v_2)dv_1dv_2 \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(W(u_1-v_1)2\pi\delta(u_2-v_2)\Big)S_{xy}(v_1,v_2)dv_1dv_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(u_1-v_1)S_{xy}(v_1,u_2)dv_1 \\
&\left(= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(u_1-v_1)W^*(v_2-u_2)S_{xx}(v_1,v_2)dv_1dv_2\right).
\end{aligned}
$$

Windowing

Hence,

$$\begin{aligned} R_{YY}(u_1,u_2) &= S_{yy}(u_1,-u_2) \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(u_1-v_1)W^*(v_2+u_2)S_{xx}(v_1,v_2)dv_1dv_2. \end{aligned}$$

For WSS $\boldsymbol{x}(t)$, $S_{xx}(v_1,v_2)=2\pi S_{xx}(v_1)\delta(v_1+v_2)$ (cf. Slide 11-81); this reduces the formula of $R_{YY}(u_1,u_2)$ to:

$$\begin{aligned} & R_{YY}(u_1,u_2) \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(u_1-v_1)W^*(v_2+u_2)2\pi S_{xx}(v_1)\delta(v_1+v_2)dv_1dv_2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(u_1-v_1)W^*(u_2-v_1)S_{xx}(v_1)dv_1. \end{aligned}$$

Example 11-11 $w(t) = \mathbf{1}\{|t| \le T\}$ for WSS $\boldsymbol{x}(t)$. Determine $R_{YY}(u,u)$.

Answer: We know that $W(\omega) = 2\sin(T\omega)/\omega$.
Hence,

$$\begin{aligned}
R_{YY}(u,u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(u-v)W^*(u-v)S_{xx}(v)dv \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |W(u-v)|^2 S_{xx}(v)dv \\
&= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(T(u-v))}{(u-v)^2} S_{xx}(v)dv.
\end{aligned}$$

□

Fourier-Stieltjes Representation of WSS processes

Define

$$\boldsymbol{Z}(\omega) \triangleq \int_{-\infty}^{\omega} \boldsymbol{X}(\alpha)d\alpha$$

where $\boldsymbol{X}(\omega)$ is the Fourier transform of a WSS process $\boldsymbol{x}(t)$.

- By the Fourier-Stieltjes notation,

$$d\boldsymbol{Z}(\omega) = \boldsymbol{X}(\omega)d\omega.$$

Hence,

$$\boldsymbol{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\boldsymbol{X}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}d\boldsymbol{Z}(\omega).$$

Properties of $\boldsymbol{Z}(\omega)$

- That $\boldsymbol{x}(t)$ is WSS implies

$$R_{XX}(u,v) = 2\pi S_{xx}(u)\delta(u-v),$$

where $S_{xx}(u) \geq 0$, namely, $\boxed{\boldsymbol{X}(u) \text{ is a nonstationary white process}}$ (cf. Slide 11-81).

- Integration of a nonstationary white process is a process with **orthogonal increments**.

Proof:

$$\begin{aligned}
& E\{[\boldsymbol{Z}(\omega_2) - \boldsymbol{Z}(\omega_1)][\boldsymbol{Z}(\omega_4) - \boldsymbol{Z}(\omega_3)]^*\} \\
&= E\left\{\int_{\omega_1}^{\omega_2} \boldsymbol{X}(\alpha)d\alpha \cdot \int_{\omega_3}^{\omega_4} \boldsymbol{X}^*(\beta)d\beta\right\} \\
&= \int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4} R_{XX}(\alpha,\beta)d\beta d\alpha \\
&= \int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4} 2\pi S_{xx}(\alpha)\delta(\alpha-\beta)d\beta d\alpha \\
&= \int_{\omega_1}^{\omega_2} 2\pi S_{xx}(\alpha)\mathbf{1}\{\omega_3 < \alpha < \omega_4\}d\alpha \\
&= 0,\ \text{if } (\omega_1,\omega_2)\cap(\omega_3,\omega_4) = \emptyset.
\end{aligned}$$

Wold's Decomposition

Theorem (Wold's decomposition for continuous processes) An arbitrary WSS process $\boldsymbol{x}(t)$ can be decomposed into sum of a *regular* process $\boldsymbol{x}_r(t)$ and a *predictable* process $\boldsymbol{x}_p(t)$, for which $\boldsymbol{x}_r(t)$ and $\boldsymbol{x}_p(t)$ are orthogonal.

Definition (Predictable process) A process is called *predictable* if its present value can be determined by its past.

