國立陽明交通大學 OCW 課程講義

Linear Time-Invariant Systems

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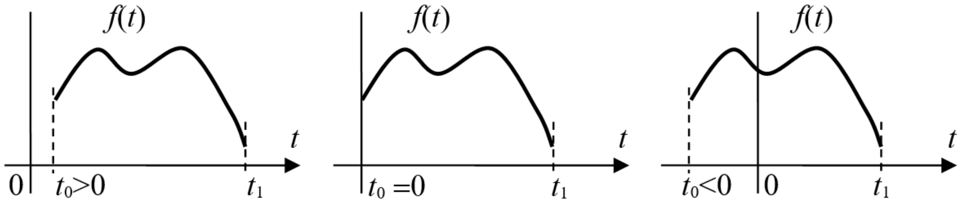
Outlines **Pages**

1. Continuous-Time Signals

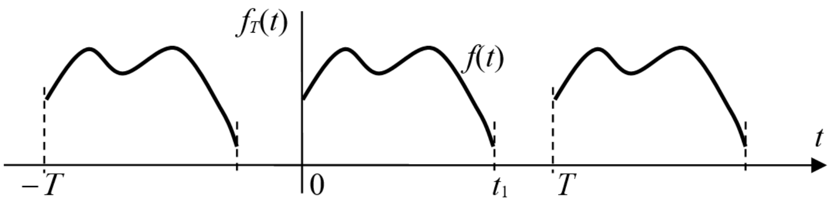
➢ Finite-duration Signals



We often choose the initial time at $t_0 = 0$ for real signals.

If $t_1 < \infty$ and $f(t) = 0$ for $t \notin [t_0, t_1]$, then $f(t)$ is called a finite-duration signal.

➢ Periodic Signals



Let $f(t) \in R$ be a finite-duration signal within $0 \le t \le t_1$, then

(1-1) $$f_T(t) = \sum_{k=-\infty}^{\infty} f(t - kT), \qquad -\infty < t < \infty, \quad T > t_1$$

is called a periodic function with period *T*, i.e., repeating itself every interval *T*.

➢ Fourier Series

Fourier series is an important mathematical tool, usually used to represent periodic functions satisfying the Dirichlet conditions:

(C1) $f_T(t)$ has finite number of discontinuous points in a period.

(C2) $f_T(t)$ has finite number of maximum and minimum in a period.

(C3) $f_T(t)$ is absolutely integrable in a period, i.e., $\int_T f_T(t)dt < \infty$.

There are two types of Fourier series often used, one in sine-cosine form and the other in exponential form.

The sine-cosine form of $f_T(t)$ is expressed as

(1-2) $$f_T(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k \cos\frac{2k\pi t}{T} + B_k \sin\frac{2k\pi t}{T}\right), \qquad -\infty < t < \infty$$

where $$A_0 = \frac{1}{T}\int_a^b f_T(t)dt = \frac{1}{T}\int_T f_T(t)dt$$

$$A_k = \frac{2}{T}\int_T f_T(t)\cos\frac{2k\pi t}{T}dt$$

$$B_k = \frac{2}{T}\int_T f_T(t)\sin\frac{2k\pi t}{T}dt$$

The exponential form of $f_T(t)$ is expressed as

(1-3) $$f_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad -\infty < t < \infty$$

where $\omega_0 = \dfrac{2\pi}{T}$ and $c_k = \dfrac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt = |c_k|\, e^{j\phi_k}$.

It is noticed that finite-duration signals can be expressed by Fourier series.
For example, $f(t) = f_T(t)$, in the interval $0 \le t \le t_1$, not $-\infty < t < \infty$.

The expression $f(t) = f_T(t)$ for $0 \le t \le t_1$ is not unique since the period T can be any number larger than t_1. To avoid such drawback, we use Fourier series with $T \to \infty$ as a replacement, such that the expression becomes unique and equivalent to Fourier transform.

Example

Given $f_T(t) = \cos 2t + \sin^2 t - \sin t \cdot \cos 3t$, determine its Fourier series and calculate A_0, A_k and B_k, $k = 1,2,3$.

Solution:

Since $f_T(t) = \cos 2t + \sin^2 t - \sin t \cdot \cos 3t = \frac{1}{2} + \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t - \frac{1}{2}\sin 4t$, we can rewrite it as

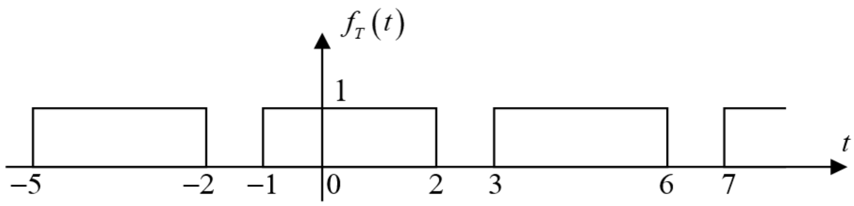
$$f_T(t) = \frac{1}{2} + \frac{1}{2}cos\frac{2\pi t}{T} + \frac{1}{2}sin\frac{2\pi t}{T} - \frac{1}{2}sin\frac{2\cdot 2\pi t}{T}$$
$$= A_0 + A_1 cos\frac{2\pi t}{T} + B_1 sin\frac{2\pi t}{T} + B_2 sin\frac{2\cdot 2\pi t}{T}$$

Clearly, $T = \pi$ and $A_0 = \frac{1}{2}$, $A_1 = \frac{1}{2}$, $B_1 = \frac{1}{2}$, $B_2 = -\frac{1}{2}$.

The remaining coefficients are all 0.

Example

Given $f_T(t) = f_T(t+4)$ and $f_T(t) = \begin{cases} 1, & -1 \le t \le 2 \\ 0, & 2 < t \le 3 \end{cases}$, determine its Fourier series and calculate A_0, A_k and B_k, for $k = 1,2,3$.



Solution:

Since $f_T(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi t}{T} + B_k sin\frac{2k\pi t}{T}\right)$, where $T = 4$, we have

$$A_0 = \frac{1}{4}\int_{-1}^{2} dt = \frac{3}{4} = 0.75, \qquad A_k = \frac{1}{2}\int_{-1}^{2} cos\frac{k\pi t}{2}dt = \frac{1}{k\pi}sin\frac{k\pi}{2}$$

$$B_k = \frac{1}{2}\int_{-1}^{2} sin\frac{k\pi t}{2}dt = -\frac{1}{k\pi}cos\,k\pi + \frac{1}{k\pi}cos\frac{k\pi}{2}$$

Hence, $A_1 = \frac{1}{\pi} = 0.3183$, $A_2 = 0$, $A_3 = -\frac{1}{3\pi} = -0.1061$

$$B_1 = \frac{1}{\pi} = 0.3183, \quad B_2 = -\frac{1}{\pi} = -0.3183, \quad B_3 = \frac{1}{3\pi} = 0.1061$$

➢ Fourier Transform

Mathematically, the Fourier transform of $f(t) \in R$ is defined as

(1-4) $\Im\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

Then, the inverse Fourier transform of $F(\omega)$ can be determined as

(1-5) $\Im^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

which implies $f(t)$ consists of an infinite number of terms $F(\omega)e^{j\omega t}$.

Since the Fourier transform $F(\omega)$ is a complex number, we can write it as

(1-6) $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = |F(\omega)|\, e^{j\phi(\omega)}$

From $F(-\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt = F^*(\omega)$, we have

(1-7) $|F(-\omega)|\, e^{j\phi(-\omega)} = |F(\omega)|\, e^{-j\phi(\omega)}$

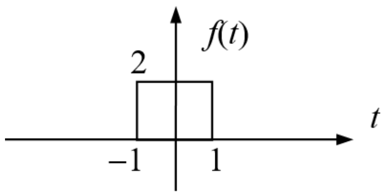
Hence, $|F(-\omega)| = |F(\omega)|$ and $\phi(-\omega) = -\phi(\omega)$. Clearly, $F(\omega)$ is an even function and $\phi(\omega)$ is an odd function.

For example, consider a pulse function

$$f(t) = 2, \qquad |t| \le 1$$

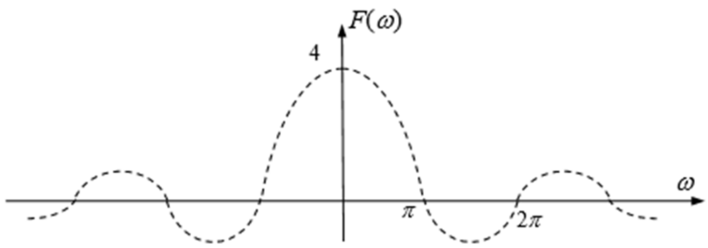
whose fourier transform is

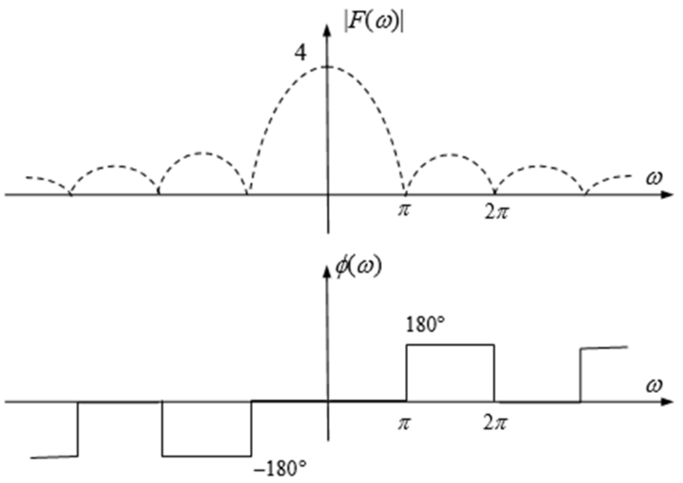
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-1}^{1} 2e^{-j\omega t} dt$$
$$= \frac{4\sin\omega}{\omega} = 4\,sinc(\omega)$$



Since $sinc(\omega) \equiv \frac{\sin\omega}{\omega}$ is a real number,

we have $|F(\omega)| = 4|sinc(\omega)|$ and $\phi(\omega) = \pm 180°$.





➢ **Fourier transform of a periodic function**

A periodic function can be represented by Fourier series, shown as

(1-8) $$f(t)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$$

whose Fourier transform can be expressed as

(1-9) $$\Im\{f(t)\}=\Im\left\{\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}\right\}=\sum_{k=-\infty}^{\infty}c_k\Im\left\{e^{jk\omega_0 t}\right\}$$

From the truth of

(1-10) $$\Im^{-1}\{\delta(\omega-k\omega_0)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-k\omega_0)e^{j\omega t}d\omega=\frac{1}{2\pi}e^{jk\omega_0 t}$$

we have

(1-11) $$\Im\left\{e^{jk\omega_0 t}\right\}=\int_{-\infty}^{\infty}e^{jk\omega_0 t}e^{-j\omega t}d\omega=\int_{-\infty}^{\infty}e^{-j(\omega-k\omega_0)t}d\omega=2\pi\delta(\omega-k\omega_0)$$

Hence, (1-9) can be rewritten as

(1-12) $$\Im\{f(t)\}=\sum_{k=-\infty}^{\infty}c_k\Im\left\{e^{jk\omega_0 t}\right\}=\sum_{k=-\infty}^{\infty}2\pi c_k\delta(\omega-k\omega_0)$$

which consists of a sequence of impulses $\delta(\omega-k\omega_0)$ weighted with $2\pi c_k$.

Therefore, a periodic function has a discrete frequency spectra.

➢ Properties of Fourier transform

Linearity

(1-13) $$\Im\{af(t)+bg(t)\}=aF(\omega)+bG(\omega)$$

Time-shifting

(1-14) $$\Im\{f(t-t_0)\}=e^{-j\omega t_0}\ F(\omega)$$

Frequency-shifting

(1-15) $$\Im\{e^{j\omega_0 t}f(t)\}=F(\omega-\omega_0)$$

Time compression and expansion

(1-16) $$\Im\{f(at)\}=\frac{1}{a}F\left(\frac{\omega}{a}\right)$$

Convolution in **time**

(1-17) $$\Im\left\{\int_{-\infty}^{\infty}f(t-\tau)g(\tau)d\tau\right\}=\Im\{f(t)*g(t)\}=F(\omega)G(\omega)$$

Multiplication in time

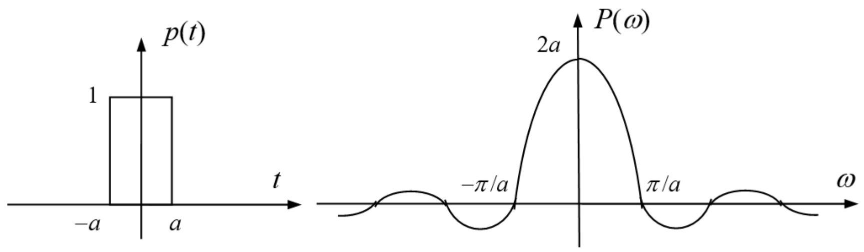
(1-18) $$\Im\{f(t)g(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega-\Omega)G(\Omega)d\Omega=\frac{1}{2\pi}F(\omega)*G(\omega)$$

Time bandwidth product

Consider a pulse $p(t)=1,\ -a\le t\le a$, whose time duration is $TD=2a$.

