高微第十三週作業

1. Prove that Let $E \subseteq R^3$ and suppose the functions $g: E \to R$ and $h: E \to R$ are continuously differentiable. Assume that $(x_0, y_0, z_0) \in E$ at which

$$\begin{cases} g(x_0, y_0, z_0) = 0 \\ h(x_0, y_0, z_0) = 0 \end{cases} \text{ and } \nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0) \neq \mathbf{0}.$$

Then there exists a neighborhood of the point (x_0, y_0, z_0) in which the solutions of the system of the equations

$$\begin{cases} g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}, \text{ where } (x, y, z) \in E$$

consists of a path that at the point (x_0, y_0, z_0) has $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ as a tangent vector.

2. Let $E \subseteq R^3$ and suppose the function $f: E \to R$, $g: E \to R$ are continuously differentiable. Let $S = \{\mathbf{x} \in E \mid g(\mathbf{x}) = 0\}$, suppose that $f|_S : S \to R$ has an extremum at $\mathbf{u} \in S$ and $\nabla g(\mathbf{u}) \neq \mathbf{0}$, prove that there exists a real number λ such that

$$\nabla f(\mathbf{u}) = \lambda \nabla g(\mathbf{u})$$

3. Let $E \subseteq R^3$ and suppose the function $f: E \to R$, $g: E \to R$ and $h: E \to R$ are continuously differentiable. Let $C = \{\mathbf{x} \in E \mid g(\mathbf{x}) = 0 \text{ and } h(\mathbf{x}) = 0\}$, suppose that $f|_C : C \to R$ has an extremum at $\mathbf{u} \in C$ and $\nabla g(\mathbf{u}) \times \nabla h(\mathbf{u}) \neq \mathbf{0}$, prove that there exists real number λ_1, λ_2 such that

$$\nabla f(\mathbf{u}) = \lambda_1 \nabla g(\mathbf{u}) + \lambda_2 \nabla h(\mathbf{u}).$$