- A (WSS) process is predictable if, and only if, **its spectrum consists of lines**.
- An example of a predictable process is the discrete AR process with line spectra. See Slide 11-51:

$$\boldsymbol{x}[t] + a_1\boldsymbol{x}[t-1] + a_2\boldsymbol{x}[t-2] + \cdots + a_n\boldsymbol{x}[t-n] \;=\; 0.$$

Theorem (Wold's decomposition for discrete processes) An arbitrary WSS process $\boldsymbol{x}[t]$ can be decomposed into sum of a *regular* process $\boldsymbol{x}_r[t]$ and a *predictable* process $\boldsymbol{x}_p[t]$, for which $\boldsymbol{x}_r[t]$ and $\boldsymbol{x}_p[t]$ are orthogonal.

Wold's Decomposition

Proof:

- Form the predictor of $\boldsymbol{x}[t]$ based on its past as:

$$\hat{\boldsymbol{x}}[t] = \sum_{k=1}^{\infty} a_k \boldsymbol{x}[t-k].$$

The optimal $\{a_k\}_{k=1}^{\infty}$ in the MS sense can be obtained through the fact that the MS prediction error

$$\boldsymbol{e}[t] = \boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t]$$

is orthogonal to the data, i.e.,

$$\begin{aligned} E\{\boldsymbol{e}[t]\boldsymbol{x}^*[t-m]\} &= E\left\{\left(\boldsymbol{x}[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}[t-k]\right)\boldsymbol{x}^*[t-m]\right\} \\ &= 0 \text{ for any } m \geq 1. \end{aligned}$$

This leads to the discrete Wiener-Höpe equation:

$$R_{xx}[m] = \sum_{k=1}^{\infty} a_k R_{xx}[m-k] \text{ for } m > 0.$$

Wold's Decomposition

In addition, it can be shown that $\boldsymbol{e}[t]$ is a white process.

For $\tau > 0$,

$$E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{e}^*[t]\right\} = \underbrace{\cancel{E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{x}^*[t]\right\}}}_{=0} - \sum_{m=1}^{\infty} a_m \underbrace{\cancel{E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{x}^*[t-m]\right\}}}_{=0} = 0.$$

For $\tau < 0$,

$$E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{e}^*[t]\right\} = \left(E\left\{\boldsymbol{e}[t]\boldsymbol{e}^*[t+\tau]\right\}\right)^* = 0.$$

Hence, $\boldsymbol{e}[t]$ is white.

In summary,

$\hat{\boldsymbol{x}}[t]$ is the best MS estimate of $\boldsymbol{x}[t]$ in terms of the past of $\boldsymbol{x}[t]$.

$\boldsymbol{e}[t] = \boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t]$ is the part of $\boldsymbol{x}[t]$ that remains "unestimated."

Wold's Decomposition

- Form the best MS estimator of $\boldsymbol{x}[t]$ in terms of $\boldsymbol{e}[t]$ and its past:

$$\boldsymbol{x}_r[t] = \sum_{k=0}^{\infty} w_k \boldsymbol{e}[t-k].$$

Again, the error

$$\boldsymbol{x}_p[t] = \boldsymbol{x}[t] - \boldsymbol{x}_r[t] \quad = \boldsymbol{x}[t] - \sum_{k=0}^{\infty} w_k \underbrace{\left(\boldsymbol{x}[t-k] - \sum_{\ell=1}^{\infty} \boldsymbol{x}[t-k-\ell] \right)}_{\boldsymbol{e}[t-k]}$$

should be orthogonal to $\{\boldsymbol{e}[t-k]\}_{k=0}^{\infty}$. Since $\boldsymbol{x}_p[t]$ is a linear combination of $\boldsymbol{x}[t]$ and its past, $\boldsymbol{x}_p[t]$ is orthogonal to $\boldsymbol{e}[t+m]$ for $m > 0$.

In summary,

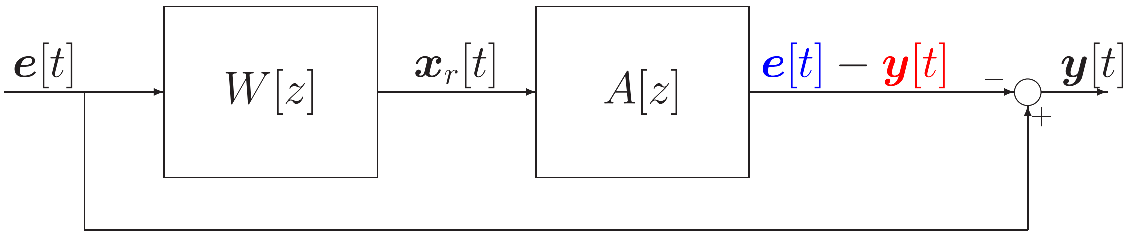
$$\begin{cases} \boldsymbol{x}_p[t] \perp \boldsymbol{e}[t-k] \quad \text{for every integer } k \\ \boldsymbol{x}_r[t] = \sum_{k=0}^{\infty} w_k \boldsymbol{e}[t-k] \end{cases}$$

implies $\boldsymbol{x}_p[t] \perp \boldsymbol{x}_r[t]$.