The Fourier transform of $p(t)$ is

(1-19) $$P(\omega)=\int_{-a}^{a}p(t)e^{-j\omega t}dt=\int_{-a}^{a}e^{-j\omega t}dt=2a\frac{\sin\omega a}{\omega a}=2a\,sinc(\omega a)$$



Since most of the power of $P(\omega)$ is concentrated in $-\frac{\pi}{a}\le\omega\le\frac{\pi}{a}$, the frequency bandwidth is regarded as $BW=2\pi/a$. Hence, the time-bandwidth product (*TBP*) is

(1-20) $$TD \cdot BW = 2a \cdot \frac{2\pi}{a} = 4\pi$$

Obviously, the *TBP* is a constant, i.e., *TD* and *BW* are inversely proportional.

According to *TBP*, if we want to transmit a pulse train with *TD* through a channel, then the bandwidth *BW* of the channel should be proportional to the inverse of *TD*.
In other words, the faster the pulse train (with smaller *TD*), the wider the bandwidth (larger *BW* selected).

- **Parseval's theorem**

The instantaneous power of a function $f(t)$ is often expressed as $f^2(t)$ or $f(t)f^*(t)$. Then, the total energy is given as

(1-21)

$$E = \int_{-\infty}^{\infty} f(t) f^*(t) dt = \int_{-\infty}^{\infty} \left| f(t) \right|^2 (t) dt$$

which also can be expressed as

(1-22)

$$\begin{aligned} E &= \int_{-\infty}^{\infty} f(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega \right) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| F(\omega) \right|^2 d\omega \end{aligned}$$

Therefore,

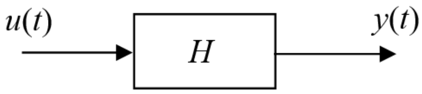
(1-23)

$$\int_{-\infty}^{\infty} \left| f(t) \right|^2 (t) d\,t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| F(\omega) \right|^2 d\,\omega$$

which is the so-called **parseval theorem**.

2. Linear Time-Invariant Systems

➢ Causal systems



A system operating in time domain must be a causal system, whose output $y(t)$ cannot be received earlier than the time $u(t)$ is sent to the system. For example, if input $u(t)$ starts at t=0, then $y(t)$=0 for t<0.

➢ Linear time-invariant systems

Let a linear time-invariant (LTI) system be described as

(2-1) $\quad y(t) = H[u(t)]$

For linearity, the system should satisfy the superposition theorem, i.e.,

if $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$, then

(2-2) $$y(t) = H\left[u(t)\right] = H\left[\alpha_1 u_1(t) + \alpha_2 u_2(t)\right]$$
$$= \alpha_1 H\left[u_1(t)\right] + \alpha_2 H\left[u_2(t)\right] = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

where $y_1(t) = H[u_1(t)]$ and $y_2(t) = H[u_2(t)]$. It can be further extended to the case that if $u(t) = \sum_{i=1}^{m} \alpha_i u_i(t)$, then

(2-3) $$y(t) = H\left[\sum_{i=1}^{m} \alpha_i u_i(t)\right] = \sum_{i=1}^{m} \alpha_i H\left[u_i(t)\right] = \sum_{i=1}^{m} \alpha_i y_i(t)$$

where $y_i(t) = H[u_i(t)]$, i=1,2,…,m.

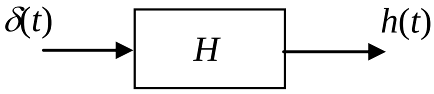
As for the **time-invariance**, it states that if the the input is shifted in time by τ, then the output is also shifted by τ, i.e.,

(2-4) $\quad y(t-\tau) = H[u(t-\tau)]$

To sum up, an LTI system should satisfy superposition theorem and be independent of time shift.

An LTI system can be represented by its impulse response $h(t)$, i.e., $y(t)=h(t)$ if $u(t)=\delta(t)$, or expressed as

(2-5) $\quad h(t)=H[\delta(t)]$



It is known that

(2-6) $\quad u(t)=\int_{-\infty}^{\infty}u(\tau)\delta(t-\tau)d\tau$

which can be changed into the following discrete form:

(2-7) $\quad u(t)=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty}u(k\,\Delta\tau)\delta(t-k\,\Delta\tau)\Delta\tau=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty}\alpha_k u_k(t)$

where $\alpha_k=u(k\,\Delta\tau)\Delta\tau$ and $u_k(t)=\delta(t-k\,\Delta\tau)$. Hence,

(2-8) $\quad y(t)=H\left[u(t)\right]=\lim_{\Delta\tau\to 0}H\left[\sum_{k=-\infty}^{\infty}\alpha_k u_k(t)\right]$

Since the system is linear, we have $H\left[\sum_{k=-\infty}^{\infty}\alpha_k u_k(t)\right]=\sum_{k=-\infty}^{\infty}\alpha_k H\left[u_k(t)\right]$.

Hence,

(2-9) $\quad y(t)=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty}u(k\,\Delta\tau)H\left[\delta(t-k\,\Delta\tau)\right]\Delta\tau$

which can be rewritten into the integral form as

(2-10) $\quad y(t)=\int_{-\infty}^{\infty}u(\tau)H[\delta(t-\tau)]d\,\tau$

According to time-invariant property, we have $h(t-\tau)=H[\delta(t-\tau)]$. Therefore,

(2-11) $\quad y(t)=\int_{-\infty}^{\infty}h(t-\tau)u(\tau)d\,\tau=h(t)*u(t)$

In other words, the output $y(t)$ of an LTI system is equal to the convolution of impulse response $h(t)$ and input $u(t)$

According to the property of convolution in time of Fourier transform, we have

(2-12) $\quad \Im\{y(t)\}=\Im\{h(t)*u(t)\}=\Im\{h(t)\}\Im\{u(t)\}$

Define $\Im\{y(t)\}=Y(\omega)$, $\Im\{h(t)\}=H(\omega)$ and $\Im\{u(t)\}=U(\omega)$, then

(2-13) $\quad Y(\omega)=H(\omega)U(\omega)$

which has been widely used in filter design.

➢ **Causal LTI systems**

For a causal system, if $u(t)=0$ for $t<0$, then

(2-14) $$y(t)=\int_{-\infty}^{\infty}u(\tau)h(t-\tau)d\tau=\int_{0}^{\infty}u(\tau)h(t-\tau)d\tau$$

Because the impulse response must satisfiy $h(t)=0$ for $t<0$, i.e., $h(t-\tau)=0$ for $\tau>t$, we have

(2-15) $$y(t)=\int_{0}^{\infty}u(\tau)h(t-\tau)d\tau=\int_{0}^{t}u(\tau)h(t-\tau)d\tau$$

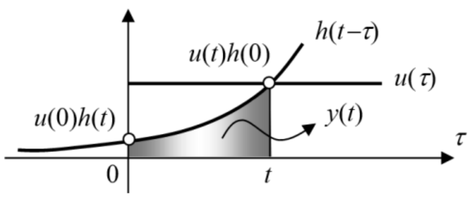
which is a kind of running integral, the integral result "*running*" with the upper limit *t*.

Clearly, the present $y(t)$ only depends on the input $u(\tau)$ for $0\le\tau\le t$ before the present time *t*, which means (2-15) indeed represents a causal system. Most importantly, $y(t)$ seems to have '*memory*' that records the past of $u(t)$.

Later, we will learn that the impulse response of a stable LTI system often consists of decreased exponential function e^{-at} where $a>0$. To demonstrate how to calculate convolution, let's use an example with $u(t)=1$ and $h(t)=e^{-2t}$ for $t\ge 0$, then

(2-16) $$y(t)=\int_{0}^{t}u(\tau)h(t-\tau)d\tau=\int_{0}^{t}e^{-2(t-\tau)}d\tau=0.5\left(1-e^{-2t}\right)$$

which is depicted in the following figure.



3. Laplace Transform

➢ Definition of Laplace transform

The Fourier transform of a real signal $f(t)$ starting from initial time $t=0$ is

(3-1) $$\Im\{f(t)\}=F(\omega)=\int_{0^-}^{\infty} f(t)e^{-j\omega t}dt$$

where the lower limit $t=0^-$ is used to include the case that an impulse $\delta(t)$ happens at $t=0$. However, the Fourier transform is a tool suitable for signal processing, not for system analysis. Instead, the so-called Laplace transform is usually selected as a tool for LTI systems.

Let $f(t)$ initially start from $t=0$ and define its Laplace transform as

(3-2) $$\mathcal{L}\{f(t)\}=F(s)=\int_{0^-}^{\infty} f(t)e^{-st}dt$$

where $s=\sigma+j\omega$ is a complex variable. If $s=j\omega$, then (3-2) is expressed as

(3-3) $$F(s)\big|_{s=j\omega}=F(j\omega)=\int_{0^-}^{\infty} f(t)e^{-j\omega t}dt$$

From (3-1) and (3-3), we know that Fourier transform $F(\omega)$ is the same as Laplace trasform $F(s)\big|_{s=j\omega}=F(j\omega)$.

➢ Region of Convergence

According to the definition of Laplace transform, we have

(3-4) $$|F(s)|=\left|\int_{0^-}^{\infty} f(t)e^{-st}dt\right|=\left|\int_{0^-}^{\infty} f(t)e^{-\sigma t-j\omega t}dt\right|$$
$$\le\int_{0^-}^{\infty}\left|f(t)e^{-\sigma t-j\omega t}\right|dt=\int_{0^-}^{\infty}\left|f(t)e^{-\sigma t}\right|\cdot\left|e^{-j\omega t}\right|dt=\int_{0^-}^{\infty}\left|f(t)e^{-\sigma t}\right|dt$$

where $\left|e^{-j\omega t}\right|=1$. Note that the integral $\int_{0^-}^{\infty}\left|f(t)e^{-\sigma t}\right|dt$ is converged when the real part $Re(s)=\sigma>0$ is large enough. For example, if $|f(t)|<Ae^{\alpha t}$, then we will take the region $Re(s)=\sigma>\alpha$ to ensure the convergence of $F(s)$ and call it the region of convergence, or ***ROC*** in short.

In addition, $F(s)$ is unique for an ROC; which implies $f(t)$ and $F(s)$ form a special mapping pair, shown as

(3-5) $\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt, \qquad Re(s) = \sigma > \alpha$

(3-6) $f(t) = \mathcal{L}^{-1}\{F(s)\} = \dfrac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds, \qquad Re(s) = \sigma > \alpha$

➢ Properties of Laplace transform

Linearity

(3-7) $\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1F_1(s) + a_2F_2(s)$

Time shifting

(3-8) $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad a>0$

Frequency shifting

(3-9) $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$

Time differentiation

(3-10) $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dot{f}(0^-) - \cdots - f^{(n-1)}(0^-)$

Frequency differentiation

(3-11) $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

Initial value theorem

(3-12) $f(0^+) = \lim_{s\to\infty} sF(s)$

Final value theorem

(3-13) $f(\infty) = \lim_{s\to 0} sF(s)$

where $f(\infty)$ exists and must be a constant finite value.

Convolution in time

(3-14) $\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s)$

➢ Pairs of Laplace transform

(3-15) $\mathcal{L}\{u(t)\} = \dfrac{1}{s}$

(3-16) $\mathcal{L}\{e^{-\alpha t}u(t)\} = \dfrac{1}{s+\alpha}$

(3-17) $\mathcal{L}\{t^n u(t)\} = \dfrac{n!}{s^{n+1}}$

(3-18) $\mathcal{L}\{cos\,\beta t \cdot u(t)\} = \dfrac{s}{s^2+\beta^2}$

(3-19) $\mathcal{L}\{sin\,\beta t \cdot u(t)\} = \dfrac{\beta}{s^2+\beta^2}$

(3-20) $\mathcal{L}\{e^{-\alpha t}\,cos\,\beta t \cdot u(t)\} = \dfrac{s+\alpha}{(s+\alpha)^2+\beta^2}$

(3-21) $\mathcal{L}\{e^{-\alpha t}\,sin\,\beta t \cdot u(t)\} = \dfrac{\beta}{(s+\alpha)^2+\beta^2}$

➢ Partial fraction expansion

A method called **partial fraction expansion** has been widely adopted to find $f(t)$ from the inverse of $F(s)$. First, we partition $F(s)$ into $\sum_{k=1}^{n} F_k(s)$, and then we obtain $f(t)=\sum_{k=1}^{n} f_k(t)$ from the table of mapping pairs $f_k(t)$-$F_k(s)$.

