NCTU Open Course Ware 國立交通大學開放式課程

高微第十週作業

1. Let $E \subseteq R^n$ be an open convex set and suppose that $f : E \to R$ is differentiable. Prove that for all $\mathbf{a}, \mathbf{b} \in E$, there exists such that $0 < \theta < 1$ such that
$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f((1-\theta)\mathbf{a} + \theta\mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}).$$

2. Let $E \subseteq R^2$ be an open convex set and $f : E \to R$. Suppose that $\frac{\partial f}{\partial x}(x, y) = 0$ for all $(x, y) \in E$, prove that $f(x_1, y) = f(x_2, y)$ for all $(x_1, y), (x_2, y) \in E$.

3. Let
$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{, if } (x,y) \neq (0,0) \\ 0 & \text{, if } (x,y) = (0,0) \end{cases}$$
Prove that the 2nd-order partial derivatives of f exist for all $(x, y) \in \mathbb{R}^2$, but $\partial_y \partial_x f(0,0) \neq \partial_x \partial_y f(0,0)$.

4. Let $E \subseteq R^n$ be an open set and suppose $f : E \to R$ has continuous 2nd -order derivatives. Suppose that $\{\mathbf{x} \mid |\mathbf{x} - \mathbf{a}| < r\} \subseteq E$, prove that
$$\varphi''(t) = \left(f''(\mathbf{x} + t\mathbf{h})\mathbf{h}\right) \cdot \mathbf{h}$$

Definition Let $A \in L(R^n)$ is said to be positive definite provided that
$$A\mathbf{u} \cdot \mathbf{u} > 0 \text{ for all } \mathbf{0} \neq \mathbf{u} \in R^n$$
and is said to be negative definite provided that
$$A\mathbf{u} \cdot \mathbf{u} < 0 \text{ for all } \mathbf{0} \neq \mathbf{u} \in R^n.$$

5. Let $A \in L(R^2)$ and $[A] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
   (a) Prove that A is positive definite if and only if $a > 0$ and $ac - b^2 > 0$.
   (b) Prove that if $ac - b^2 < 0$ then there exist $\mathbf{u}_1, \mathbf{u}_2 \in R^2$ such that $A\mathbf{u}_1 \cdot \mathbf{u}_1 > 0$ and $A\mathbf{u}_2 \cdot \mathbf{u}_2 < 0$.