- $\boldsymbol{x}_r[t]$ is obtained by feeding a white input to a causal (and stable) filter; hence, it is regular.

Wold's Decomposition

- It remains to prove that $\boldsymbol{x}_p[t]$ is predictable.
  - Define two filters $A[z] = 1 - \sum_{k=1}^{\infty} a_k z^{-k}$ and $W[z] = \sum_{k=0}^{\infty} w_k z^{-k}$.
  - Define $\boldsymbol{y}[t] = \boldsymbol{x}_p[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}_p[t-k]$.
  - Then, by that $\boldsymbol{e}[t]$ and $\boldsymbol{y}[t]$ are respectively the outputs due to inputs $\boldsymbol{x}[t]$ and $\boldsymbol{x}_p[t]$ through linear filter $A[z]$, we learn that $\boldsymbol{e}[t] - \boldsymbol{y}[t]$ is the output due to input $\boldsymbol{x}_r[t] = \boldsymbol{x}[t] - \boldsymbol{x}_p[t]$ through filter $A[z]$. Together with that $\boldsymbol{x}_r[t]$ is the output due to input $\boldsymbol{e}[t]$ through filter $W[z]$, we have:



  - This summarizes to that $\boldsymbol{y}[t]$ is the output due to input $\boldsymbol{e}[t]$ through filter $1 - A[z]W[z]$. So, $\boldsymbol{y}[t]$ is completely determined by $\boldsymbol{e}[t]$ and its past.
  - However, the definition of $\boldsymbol{y}[t] = \boldsymbol{x}_p[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}_p[t-k]$ indicates that $\boldsymbol{y}[t]$ is also completely determined by $\boldsymbol{x}_p[t]$ and its past.
  - Finally, $\boldsymbol{x}_p[t] \perp \boldsymbol{e}[t-k]$ for every integer k implies $E\{|\boldsymbol{y}[t]|^2\} = 0$. $\square$

Wold's Decomposition

Further observation on Wold's Decomposition:

- $S_{xx}[e^{j\omega}] = S_{x_r x_r}[e^{j\omega}] + S_{x_p x_p}[e^{j\omega}]$, where $S_{x_r x_r}[e^{j\omega}] = |\mathtt{L}[e^{j\omega}]|^2$ for some $\mathtt{L}[e^{j\omega}]$, and $S_p[e^{j\omega}]$ is a line spectrum.

Wold's Decomposition

Example 11-12 $\boldsymbol{y}(t) = \boldsymbol{a} \cdot \boldsymbol{x}(t)$ with $E[\boldsymbol{a}] = 0$ and WSS regular $\boldsymbol{x}(t)$ is independent of $\boldsymbol{a}$. Find Wold's decomposition $\boldsymbol{y}_r(t)$ and $\boldsymbol{y}_p(t)$ of $\boldsymbol{y}(t)$.

Answer:

$$\begin{aligned} R_{yy}(\tau) &= E[\boldsymbol{y}(t+\tau)\boldsymbol{y}^*(t)] \\ &= E\left[\boldsymbol{a}\boldsymbol{x}(t+\tau)\boldsymbol{a}^*\boldsymbol{x}^*(t)\right] \\ &= \sigma_a^2 R_{xx}(\tau), \end{aligned}$$

where $\sigma_a^2 = E[\boldsymbol{a}\boldsymbol{a}^*]$. Hence,

$$S_{yy}(\omega) = \sigma_a^2 S_{xx}(\omega) = \sigma_a^2 \left[S_{xx}^c(\omega) + 2\pi|\eta_x|^2\delta(\omega)\right],$$

where $\eta_x \triangleq E[\boldsymbol{x}(t)]$. Accordingly,

$$S_{yy,r}(\omega) = \sigma_a^2 S_{xx}^c(\omega) \quad \text{and} \quad S_{yy,p}(\omega) = 2\pi|\eta_x|^2\sigma_a^2\delta(\omega).$$

We can then set $\boldsymbol{y}_p(t) = \eta_x\boldsymbol{a}$, and $\boldsymbol{y}_r(t) = \boldsymbol{y}(t) - \eta_x\boldsymbol{a} = \boldsymbol{a}[\boldsymbol{x}(t) - \eta_x]$. $\square$