Case-A: Simple real poles

For example, $F(s)=\dfrac{s^2+12}{s(s+2)(s+3)}$ with simple poles 0, −2 and −3.

Using partial fraction expansion obtains

(3-22) $F(s)=\dfrac{s^2+12}{s(s+2)(s+3)}=\dfrac{A}{s}+\dfrac{B}{s+2}+\dfrac{C}{s+3}$

Hence,

(3-23) $s^2+12 = A(s+2)(s+3)+Bs(s+3)+Cs(s+2)$

Setting $s=0$, $s=-2$ and $s=-3$ to (3-23) yields

$$A=\left.\frac{s^2+12}{(s+2)(s+3)}\right|_{s=0}=sF(s)|_{s=0}=2$$

$$B=\left.\frac{s^2+12}{s(s+3)}\right|_{s=-2}=(s+2)F(s)|_{s=-2}=-8$$

$$C=\left.\frac{s^2+12}{s(s+2)}\right|_{s=-3}=(s+3)F(s)|_{s=-3}=7$$

i.e., $F(s)=\frac{2}{s}+\frac{-8}{s+2}+\frac{7}{s+3}$. The mapping set of Laplace transform tells that

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=2-8e^{-2t}+7e^{-3t} \quad \text{for} \quad t\geq 0.$$

Case-B: Repeated poles

For example, $F(s)=\frac{s^3+2s+6}{s(s+1)^2(s+3)}$ with a double pole $s=-1$.

From partial fraction expansion, we have

(3-24) $$F(s)=\frac{s^3+2s+6}{s(s+1)^2(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{D}{s+3}$$

Hence,

(3-25) $$s^3+2s+6=A(s+1)^2(s+3)+Bs(s+1)(s+3)+Cs(s+3)+Ds(s+1)^2$$

Applying the poles s=0, s=–1 and s=–3 obtains

$$A=sF(s)|_{s=0}=\left.\frac{s^3+2s+6}{(s+1)^2(s+3)}\right|_{s=0}=2$$

$$C=(s+1)^2F(s)|_{s=-1}=\left.\frac{s^3+2s+6}{s(s+3)}\right|_{s=-1}=-\frac{3}{2}$$

$$D=(s+3)F(s)|_{s=-3}=\left.\frac{s^3+2s+6}{s(s+1)^2}\right|_{s=-3}=\frac{9}{4}$$

To determine B, we can compare the coefficient of s^3 in (3-25), i.e., $A+B+D=1$.

Clearly, $B=1-A-D=-\frac{13}{4}$. In fact, we also can determine B by $(s+1)^2F(s)$, i.e., (3-26) $$(s+1)^2 F(s) = \frac{s^3+2s+6}{s(s+3)} = (s+1)^2\left(\frac{A}{s}+\frac{D}{s+3}\right) + B(s+1) + C$$

Taking derivative results in

(3-27) $$\frac{d}{ds}\left(\frac{s^3+2s+6}{s(s+3)}\right) = 2(s+1)\left(\frac{A}{s}+\frac{D}{s+3}\right) + (s+1)^2\frac{d}{ds}\left(\frac{A}{s}+\frac{D}{s+3}\right) + B$$

which implies $B = \left.\dfrac{d}{ds}\left(\dfrac{s^3+2s+6}{s(s+3)}\right)\right|_{s=-1} = -\dfrac{13}{4}$.

Hence, $F(s) = \dfrac{s^3+2s+6}{s(s+1)^2(s+3)} = \dfrac{2}{s} + \dfrac{-13/4}{s+1} + \dfrac{-3/2}{(s+1)^2} + \dfrac{9/4}{s+3}$ and then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2 - \frac{13}{4}e^{-t} - \frac{3}{2}te^{-t} + \frac{9}{4}e^{-3t} \quad \text{for } t \geq 0.$$

Case-C: Complex poles

For example, $F(s) = \dfrac{10}{(s+1)(s^2+4s+13)}$ with complex poles $s = -2 \pm j3$.

From partial fraction expansion, we have

(3-28) $$F(s) = \frac{10}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{B(s+2)+3C}{(s+2)^2+3^2}$$

Hence,

$$A = (s+1)F(s)\big|_{s=-1} = \left.\frac{10}{s^2+4s+13}\right|_{s=-1} = 1$$

Then, choose $s=-2$ for (3-28) to obtain

$$F(-2) = -\frac{10}{9} = -A + \frac{C}{3} = -1 + \frac{C}{3} \quad \Rightarrow \; C = -\frac{1}{3}$$

Finally, apply $s=0$ again and obtain

$$F(0) = \frac{10}{13} = A + \frac{2B+3C}{13} = 1 + \frac{2B}{13} - \frac{1}{13} \quad \Rightarrow \; B = -1$$

Therefore, $F(s) = \dfrac{1}{s+1} - \dfrac{s+2}{(s+2)^2+3^2} - \dfrac{1}{3}\dfrac{3}{(s+2)^2+3^2}$, which results in

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t} - e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t \quad \text{for } t \geq 0.$$

4. LTI System in Differential Equation

Linear ordinary differential equation with constant coefficients

In system engineering, the most common system is described an ordinary differential equation with constant coefficients, shown as below:

(4-1)
$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1\dot{y}(t)+a_0y(t)$$
$$=b_mu^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+b_1\dot{u}(t)+b_0u(t)$$

where a_k and b_l are constant, $k=1,2,...,n-1$, $l=1,2,...,m$. Note that if $n\geq m$ then the system is proper, otherwise the system is improper. In case that $n>m$, the system is called ***strictly proper***.

Taking Laplace transform of (4-1) yields

(4-2)
$$s^nY(s)-s^{n-1}y_0-s^{n-2}y_1-\cdots-sy_{n-2}-y_{n-1}$$
$$+a_{n-1}\left(s^{n-1}Y(s)-s^{n-2}y_0-s^{n-2}y_1-\cdots-sy_{n-3}-y_{n-2}\right)$$
$$+\cdots+a_1\left(sY(s)-y_0\right)+a_0Y(s)$$
$$=b_m\left(s^mU(s)-s^{m-1}u_0-s^{m-2}u_1-\cdots-su_{m-2}-u_{m-1}\right)$$
$$+b_{m-1}\left(s^{m-1}U(s)-s^{m-2}u_0-s^{m-3}u_1-\cdots-su_{m-3}-u_{m-2}\right)$$
$$+\cdots+b_1\left(sU(s)-u_0\right)+b_0U(s)$$

where $y_k=y^{(k)}(0)$ and $u_l=u^{(l)}(0)$ are the initial conditions. Hence, we have

(4-3)
$$P(s)Y(s)=Q(s)U(s)+R(s)$$

where

$$P(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0=(s-p_1)(s-p_2)\cdots(s-p_n)$$
$$Q(s)=b_ms^m+\cdots+b_1s+b_0=b_m(s-z_1)(s-z_2)\cdots(s-z_m)$$
$$R(s)=r_{n-1}s^{n-1}+\cdots+r_1s+r_0$$

We call p_1, p_2, …, p_n the poles and z_1, z_2, …, z_m the zeros of the system. It is noticed that the coefficients r_k of $R(s)$, $k=1,2,...,n-1$, are functions of the initial values y_k and u_l. Rewrite (4-3) into

(4-4) $Y(s) = Y_0(s) + H(s)U(s)$

where $H(s) = \dfrac{Q(s)}{P(s)}$ is called the transfer function and $Y_0(s) = \dfrac{R(s)}{P(s)}$. Taking inverse Laplace transform of $Y(s)$ obtains

(4-5) $y(t) = y_0(t) + y_u(t)$

where $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, $y_0(t) = \mathcal{L}^{-1}\{Y_0(s)\}$ and $y_u(t) = \mathcal{L}^{-1}\{H(s)U(s)\}$.

If $u(t) = 0$, i.e., the system is in free motion, then $y_u(t) = 0$, which implies $y(t) = y_0(t)$ only depends on its initial conditions y_k, $k = 1,2,...,n-1$.

➢ Impulse response

If a system is initially ideled, i.e., $y_k = 0$ for $k = 1,2,...,n-1$, and encounters an impulse input $u(t) = \delta(t)$, then $Y_0(s) = 0$ and $U(s) = 1$. Hence, from (4-4) we have $Y(s) = H(s)U(s) + Y_0(s) = H(s)$. Taking the inverse Laplace transform yields $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{H(s)\} = h(t)$ which is called the **impulse response**. Note that the impulse response of a system is obtained by setting $u(t) = \delta(t)$ and $y_k = 0$ for $k = 1,2,...,n-1$.

➢ Total response

From $\mathcal{L}\{h(t) * u(t)\} = H(s)U(s)$, we have

(4-6) $y_u(t) = \mathcal{L}^{-1}\{H(s)U(s)\} = h(t) * u(t) = \int_0^\infty h(t-\tau)u(\tau)d\tau$

which results in the total response

(4-7) $y(t) = y_0(t) + \int_0^\infty h(t-\tau)u(\tau)d\tau$

Such total response implies that (4-1) is not linear if initial conditions exist. Let’s explain it by a system with two inputs $u_1(t)$ and $u_2(t)$, but under the same initial conditions y_k for k=0,1,…,n–1. Hence, (4-8) $$y_1(t)=H[u_1(t)]=y_0(t)+\int_0^{\infty}h(t-\tau)u_1(\tau)d\tau$$

(4-9) $$y_2(t)=H[u_2(t)]=y_0(t)+\int_0^{\infty}h(t-\tau)u_2(\tau)d\tau$$

Now, if the system is excited by $u_3(t)=u_1(t)+u_2(t)$, then

(4-10) $$y_3(t)=H\left[u_3(t)\right]=y_0(t)+\int_0^{\infty}h(t-\tau)\big(u_1(\tau)+u_2(\tau)\big)d\tau$$

Obviously,

(4-11) $$y_3(t)\neq y_1(t)+y_2(t)=2y_0(t)+\int_0^{\infty}h(t-\tau)\big(u_1(\tau)+u_2(\tau)\big)d\tau$$

Hence, (4-1) is not a linear system since the superposition is not satisfied unless $y_0(t)$ is negligible.

Example

A second-order system with initial conditions $y(0)=1$ and $\dot{y}(0)=0$ is described as $\ddot{y}(t)+3\dot{y}(t)+a_0y(t)=u(t)$. Consider the following cases:

Case-1: $a_0=2$, $u(t)=u_1=1$

Case-2: $a_0=2$, $u(t)=u_2=2$

Case-3: $a_0=2$, $u(t)=u_1+u_2=3$

Case-4: $a_0=0$, $u(t)=u_1=1$

Case-5: $a_0=-4$, $u(t)=u_1=1$

(1) Determine the transfer function in terms of a_0.

(2) Determine the poles of Case-*k*, $k=1,2,3,4,5$.

(3) Determine the total response $y_k(t)$ of Case-*k*, $k=1,2,3,4,5$.

(4) According to Case-1, Case-2 and Case-3, show that a system with poles all located in the left-half *z*-plane is not linear unless $t\to\infty$.

(5) From Case-4 and Case-5, show that if a system possesses a pole not in the left-half *z*-plane, then the total response is unbounded as $t\to\infty$.

Solution:

Taking Laplace transform yields

$$s^2Y(s)-sy(0)-\dot{y}(0)+3\big(sY(s)-y(0)\big)+a_0Y(s)=U(s)$$

i.e., $s^2Y(s)-s\alpha+3\big(sY(s)-\alpha\big)+a_0Y(s)=U(s)$

or $Y(s)=\dfrac{1}{s^2+3s+a_0}U(s)+\dfrac{s\alpha+3\alpha}{s^2+3s+a_0}=H(s)U(s)+Y_0(s)$

(1) The transfer function is $H(s)=\dfrac{1}{s^2+3s+a_0}$.

(2) For Case-1 to Cases-3, we have $s^2+3s+a_0=s^2+3s+2=(s+1)(s+2)$, i.e., the poles are -1 and -2 and both are located in the left-half z-plane.
For Case-4, we have $s^2+3s+a_0=s^2+3s=s(s+3)$, i.e., the poles are 0 and -3. For Case-5, we have $s^2+3s+a_0=s^2+3s-4=(s-1)(s+4)$, i.e., the poles are 1 and -4. Note that the pole 0 in Case-4 and the pole 1 in Case-5 are not in the left-half z-plane.

