高微第九週作業

Rudin : p.79 # 9.

Extra problems

1. Given $\sum_{n=1}^{\infty} a_n$, put $\beta = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ and $\gamma = \liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Prove that

   (a) if $\beta < 1$, then $\sum_{n=1}^{\infty} a_n$ converges,

   (b) if $\gamma > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\{c_n\}$ be a sequence of positive numbers, prove that

$$\liminf_{n\to\infty}\frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty}\sqrt[n]{c_n}\ .$$

3. Given the power series $\sum_{n=0}^{\infty} c_n z^n$, put

$$\alpha = \limsup_{n\to\infty}\sqrt[n]{c_n}\ , \qquad R = \frac{1}{\alpha}.$$

   (If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$.) Prove that $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely if $|z| < R$, and diverges if $|z| > R$.

4. Given $\sum_{n=1}^{\infty} c_n$, suppose that

   (a) $|c_1| \ge |c_2| \ge |c_3| \ge \cdots$; (b) $c_{2m-1} \ge 0,\ c_{2m} \le 0 \quad (m = 1,2,3,\ldots)$; (c) $\lim_{n\to\infty} c_n = 0$.

   Prove that $\left|\sum_{n=0}^{\infty} c_n - \sum_{k=0}^{m} c_k\right| \le |c_{m+1}|,\ \forall\, m$.

5. Given the power series $\sum_{n=0}^{\infty} c_n x^n$, suppose that $\sum_{n=0}^{\infty} c_n x_1^n$ converges and $\sum_{n=0}^{\infty} c_n x_2^n$ diverges.

   (a) Prove that $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all $|x| < |x_1|$ and $\sum_{n=0}^{\infty} c_n x^n$ diverges for all $|x| > |x_2|$.

   (b) Prove that there exists a real number R such that $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely if $|z| < R$, and diverges if $|z| > R$ **without using root test**.