Examination of the selected $\boldsymbol{y}_p(t)$ and $\boldsymbol{y}_r(t)$:

- $\boldsymbol{y}_p(t) = \boldsymbol{y}_p(t-\tau)$ for any $\tau \geq 0$; hence, $\boldsymbol{y}_p(t)$ can be determined by its past.
- $E[\boldsymbol{y}_r(t+\tau)\boldsymbol{y}_r^*(t)] = \sigma_a^2 R_{xx}^c(\tau)$, and hence $S_{y_ry_r}(\omega) = \sigma_a^2 S_{xx}^c(\omega)$.
- $E[\boldsymbol{y}_r(t)\boldsymbol{y}_p^*(t)] = E\left\{\boldsymbol{a}[\boldsymbol{x}(t) - \eta_x]\eta_x^*\boldsymbol{a}^*\right\} = \sigma_a^2\eta_x^* E\left\{\boldsymbol{x}(t) - \eta_x\right\} = 0.$

Spectral Representation of Discrete Random Processes

- The Fourier transform of a discrete random process $\boldsymbol{x}[t]$ is also a random process defined as:

$$\boldsymbol{X}(u) \triangleq \sum_{t=-\infty}^{\infty} \boldsymbol{x}[t]e^{-jut},$$

which is periodic with period 2π.

Lemma

- Let $R_{XX}(u_1, u_2)$ be the autocorrelation function of $\boldsymbol{X}(t)$.
- Let $S_{xx}[f_1, f_2]$ be the two-dimensional power spectrum of discrete $\boldsymbol{x}[t]$.

Then,

$$R_{XX}(u_1, u_2) = S_{xx}[u_1, -u_2] \text{ for } -\pi \le u_1, u_2 < \pi.$$

Proof:

$$
\begin{aligned}
R_{XX}(u_1, u_2) &= E[\boldsymbol{X}(u_1)\boldsymbol{X}^*(u_2)] \\
&= \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} E\{\boldsymbol{x}[t_1]\boldsymbol{x}^*[t_2]\} e^{-j(u_1 t_1 - u_2 t_2)} \\
&= \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} R_{xx}[t_1, t_2] e^{-j[u_1 t_1 + (-u_2) t_2]} \\
&= S_{xx}[u_1, -u_2].
\end{aligned}
$$

$\square$

Example If $\boldsymbol{x}[t]$ is WSS, then for $-\pi \leq f_1, f_2 < \pi$,

$$\begin{aligned} S_{xx}[f_1, f_2] &= \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} R_{xx}[t_1 - t_2] e^{-j(f_1 t_1 + f_2 t_2)} \\ &= \sum_{t_2=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} R_{xx}[s] e^{-j(f_1 s + f_1 t_2 + f_2 t_2)} \\ &= S_{xx}[f_1] \sum_{t_2=-\infty}^{\infty} e^{-j(f_1+f_2)t_2} \\ &= 2\pi S_{xx}[f_1]\delta(f_1 + f_2). \end{aligned}$$

$$\boxed{2\pi \sum_{n=-\infty}^{\infty} \delta(x + 2\pi n) = \sum_{n=-\infty}^{\infty} e^{-jnx}}$$

Hence, for $-\pi \leq u, v < \pi$,

$$R_{XX}(u, v) = S_{xx}[u, -v] = 2\pi S_{xx}[u]\delta(u - v) \quad \left(\text{where } S_{xx}[u] \geq 0\right).$$

$\square$

Bispectra and Third Order Moments

Definition (Bispectrum) The bispectrum $\bar{S}_{xxx}(\omega_1, \omega_2)$ of a random process $\boldsymbol{x}(t)$ is the two-dimensional Fourier transform of its third order moment $\bar{R}_{xxx}(u, v) = R_{xxx}(t+u, t+v, t) \triangleq E[\boldsymbol{x}(t+u)\boldsymbol{x}(t+v)\boldsymbol{x}^*(t)]$ in u and v, where $R_{xxx}(t+u, t+v, t)$ is independent of t.

Remarks

- A case that $R_{xxx}(t+u, t+v, t)$ is independent of t is that $\boldsymbol{x}(t)$ is SSS (in which $R_{xx}(t+u, t+v, t)$ only depends on the two differences).
- When only the individual statistics of system input and system output are known, their power spectrums can only be used to determine the *system amplitude* (of $H(\omega)$)!