(3) For Case-1, since $a_0=2$ and $u(t)=u_1=1$, we have $U_1(s)=\dfrac{1}{s}$ and

$Y_1(s)=Y_{1u}(s)+Y_{10}(s)$, where

$$Y_{1u}(s)=H(s)U_1(s)=\frac{1}{s^2+3s+2}\cdot\frac{1}{s}=\frac{0.5}{s}+\frac{-1}{s+1}+\frac{0.5}{s+2}$$

$$Y_{10}(s)=\frac{s+3}{s^2+3s+2}=\frac{2}{s+1}+\frac{-1}{s+2}.$$

Hence, $y_{1u}(t)=0.5-e^{-t}+0.5e^{-2t}$, $y_{10}(t)=2e^{-t}-e^{-2t}$ and the total response is

$$y_1(t)=y_{1u}(t)+y_{10}(t)=0.5+e^{-t}-0.5e^{-2t}.$$

For Case-2, since $a_0=2$, $u(t)=u_2=2$, we have $U_2(s)=\dfrac{2}{s}$ and

$Y_2(s)=Y_{2u}(s)+Y_{20}(s)$, where

$$Y_{2u}(s)=H(s)U_2(s)=\frac{1}{s^2+3s+2}\cdot\frac{2}{s}=\frac{1}{s}+\frac{-2}{s+1}+\frac{1}{s+2}$$

$$Y_{20}(s)=\frac{s+3}{s^2+3s+2}=\frac{2}{s+1}+\frac{-1}{s+2}.$$

Hence, $y_{2u}(t)=1-2e^{-t}+e^{-2t}$, $y_{20}(t)=2e^{-t}-e^{-2t}$ and the total response is

$$y_2(t)=y_{2u}(t)+y_{20}(t)=1.$$

For Case-3, since $a_0 = 2$, $u(t) = u_1 + u_2 = 3$, we have $U_3(s) = \dfrac{3}{s}$ and

$Y_3(s) = Y_{3u}(s) + Y_{30}(s)$, where

$$Y_{3u}(s) = H(s)U_3(s) = \frac{1}{s^2+3s+2}\cdot\frac{3}{s} = \frac{1.5}{s} + \frac{-3}{s+1} + \frac{1.5}{s+2}$$

$$Y_{30}(s) = \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} + \frac{-1}{s+2}.$$

Hence, $y_{3u}(t) = 1.5 - 3e^{-t} + 1.5e^{-2t}$, $y_{30}(t) = 2e^{-t} - e^{-2t}$ and the total response is

$$y_3(t) = y_{3u}(t) + y_{30}(t) = 1.5 - e^{-t} + 0.5e^{-2t}$$

For Case-4, since $a_0 = 0$, $u(t) = u_1 = 1$, we have $U_4(s) = \dfrac{1}{s}$ and

$Y_4(s) = Y_{4u}(s) + Y_{40}(s)$, where

$$Y_{4u}(s) = H(s)U_4(s) = \frac{1}{s^2+3s}\cdot\frac{1}{s} = \frac{-0.111}{s} + \frac{0.333}{s^2} + \frac{0.111}{s+3}$$

$$Y_{40}(s) = \frac{s+3}{s^2+3s} = \frac{1}{s}.$$

Hence, $y_{4u}(t) = -0.111 + 0.333t + 0.111e^{-3t}$, $y_{40}(t) = 1$ and the total response is

$$y_4(t) = y_{4u}(t) + y_{40}(t) = 0.889 + 0.333t + 0.111e^{-3t}$$

For Case-5, since $a_0 = -4$, $u(t) = u_1 = 1$, we have $U_4(s) = \dfrac{1}{s}$ and

$Y_5(s) = Y_{5u}(s) + Y_{50}(s)$, where

$$Y_{5u}(s) = H(s)U_5(s) = \frac{1}{s^2+3s-4}\cdot\frac{1}{s} = \frac{-0.25}{s} + \frac{0.2}{s-1} + \frac{0.05}{s+4}$$

$$Y_{50}(s) = \frac{s+3}{s^2+3s-4} = \frac{0.8}{s-1} + \frac{0.2}{s+4}.$$

Hence, $y_{5u}(t) = -0.25 + 0.2e^{t} + 0.05e^{-4t}$, $y_{50}(t) = 0.8e^{t} + 0.2e^{-4t}$

and the total response is

$$y_5(t) = y_{5u}(t) + y_{50}(t) = -0.25 + e^{t} + 0.25e^{-4t}$$

(4) According to Case-1, Case-2 and Case-3, we have $y_3(t) = 1.5 - e^{-t} + 0.5e^{-2t}$ and $y_1(t) + y_2(t) = 1.5 + e^{-t} - 0.5e^{-2t}$. We know that $y_3(t) \neq y_1(t) + y_2(t)$, i.e., the system with poles all located in the left-half *z*-plane is not linear.

However, as $t \to \infty$ we can find that $y_3(\infty) = y_1(\infty) + y_2(\infty) = 1.5$, which implies the system is linear while $t \to \infty$.

(5) From Case-4 and Case-5, it is easy to find $y_4(\infty) \to \infty$ and $y_5(\infty) \to \infty$, which implies the total response of a system possessing a pole not in the left-half *z*-plane will be unbounded as $t \to \infty$.

➢ Incrementally LTI system

If a system in (4-1) at least has one pole not in the left-half z-plane, then its total response will be unbounded, referring to the Case-4 and Case-5 in the previous example. Fortunately, a system in (4-1) is usually designed to have poles all located in the left-half z-plane, i.e., the roots of $P(s) = 0$ are all possessed of negative real part. This kind of system is called a stable system. According to the Final value theorem, we have

(4-12) $$y_0(t)\Big|_{t\to\infty} = \mathcal{L}^{-1}\{Y_0(s)\}\Big|_{t\to\infty} = \lim_{s\to 0}\left[sY_0(s)\right] = \lim_{s\to 0}\left[s\frac{r_0}{a_0}\right] \to 0$$

Obviously, as $t \to \infty$ we can neglect $y_0(t)$ and obtain the total response

(4-13) $$y(t) = y_u(t) = \int_0^{\infty} h(t-\tau)u(\tau)d\tau$$

which is an LTI system and can be described as

(4-14) $$Y(s) = H(s)U(s)\Big|_{t\to\infty}$$

Since a stable system in the form of (4-1) becomes an LTI system as time t increases, we call it an incrementally LTI system.

Example

Consider the LTI system: $\dot{y}(t) + 3y(t) = u(t)$, where $y(0) = -1$, $u(t) = 3$.

Determine $y(t)$, for $t \geq 0$.

Solution:

Taking Laplace transform yields $sY(s)-y(0)+3Y(s)=U(s)$,

Since $U(s)=\dfrac{3}{s}$ and $y(0)=-1$, we have $sY(s)+1+3Y(s)=\dfrac{3}{s}$. Hence,

$$Y(s)=-\frac{s-3}{s(s+3)}=\frac{1}{s}+\frac{-2}{s+3}$$

which results in $y(t)=1-2e^{-3t}$.

Example

Consider the LTI system: $\ddot{y}(t)+2\dot{y}(t)+2y(t)=\dot{u}(t)-2u(t)$, where $y(0)=1$, $\dot{y}(0)=0$, $u(t)=cos3t$. Determine $y(t)$, for $t\geq 0$.

Solution:

Taking Laplace transform yields

$$s^2Y(s)-sy(0)-\dot{y}(0)+2sY(s)-2y(0)+2Y(s)=sU(s)-u(0)-2U(s)$$

Since $y(0)=1$, $\dot{y}(0)=0$, $u(0)=1$ and $U(s)=\dfrac{s}{s^2+9}$, we have

$$\left(s^2+2s+2\right)Y(s)-s-2=(s-2)\frac{s}{s^2+9}-1$$

Hence, $$Y(s)=\frac{s^3+2s^2+7s+9}{\left(s^2+9\right)\left(s^2+2s+2\right)}=\frac{As+3B}{s^2+9}+\frac{C(s+1)+D}{(s+1)^2+1}$$

and $y(t)=Acos3t+Bsin3t+e^{-t}(Ccost+Dsint)$. It can be obtained that $A=\dfrac{32}{85}$, $B=-\dfrac{9}{85}$, $C=\dfrac{53}{85}$ and $D=\dfrac{38}{85}$. Therefore,

$$y(t)=\frac{32}{85}cos3t-\frac{9}{85}sin3t+e^{-t}\left(\frac{53}{85}cost+\frac{38}{85}sint\right)$$

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5. Signal Filtering

➢ Filter

Signal filtering is an important function for an LTI system when it is operated around a noisy environment. If a system is designed to filter out noise or to achieve desired signals, we call it a filter. In system enegineering, a filter is often designed as a stable LTI system, described as

(5-1) $$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1\dot{y}(t)+a_0y(t)$$
$$=b_mu^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+b_1\dot{u}(t)+b_0u(t)$$

Since it is stable, we neglect initial conditions and take Laplace transform to obtain

(5-2) $$Y(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s)=H(s)U(s)$$

where $\mathcal{L}\{u(t)\}=U(s)$, $\mathcal{L}\{y(t)\}=Y(s)$ and $H(s)$ is the transfer function of the filter expressed as

(5-3) $$H(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{Q(s)}{P(s)}=\frac{b_m(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

where $P(s)=(s-p_1)\cdots(s-p_n)$ and $Q(s)=b_m(s-z_1)\cdots(s-z_m)$. The n roots of $P(s)=0$, $p_k|_{k=1,2,\ldots,n}$, are the poles and the m roots of $Q(s)=0$, $z_i|_{i=1,2,\ldots,m}$, are the zeros. All the poles and zeros are complex numbers. Most importantly, all the poles p_k should be located in the left-half z-plane to guarantee the stability of the filter.

Before filter design, let's discuss the frequency response of (5-2) with $s=j\omega$, i.e.,

(5-4) $$Y(j\omega)=H(j\omega)U(j\omega)$$

where $U(j\omega)$ and $Y(j\omega)$ are the Fourier transforms of $u(t)$ and $y(t)$. Using the inverse Fourier transform, we have

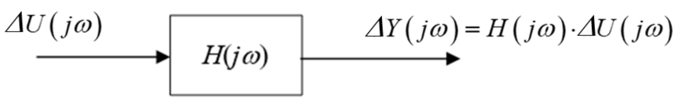
(5-5) $$u(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(j\omega)e^{j\omega t}d\omega=\int_{-\infty}^{\infty}e^{j\omega t}\cdot\underbrace{\frac{1}{2\pi}U(j\omega)d\omega}_{\Delta U(j\omega)}$$

(5-6) $\quad y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega=\int_{-\infty}^{\infty}e^{j\omega t}\cdot\underbrace{\frac{1}{2\pi}Y(j\omega)d\omega}_{\Delta Y(j\omega)}$

which implies $u(t)$ and $y(t)$ can be decomposed into an infinite number of components $\Delta U(j\omega)=\frac{1}{2\pi}U(j\omega)d\omega$ and $\Delta Y(j\omega)=\frac{1}{2\pi}Y(j\omega)d\omega$ at ω. Then, further multiplying $\frac{1}{2\pi}d\omega$ into (5-4) will yield

(5-7) $\quad \frac{1}{2\pi}Y(j\omega)d\omega=H(j\omega)\left(\frac{1}{2\pi}U(j\omega)d\omega\right)$

i.e., $\Delta Y(j\omega)=H(j\omega)\cdot\Delta U(j\omega)$. Below shows the block diagram.



Obviously, if $|H(j\omega)|\ll 1$, then output component $\Delta Y(j\omega)$ will be greatly reduced, which also means input component $\Delta U(j\omega)$ is filtered out and will not appear in the output.

In fact, we can show the filter (5-4) in two aspects, magnitude and phase, i.e.,

(5-8) $\quad |Y(j\omega)|=|H(j\omega)||U(j\omega)|$

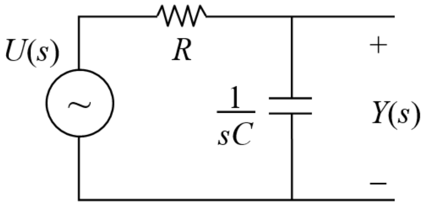
(5-9) $\quad \angle Y(j\omega)=\angle H(j\omega)+\angle U(j\omega)$

- ***Bode-plot***

In filter design, we often adopt the Bode-plot of $H(j\omega)=\left.\frac{Q(s)}{P(s)}\right|_{s=j\omega}$, which consists of magnitude $|H(j\omega)|_{dB}\left(=20log_{10}|H(j\omega)|\right)$ and phase $\angle H(j\omega)$. The reason to use $|H(j\omega)|_{dB}$ is because the multiplication in (5-8) can be simply implemented by addition as below:

(5-10) $\quad |Y(j\omega)|_{dB}=|H(j\omega)|_{dB}+|U(j\omega)|_{dB}$

➢ ***Lowpass filter***



The simplest lowpass filter is an RC circuit whose transfer function is

(5-11) $$H(s)=\frac{Y(s)}{U(s)}=\frac{1}{1+sRC}$$

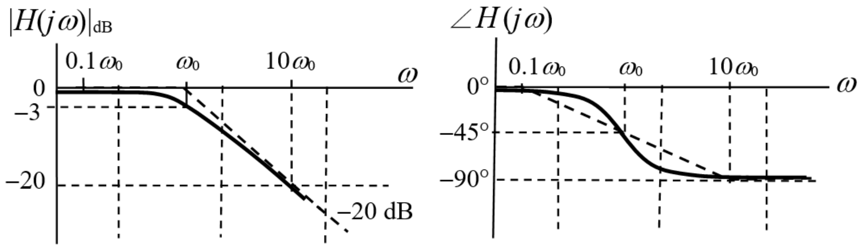
where $U(s)$ and $Y(s)$ are the input and output voltage signals. Let $\omega_0=\frac{1}{RC}$,

then $H(s)=\frac{1}{1+(s/\omega_0)}$ and its Fourier transform is

(5-12) $$H(j\omega)=\frac{1}{1+j(\omega/\omega_0)}=\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}e^{-j\angle tan^{-1}(\omega/\omega_0)}$$

with magnitude $\left|H(j\omega)\right|=\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$ and phase $\angle H(j\omega)=-\angle tan^{-1}(\omega/\omega_0)$.