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega).$$

- In light of the third-order moments, the *system phase* can be identified.

$$\bar{S}_{yyy}(\omega_1, \omega_2) = \bar{S}_{xxx}(\omega_1, \omega_2) H(\omega_1) H(\omega_2) H^*(\omega_1 + \omega_2).$$

Bispectra and Third Order Moments

$$
\begin{aligned}
\bar{S}_{yyy}(\omega_1,\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{R}_{yyy}(u,v)e^{-j(u\omega_1+v\omega_2)}dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\boldsymbol{y}(t+u)\boldsymbol{y}(t+v)\boldsymbol{y}^*(t)\right]e^{-j(u\omega_1+v\omega_2)}dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\left(\int_{-\infty}^{\infty} h(\tau_1)\boldsymbol{x}(t+u-\tau_1)d\tau_1\right)\right. \\
&\quad \left.\left(\int_{-\infty}^{\infty} h(\tau_2)\boldsymbol{x}(t+v-\tau_2)d\tau_2\right)\left(\int_{-\infty}^{\infty} h^*(\tau_3)\boldsymbol{x}^*(t-\tau_3)d\tau_3\right)\right] \\
&\quad e^{-j(u\omega_1+v\omega_2)}dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)h^*(\tau_3)\bar{R}_{xxx}(u-\tau_1+\tau_3,v-\tau_2+\tau_3) \\
&\quad e^{-j(u\omega_1+v\omega_2)}dudvd\tau_1d\tau_2d\tau_3 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)h^*(\tau_3)\bar{R}_{xxx}(u',v') \\
&\quad e^{-j(u'\omega_1+\tau_1\omega_1-\tau_3\omega_1+v'\omega_2+\tau_2\omega_2-\tau_3\omega_2)}du'dv'd\tau_1d\tau_2d\tau_3 \\
&= \bar{S}_{xxx}(\omega_1,\omega_2)H(\omega_1)H(\omega_2)H^*(\omega_1+\omega_2).
\end{aligned}
$$

Example If $\boldsymbol{x}(t)$ is a SSS white process, where "white" implies $\bar{R}_{xxx}(u,v) = Q\delta(u)\delta(v)$ and $\bar{S}_{xxx}(\omega_1,\omega_2) = Q$, then

$$\bar{S}_{yyy}(\omega_1,\omega_2) \;=\; Q\cdot H(\omega_1)H(\omega_2)H^*(\omega_1+\omega_2),$$

which implies

$$\begin{aligned}\theta(\omega_1,\omega_2) \triangleq \angle\bar{S}_{yyy}(\omega_1,\omega_2) &= \angle H(\omega_1) + \angle H(\omega_2) - \angle H(\omega_1+\omega_2)\\ &\triangleq \varphi(\omega_1) + \varphi(\omega_2) - \varphi(\omega_1+\omega_2).\end{aligned}$$

Then

$$\left.\frac{\partial\theta(\omega_1,\omega_2)}{\partial\omega_2}\right|_{\omega_2=0} \;=\; \varphi'(0) - \varphi'(\omega_1),$$

and

$$\begin{aligned}\varphi(\omega) - \varphi(0) &= \int_0^{\omega}\varphi'(\omega_1)d\omega_1\\ &= \varphi'(0)\omega - \int_0^{\omega}\left.\frac{\partial\theta(\omega_1,\omega_2)}{\partial\omega_2}\right|_{\omega_2=0} d\omega_1.\end{aligned}$$

Note that for a real system, $\varphi(0) = 0$. However, $\varphi'(0)$ may not be zero!

Theorem 11-4 For a real SSS process $\boldsymbol{x}(t)$,

$$R_{XXX}(u,v,\omega) = E[\boldsymbol{X}(u)\boldsymbol{X}(v)\boldsymbol{X}^*(\omega)] = 2\pi\bar{S}_{xxx}(u,v)\delta(u+v-\omega).$$

Proof:

$$\begin{aligned}
E[\boldsymbol{X}(u)\boldsymbol{X}(v)\boldsymbol{X}^*(\omega)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)\boldsymbol{x}^*(t_3)]e^{-j(ut_1+vt_2-\omega t_3)}dt_1dt_2dt_3 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{R}_{xxx}(t_1-t_3,t_2-t_3)e^{-j(ut_1+vt_2-\omega t_3)}dt_1dt_2dt_3 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{R}_{xxx}(s_1,s_2)e^{-j(us_1+ut_3+vs_2+vt_3-\omega t_3)}ds_1ds_2dt_3 \\
&= 2\pi\bar{S}_{xxx}(u,v)\delta(u+v-\omega).
\end{aligned}$$

$\square$

The end of Section 11-4 Spectral Representation of Random Processes