The Bode-plot of magnitude and phase is shown below:



From the curve of $\left|H(j\omega)\right|_{dB}$, since

$$\left|H(j\omega_0)\right|=\left.\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}\right|_{\omega=\omega_0}=2^{-\frac{1}{2}}=0.707$$

$$\left|H(j10\omega_0)\right|=\left.\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}\right|_{\omega=10\omega_0}=\frac{1}{\sqrt{101}}\approx 10^{-1}$$

we have

$$\left|H\left(j\omega_0\right)\right|_{dB}=20log_{10}2^{-\frac{1}{2}}=-10log_{10}2=-3.01dB\approx-3d\mathrm{B}$$

$$\left|H\left(j10\omega_0\right)\right|_{dB}\approx 20log_{10}10^{-1}=-20dB$$

It is notice that $Y\left(j\omega\right)=H\left(j\omega\right)U\left(j\omega\right)<0.01\cdot U\left(j\omega\right)$ for $\omega>10\omega_0$. The frequency $\omega=\omega_0$ is called the **cutoff frequency** or the **3dB point**. Obviously, all the output power is concentrated in $0\le\omega\le\omega_0$, so we say that the filter is a lowpass filter and the **bandwidth** is $BW=\omega_0$.

Sometimes, it is required to design a filter of higher order, for example a 2nd order filter $H(s)=\dfrac{1}{\left(1+\dfrac{s}{10}\right)\left(1+\dfrac{s}{40}\right)}$.

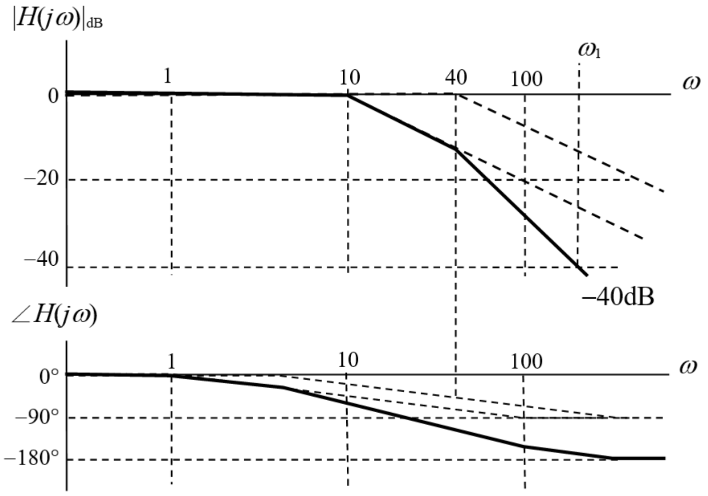
Since $H(j\omega)=\dfrac{1}{\left(1+j\dfrac{\omega}{10}\right)\left(1+j\dfrac{\omega}{40}\right)}$, the magnitude and phase are

$$\left|H\left(j\omega\right)\right|=\frac{1}{\left|1+j\dfrac{\omega}{10}\right|\cdot\left|1+j\dfrac{\omega}{40}\right|}=\frac{1}{\sqrt{\left(1+\dfrac{\omega^2}{100}\right)\left(1+\dfrac{\omega^2}{1600}\right)}}$$

or $$\left|H\left(j\omega\right)\right|_{dB}=20log_{10}\left|H\left(j\omega\right)\right|=-10log_{10}\left[\left(1+\frac{\omega^2}{100}\right)\left(1+\frac{\omega^2}{1600}\right)\right]$$

$$\angle H\left(j\omega\right)=-\angle tan^{-1}\frac{\omega}{10}-\angle tan^{-1}\frac{\omega}{40}$$

The Bode plot is shown as below:

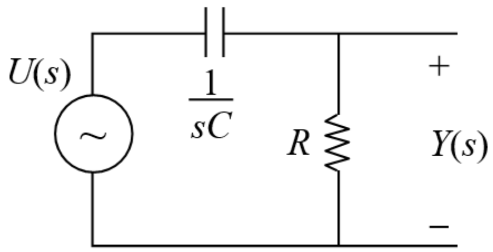


From the Bode-plot, at $\omega = \omega_1 \approx 200$ rad/sec we know that $\left|H(j\omega_1)\right|_{dB} = -40dB$, i.e., $\left|H(j\omega_1)\right| = 0.01$. Hence, we have

$$\left|H(j\omega_1)\right| = \frac{1}{\sqrt{\left(1+\frac{\omega_1^2}{100}\right)\left(1+\frac{\omega_1^2}{1600}\right)}} = 0.01$$

and the resullt is $\omega_1 = 198$ rad/s, indeed around 200 rad/sec.

- ***Highpass filter***



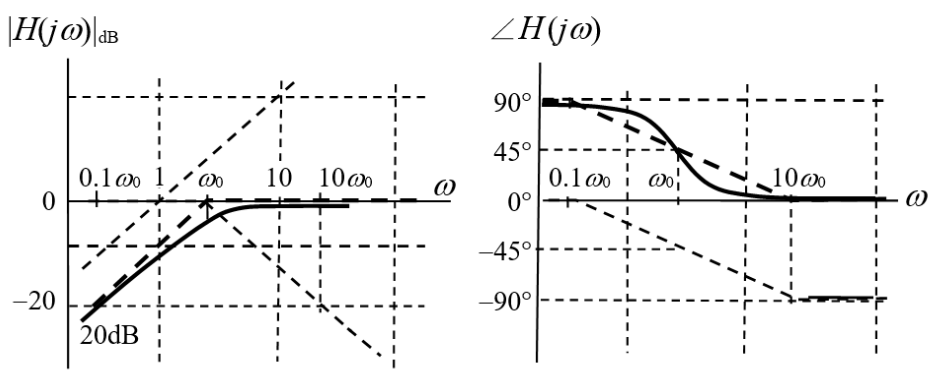
The *RC* circuit is a **highpass filter**, whose transfer function is

(5-13) $$H(s) = \frac{Y(s)}{U(s)} = \frac{sRC}{1+sRC}$$

Let $\omega_0 = \dfrac{1}{RC}$, then $H(s) = \dfrac{s/\omega_0}{1+(s/\omega_0)}$ and its Fourier transform is

(5-14) $$H(j\omega) = \frac{j\omega/\omega_0}{1+(j\omega/\omega_0)} = \frac{\omega/\omega_0}{\sqrt{1+\omega/\omega_0^{\,2}}}\angle\left(90^\circ - tan^{-1}\frac{\omega}{\omega_0}\right)$$

Assume $10 > \omega_0 > 1$, then the Bode plot can be obtained as below:

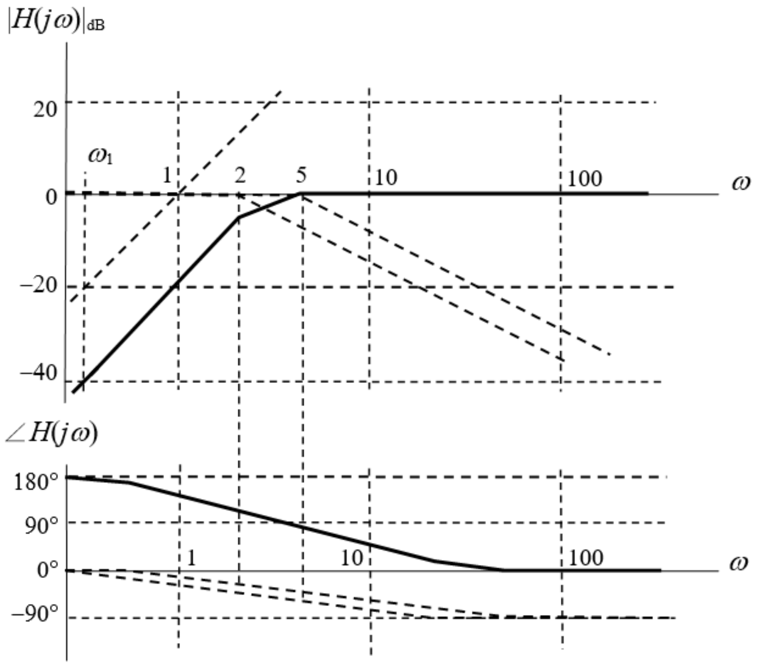


Sometimes, we will use a highpass filter of higher order, such as a 2nd order filter expressed as $H(s)=\dfrac{bs^2}{1+as+bs^2}$.

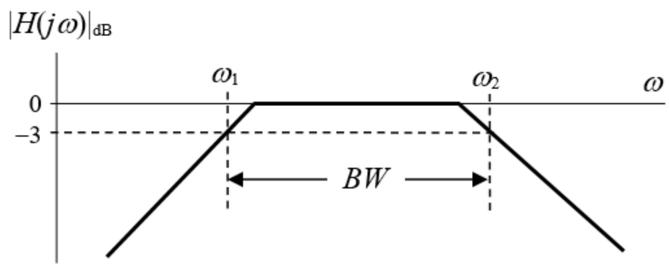
Let's take $H(s)=\dfrac{s^2/10}{\left(1+\dfrac{s}{2}\right)\left(1+\dfrac{s}{5}\right)}$ as an example.

Hence, $H(j\omega)=\dfrac{-\omega^2/10}{\left(1+j\dfrac{\omega}{2}\right)\left(1+j\dfrac{\omega}{5}\right)}=\dfrac{\omega^2/10}{\sqrt{1+\dfrac{\omega^2}{4}}\sqrt{1+\dfrac{\omega^2}{25}}}e^{j\left(180°-\angle\tan^{-1}\frac{\omega}{2}-\angle\tan^{-1}\frac{\omega}{5}\right)}$

The Bode plot is shown as below:



Since $\dfrac{1}{1+as+bs^2}$ and $\dfrac{bs^2}{1+as+bs^2}$ respectively represent a lowpass filter and a highpass filter, we know that $\dfrac{as}{1+as+bs^2}$ will be a bandpass filter and its Bode plot is roughly depicted as below:



6. System Realization

After a filter is designed by the use of an LTI system, one question is raised: How to implement the LTI system? In system engineering, it's a technique called system realiztion.

➢ *System realization*

Consider an LTI system

(6-1) $$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1\dot{y}(t)+a_0y(t)$$
$$=b_mu^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+b_1\dot{u}(t)+b_0u(t)$$

which of course is proper $(n\geq m)$ and stable. By neglecting the initial conditions, we obtain the frequency response as

(6-2) $$Y(s)=H(s)U(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s)$$

where $U(s)=\mathcal{L}\{u(t)\}$ and $Y(s)=\mathcal{L}\{y(t)\}$. The transfer function is

(6-3) $$H(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{b_m(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_m)}$$

where p_k, k=1,2,...,n, are the poles located in the left-half complex plane, i.e., $Re(p_k)<0$, and z_j, j=1,2,...,m, are the zeros.

It is known that there are a lot of methods to realize the LTI system in (6-2).
Here, let's consider a strictly proper system $(n>m)$ and introduce a method based on state variables chosen as

(6-4) $$X_k(s)=\frac{s^{k-1}}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s),\quad k=1,2,\ldots,n$$

It is easy to check that

(6-5) $$X_1(s)=\frac{1}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s)$$

(6-6) $$X_k(s)=sX_{k-1}(s)=s^{k-1}X_1,\qquad k=2,3,\ldots,n$$

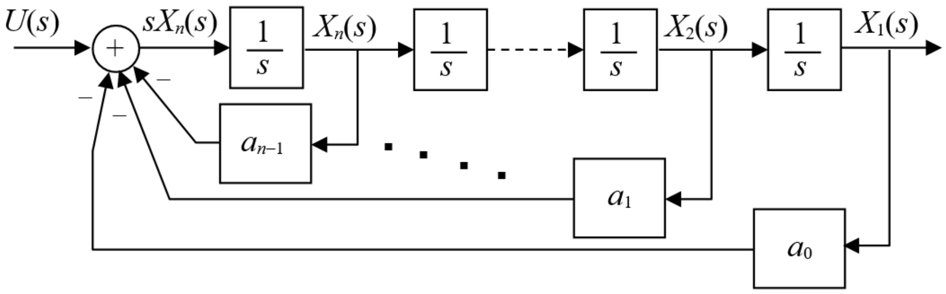
From (6-5) and (6-6), we have

(6-7) $$s^n X_1(s)+a_{n-1}s^{n-1}X_1(s)+\cdots+a_1 sX_1(s)+a_0X_1(s)=U(s)$$

and thus, $sX_n(s)+a_{n-1}X_n(s)+\cdots+a_1X_2(s)+a_0X_1(s)=U(s)$, i.e.,

(6-8) $$sX_n(s)=-a_0X_1(s)-a_1X_2(s)-\cdots-a_{n-1}X_n(s)+U(s)$$

The block diagram to realize (6-8) is shown as below:



➢ ***State equation***

Let $x_k(t)=\mathcal{L}^{-1}\{X_k(s)\}$, then from (6-6) and (6-8) we have

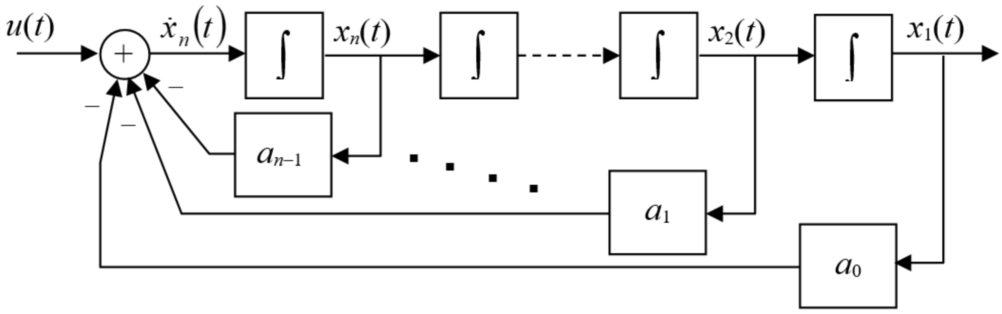
(6-9) $$\begin{cases}\dot{x}_1(t)=x_2(t)\\ \dot{x}_2(t)=x_3(t)\\ \quad\vdots\\ \dot{x}_{n-1}(t)=x_n(t)\\ \dot{x}_n(t)=-a_0x_1(t)-a_1x_2(t)-\cdots-a_{n-1}x_n(t)+u(t)\end{cases}$$

or in matrix form

(6-10) $$\underbrace{\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \vdots\\ \dot{x}_{n-1}(t)\\ \dot{x}_n(t)\end{bmatrix}}_{\dot{\boldsymbol{x}}(t)}=\underbrace{\begin{bmatrix}0&1&0&\cdots&0&0\\ 0&0&1&\cdots&0&0\\ 0&0&0&\cdots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&0&\cdots&0&1\\ -a_0&-a_1&-a_2&\cdots&-a_{n-2}&-a_{n-1}\end{bmatrix}}_{\boldsymbol{A}}\cdot\underbrace{\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ \vdots\\ x_{n-1}(t)\\ x_n(t)\end{bmatrix}}_{\boldsymbol{x}(t)}$$

where $\boldsymbol{x}(t)=\begin{bmatrix}x_1(t) & x_2(t) & \cdots & x_{n-1}(t) & x_n(t)\end{bmatrix}^T$ is called the state vector.

Correspondingly, the block diagram is shown as below:



➢ ***Output equation***

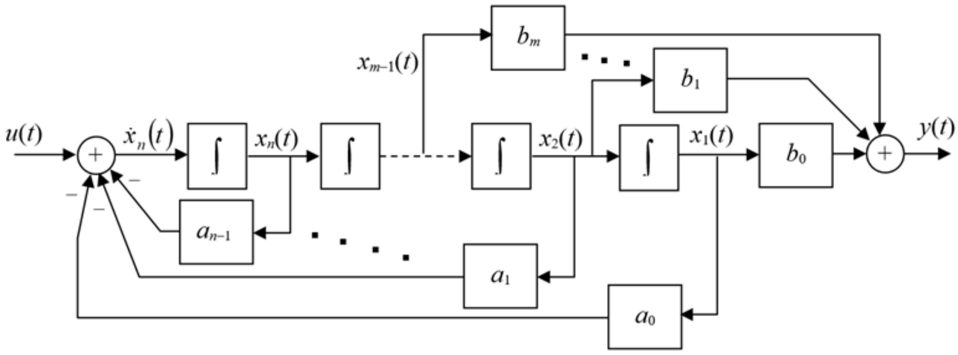
From (6-2) and (6-4), the output equation can be derived as

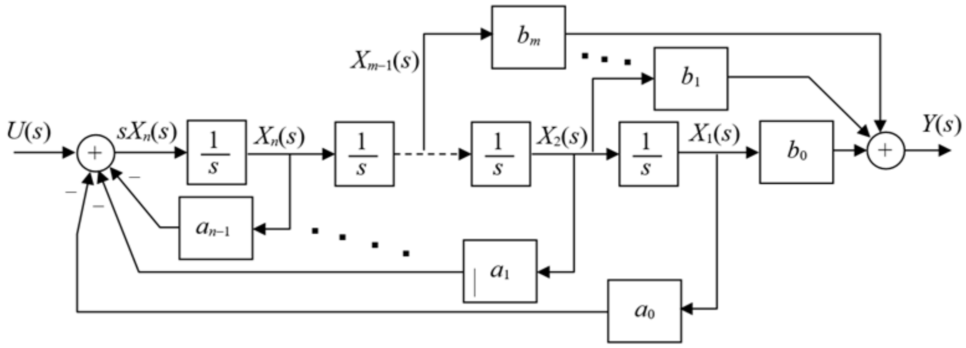
(6-11) $$Y(s) = b_m X_{m-1}(s) + \cdots + b_2 X_3(s) + b_1 X_2(s) + b_0 X_1(s)$$

i.e.,

(6-12) $$y(t) = b_m x_{m-1}(t) + \cdots + b_2 x_3(t) + b_1 x_2(t) + b_0 x_1(t)$$

As a result, the total system (6-1) can be implemented in time domain or in frequency domain, both shown as below:





In the case of $n = m$, such as a highpass filter, the transfer function is given as

(6-13) $$H(s) = \frac{b_n s^n + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

i.e.,

(6-14) $$H(s)=b_n+\frac{\beta_{n-1}s^{n-1}+\cdots+\beta_1 s+\beta_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

where $\beta_k=b_k-b_n a_k$, k=0,1,2,…,n–1. Therefore,

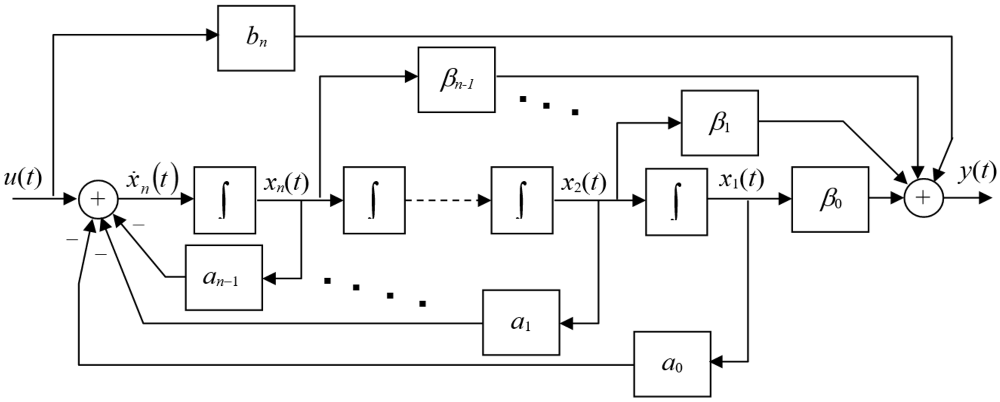
(6-15) $$Y(s)=H(s)U(s)=\left(b_n+\frac{\beta_{n-1}s^{n-1}+\cdots+\beta_1 s+\beta_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}\right)U(s)$$

i.e.,

(6-16) $$Y(s)=b_n U(s)+\beta_{n-1}X_n(s)+\cdots+\beta_2 X_3(s)+\beta_1 X_2(s)+\beta_0 X_1(s)$$

or

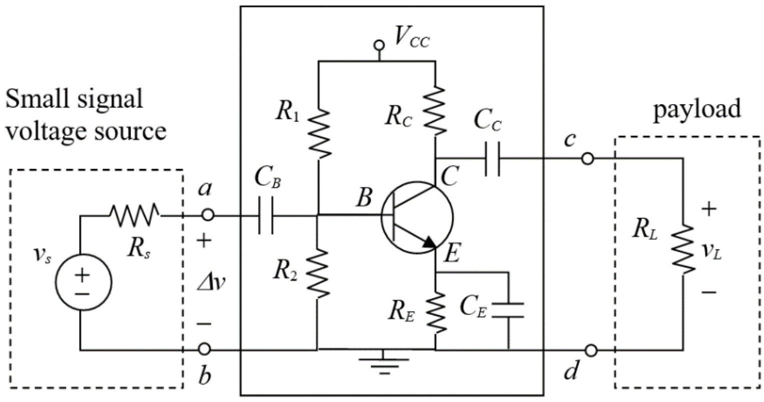
(6-17) $$y(t)=b_n u(t)+\beta_{n-1}x_n(t)+\cdots+\beta_2 x_3(t)+\beta_1 x_2(t)+\beta_0 x_1(t)$$



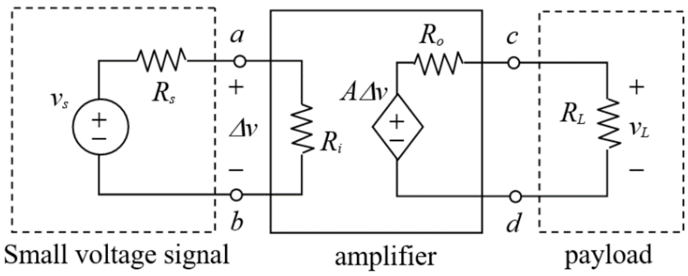
In addition to the above realization, the system can be changed into series, parallel, or a lot of different forms. Most importantly, all of them are often realized by the use of operational amplifiers (OpAmps).

➢ ***Ideal Operational Amplifiers***

In practice, we can use a transistor to implement a voltage amplifier shown as below:



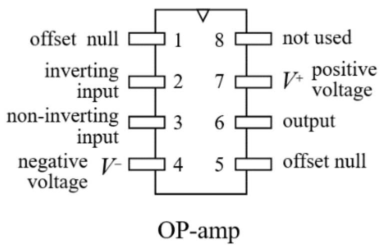
which is equivalent to

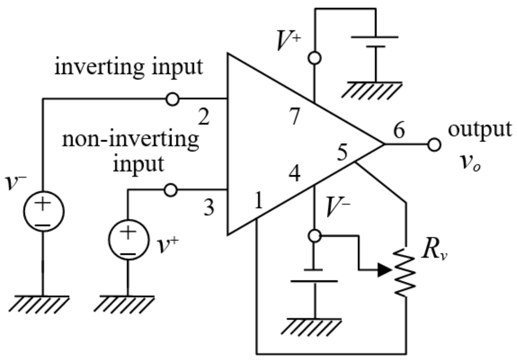


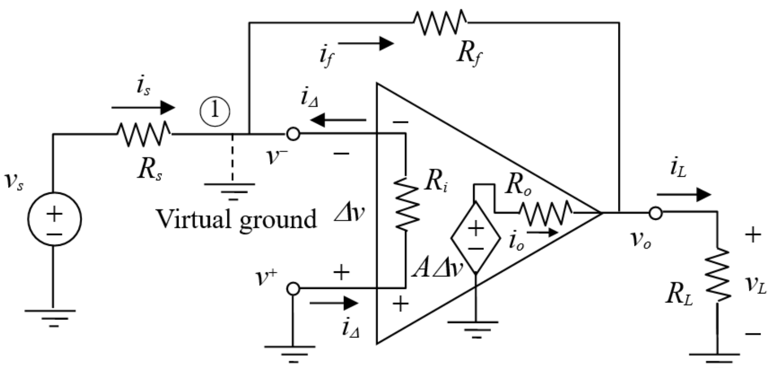
It can be obtained that

$$v_L(t) = \frac{R_L}{R_o + R_L} A\Delta v(t) = \left(\frac{R_L}{R_o + R_L}\right)\left(\frac{R_i}{R_i + R_s}\right) A v_s(t)$$

If $R_i \to \infty$ and $R_o \approx 0$, then $v_L(t) = Av_s(t)$, not affected by R_s and R_L. This is the reason that we have to use a chip of OpAmps, shown as below:



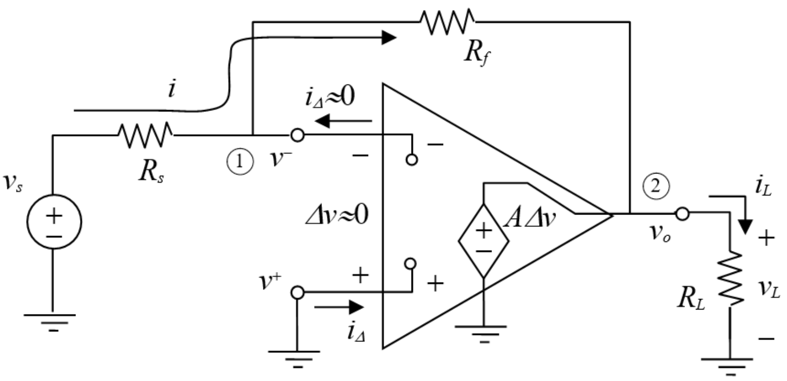


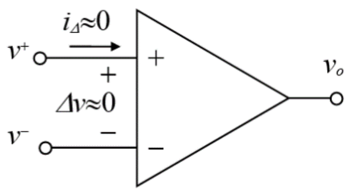


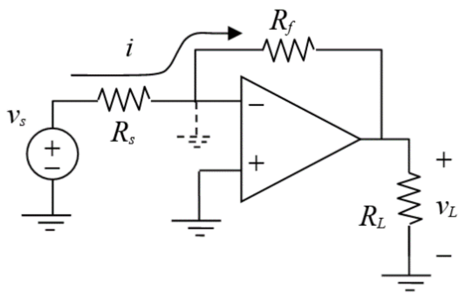
Above shows the equivalent circuit of an OpAmp. For an ideal OpAmp, we have

$$R_i \to \infty,\ \ R_o \approx 0,\ \ A \to \infty$$

which implies $v_o(t) = A\Delta v(t)$, and thus, $\Delta v(t) \approx 0$ and $i_\Delta \to 0$. In general, we can assume node ① is a virtual ground. Below shows a circuit with an ideal OpAmp.

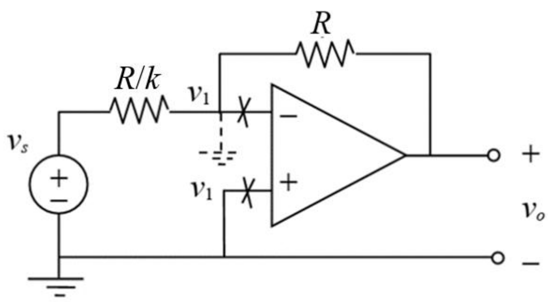






It is easy to check that $i(t)=\frac{v_s(t)}{R_s}=-\frac{v_L(t)}{R_f}$, i.e., $v_L(t)=-\frac{R_f}{R_s}v_s(t)$. Clearly, the circuit is an amplifier with a $-\frac{R_f}{R_s}$ independent to the payload R_L, as expected. Below introduce three basic circuits with an ideal OpAmp for system realization.

➢ ***Inverter***



Using Kirchhoff's current law (KCL) on the virtual ground, we have

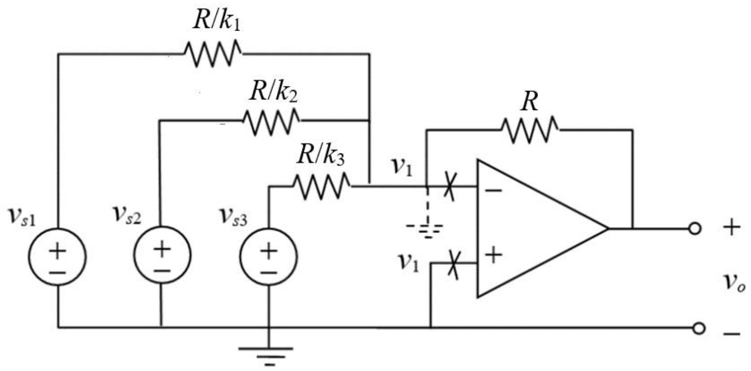
(6-18) $$\frac{v_1(t)-v_s(t)}{R/k}+\frac{v_1(t)-v_o(t)}{R}=\frac{0-v_s(t)}{R/k}+\frac{0-v_o(t)}{R}=0$$

which results in

(6-19) $v_o(t) = -k\,v_s(t) = A_v\,v_s(t)$

Clearly, the gain $A_v = -k$ is negative, so the circuit is an **inverter**.

➢ *Inverting Adder*



Using KCL yields

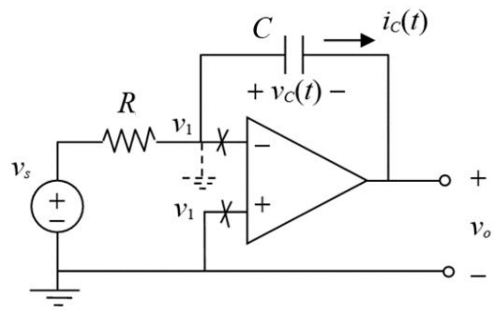
(6-20) $$\frac{0 - v_{s1}(t)}{R/k_1} + \frac{0 - v_{s2}(t)}{R/k_2} + \frac{0 - v_{s3}(t)}{R/k_3} + \frac{0 - v_o(t)}{R} = 0$$

i.e.,

(6-21) $v_o(t) = -\left(k_1\,v_{s1}(t) + k_2\,v_{s2}(t) + k_3\,v_{s3}(t)\right)$

which is circuit to perform **inverting adder**.

➢ *Inverting Integrator*



It is known that

(6-22) $$i_C(t)=C\frac{dv_C(t)}{dt}=C\frac{d}{dt}\left(v_1(t)-v_o(t)\right)=-C\frac{d}{dt}v_o(t)$$

Based on KCL, we have

(6-23) $$\frac{v_1(t)-v_s(t)}{R}+i_C(t)=-\frac{v_s(t)}{R}-C\frac{d}{dt}v_o(t)=0$$

which results in $\frac{dv_o(t)}{dt}=-\frac{v_s(t)}{RC}$, i.e., $v_o(t)=-\frac{1}{RC}\int_0^t v_s(\tau)d\tau+v_o(0)$.

If we choose $RC=1$ and neglect the initial conditon $v_o(0)$, then

(6-24) $$v_o(t)=-\int_0^t v_s(\tau)d\tau$$

It is indeed an **inverting integrator**.

Example:

Consider a linear time-invariant (LTI) system described as

(6-25) $$\ddot{y}(t)+3\dot{y}(t)+4y(t)=-\dot{u}(t)+2u(t)$$

where $u(t)$ is the input and $y(t)$ is the output. Since it is a stable system, we can neglect all the initial conditions in system realization.

Taking Laplace transform of the LTI system yields

(6-26) $$s^2Y(s)+3sY(s)+4Y(s)=-sU(s)+2U(s)$$

i.e., $(s^2+3s+4)Y(s)=(-s+2)U(s)$. Hence, we have

(6-27) $$Y(s)=\frac{-s+2}{s^2+3s+4}U(s)=H(s)U(s)$$

where $H(s)=\frac{-s+2}{s^2+3s+4}$ is the transfer function.

To implement the LTI system, we choose two state variables $x_1(t)$ and $x_2(t)$.

Define

(6-28) $\mathcal{L}\{x_1(t)\} = X_1(s) = \dfrac{1}{s^2+3s+4}U(s)$

(6-29) $\mathcal{L}\{x_2(t)\} = X_2(s) = \dfrac{s}{s^2+3s+4}U(s) = sX_1(s)$

From (6-29), we have $sX_1(s) = X_2(s)$, which is equivalent to

(6-30) $\dot{x}_1(t) = x_2(t)$

From (6-28), we have $(s^2+3s+4)X_1(s) = sX_2(s)+3X_2(s)+4X_1(s) = U(s)$, i.e.,

(6-31) $sX_2(s) = -4X_1(s) - 3X_2(s) + U(s)$

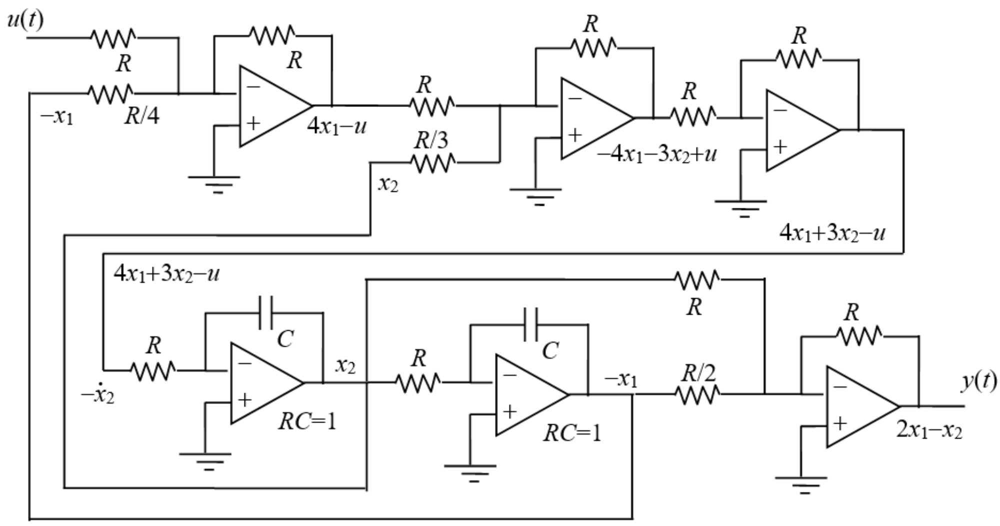
which is equivalent to

(6-32) $\dot{x}_2(t) = -4x_1(t) - 3x_2(t) + u(t)$

Since $Y(s) = 2\dfrac{1}{s^2+3s+4}U(s) - \dfrac{s}{s^2+3s+4}U(s) = 2X_1(s) - X_2(s)$, we have

(6-33) $y(t) = 2x_1(t) - x_2(t)$

Next, let's construct a circuit with inverters, inverting adders and inverting integrators to impement the LTI system, represented by the state equatios (6-30) and (6-32), and the output equation (6-33).

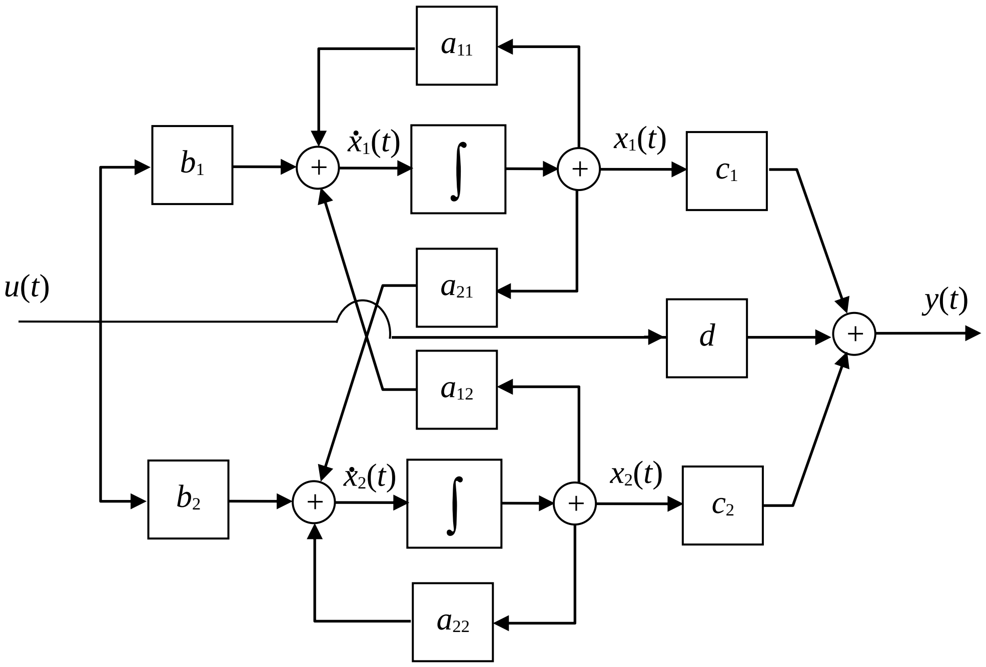


7. System in State-Space Description

It has been introduced that an LTI system can be expressed by an ordinary differential equation to describe the relation between input and output. Usually, we say that the system is in **input-output description**. However, the input-output description is unavailable to express the whole features of a system. If a system is known all in detail, not just the relation between input and output, we often write it in **state-space description**.

➢ State-Space Descrition

For simplicity, we present an example of a 2nd order system in state-space description here, and all the properties discussed below are also available for higher order systems.



In general, an LTI system in state space description is expressed by two 1st order differential equations, called the **state equations**, given as below:

(7-1) $\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t), \qquad x_1(0) = x_{10}$

(7-2) $\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t), \qquad x_2(0) = x_{20}$

and its **output eqation** is

(7-3) $y(t) = c_1x_1(t) + c_2x_2(t) + du(t)$

The block diagram is shown in the figure.

For convenience, rearrange the 2nd order LTI system into a vector form, where the state equation is

(7-4) $$\underbrace{\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}}_{\dot{\boldsymbol{x}}(t)}=\underbrace{\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}_{\boldsymbol{A}}\cdot\underbrace{\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}}_{\boldsymbol{x}(t)}+\underbrace{\begin{bmatrix}b_1\\ b_2\end{bmatrix}}_{\boldsymbol{b}}u(t),\qquad \underbrace{\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}}_{\boldsymbol{x}(0)}=\underbrace{\begin{bmatrix}x_{10}\\ x_{20}\end{bmatrix}}_{\boldsymbol{x}_0}$$

and the output equation is

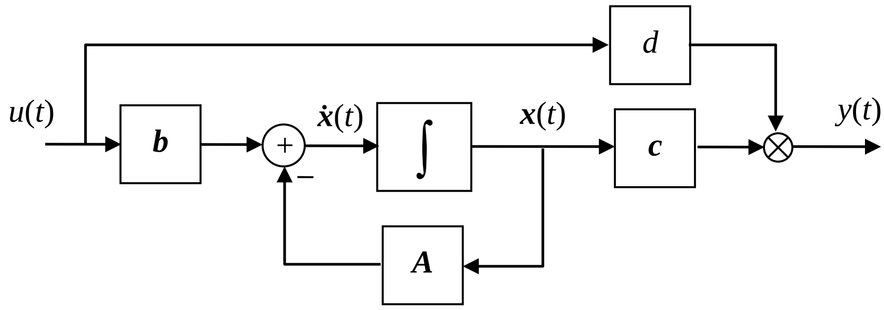
(7-5) $$y(t)=\underbrace{\begin{bmatrix}c_1 & c_2\end{bmatrix}}_{\boldsymbol{c}}\cdot\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+du(t)$$

Both are rewritten as below:

(7-6) $$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\cdot\boldsymbol{x}(t)+\boldsymbol{b}u(t),\qquad \boldsymbol{x}(0)=\boldsymbol{x}_0$$

(7-7) $$y(t)=\boldsymbol{c}\cdot\boldsymbol{x}(t)+du(t)$$

Below shows the block diagram correspondingly.



It is known that the **system state** $\boldsymbol{x}(t)$ and **output** $y(t)$ can be obtained as

(7-8) $$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\,\boldsymbol{x}_0+\int_0^t e^{\boldsymbol{A}(t-\tau)}\,\boldsymbol{b}\,u(\tau)d\tau$$

(7-9) $$y(t)=\boldsymbol{c}\boldsymbol{x}(t)+du(t)=\boldsymbol{c}e^{\boldsymbol{A}t}\boldsymbol{x}_0+\int_0^t \boldsymbol{c}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)d\tau+du(t)$$

However, it is not an easy work to determine $x(t)$ by directly calculating (8) and (9). Instead, we often apply numerical tools to find the numerical results, such as the software MATLAB.

Let's take the following LTI system as an example

(7-10) $$\dot{x}_1(t)=x_2(t)$$

(7-11) $$\dot{x}_2(t)=-3x_1(t)-4x_2(t)+u(t)$$

(7-12) $$y(t)=2x_1(t)-x_2(t)$$

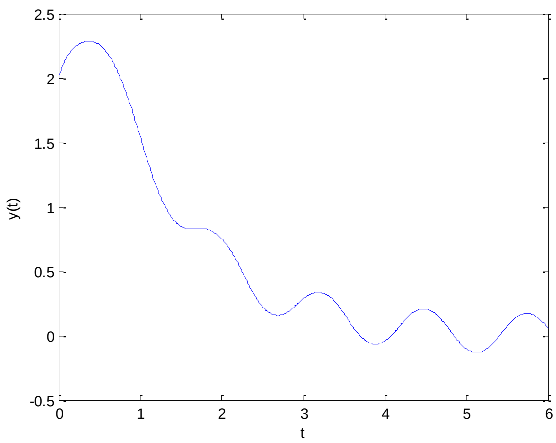
with initial conditions $x_1(0)=1$ and $x_2(0)=0$. If $u(t)=sin(cos(5t))$, then the system response $y(t)$ can be obained by the instruction ode45 in MATLAB, or by Simulink.

==

Create m-file: second.m

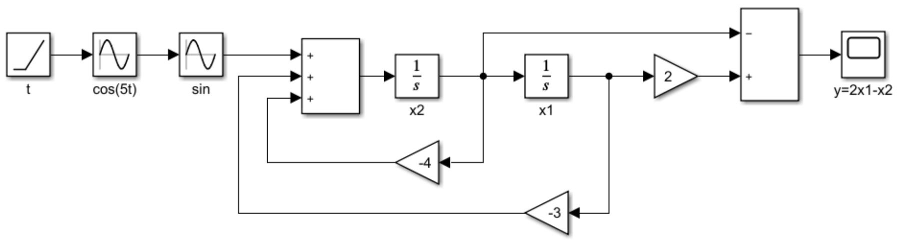
```
function dx=second(t,x)
dx=zeros(2,1);
dx(1)=x(2);
dx(2)=-3*x(1)-4*x(2)+sin(cos(5*t));
```

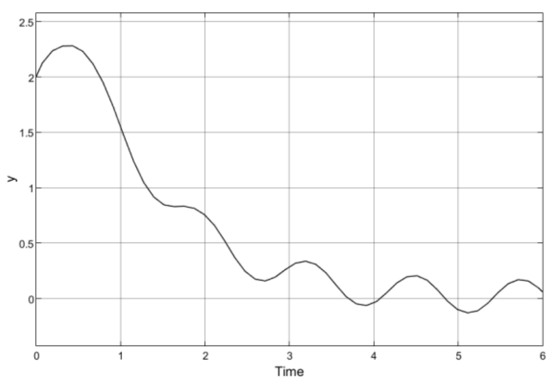
```
>> % key in the following instructions
>> [t,x]=ode45(@second,[0:0.01:6],[1 0])
>> y=2*x(:,1)-x(:,2);
>> plot(t,y); xlabel('t'); ylabel('y(t)')
```



==

Simulink:





If an LTI system is given in **state space description (SSD)**, then we can change it into **input-output description (IOD)**.

➢ Input-Output Descrition

Now, let's present the procedure to derive the **input-output description (IOD)** from the **state space description (SSD)**. First, find the **characteristic equation** of system matrix $\boldsymbol{A}$, which is defined as

(7-13) $$p(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = 0$$

i.e., $p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$, where $p(\lambda)$ is the characteristic polynomial. Let $a_1 = -(a_{11} + a_{22})$ and $a_0 = a_{11}a_{22} - a_{12}a_{21}$, then

(7-14) $$p(\lambda) = \lambda^2 + a_1\lambda + a_0$$

According to the **Cayley-Hamilton theoreom**, we have

(7-15) $$p(\boldsymbol{A}) = \boldsymbol{A}^2 + a_1 \boldsymbol{A} + a_0 \boldsymbol{I} = \boldsymbol{0}$$

Then, calculate the following equation

(7-16) $$\begin{aligned} &\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) \\ &= \boldsymbol{c}\left(\ddot{\boldsymbol{x}}(t) + a_1\dot{\boldsymbol{x}}(t) + a_0\boldsymbol{x}(t)\right) + d\ddot{u}(t) + a_1 d\dot{u}(t) + a_0 du(t) \\ &= \boldsymbol{c}\left(\boldsymbol{A}^2\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{b}u(t) + \boldsymbol{b}\dot{u}(t) + a_1\left(\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)\right) + a_0\boldsymbol{x}(t)\right) \\ &\qquad + d\ddot{u}(t) + a_1 d\dot{u}(t) + a_0 du(t) \\ &= \boldsymbol{c}\underbrace{\left(\boldsymbol{A}^2 + a_1\boldsymbol{A} + a_0\boldsymbol{I}\right)}_{=\boldsymbol{0}}\boldsymbol{x}(t) + d\ddot{u}(t) + \left(\boldsymbol{c}\boldsymbol{b} + a_1 d\right)\dot{u}(t) \\ &\qquad + \left(\boldsymbol{c}\boldsymbol{A}\boldsymbol{b} + a_1\boldsymbol{c}\boldsymbol{b} + a_0 d\right)u(t) \end{aligned}$$

which results in the following **IOD**:

(7-17) $$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_2\ddot{u}(t) + b_1\dot{u}(t) + b_0 u(t)$$

where $b_2 = d$, $b_1 = \boldsymbol{c}\boldsymbol{b}$ and $b_0 = \boldsymbol{c}\boldsymbol{A}\boldsymbol{b} + a_1\boldsymbol{c}\boldsymbol{b} + a_0 d$.

Clearly, the SSD (7-4) and (7-5) can be changed into the IOD (7-17). However, in the previous topic we have learned the way to change the IOD (7-17) into the SSD. First, taking Laplace transform of (7-17) yields

(7-18) $$Y(s)=\frac{b_2s^2+b_1s+b_0}{s^2+a_1s+a_0}U(s)=\left(b_2+\frac{\beta_1s+\beta_0}{s^2+a_1s+a_0}\right)U(s)$$

which results in the transfer function as

(7-19) $$H(s)=\frac{Y(s)}{U(s)}=\frac{b_2s^2+b_1s+b_0}{s^2+a_1s+a_0}=b_2+\frac{\beta_1s+\beta_0}{s^2+a_1s+a_0}$$

Then, choose two state variables as

(7-20) $$X_1^*(s)=\frac{1}{s^2+a_1s+a_0}U(s)$$

(7-21) $$X_2^*(s)=\frac{s}{s^2+a_1s+a_0}U(s)=sX_1^*(s)$$

and write the output (7-18) as

(7-22) $$Y(s)=b_2U(s)+\beta_0X_1^*(s)+\beta_1X_2^*(s)$$

From (7-20), we obtain $s^2X_1^*(s)+a_1sX_1^*(s)+a_0X_1^*(s)=U(s)$, i.e.

(7-23) $$sX_2^*(s)=-a_0X_1^*(s)-a_1X_2^*(s)+U(s)$$

Hence, from (7-21) and (7-23) the state equation can be obtained as

(7-24) $$\dot{x}_1^*(t)=x_2^*(t)$$

(7-25) $$\dot{x}_2^*(t)=-a_0x_1^*(t)-a_1x_2^*(t)+u(t)$$

and from (7-22) the output equation is expressed as

(7-26) $$y(t)=\beta_0x_1^*(t)+\beta_1x_2^*(t)+b_2u(t)$$

Rewrite (7-24) to (7-26) into matrix form as below:

(7-27) $$\underbrace{\begin{bmatrix}\dot{x}_1^*(t)\\ \dot{x}_2^*(t)\end{bmatrix}}_{\dot{\boldsymbol{x}}^*(t)}=\underbrace{\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}}_{\boldsymbol{A}^*}\cdot\underbrace{\begin{bmatrix}x_1^*(t)\\ x_2^*(t)\end{bmatrix}}_{\boldsymbol{x}^*(t)}+\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{\boldsymbol{b}^*}u(t)$$

(7-28) $$y(t)=\underbrace{\begin{bmatrix}\beta_0 & \beta_1\end{bmatrix}}_{\boldsymbol{c}^*}\cdot\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+b_2u(t)$$

Comparing them with (7-4) and (7-5), we can conclude that a different SSD is achieved. That means it is not unique to transform an IOD to SSD, or more precisely, an IOD can have an infinite number of SSDs. In other words, different SSDs may have the same IOD.

➢ System in Different SSDs

Consider an LTI system of nth-order in state-space description, expressed as

(7-29) $$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ y(t) = \boldsymbol{c}\boldsymbol{x}(t) \end{cases}$$

where $u(t)$ is the input, $y(t)$ is the output and $\boldsymbol{x}(t) = \left[x_1(t)\ x_2(t)\ \cdots\ x_n(t)\right]^T$ is the system state with initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ at the initial time $t = t_0$.

Define a new state $\boldsymbol{z}(t) = \boldsymbol{P}\boldsymbol{x}(t)$, where $\boldsymbol{P} \in R^{n\times n}$ is invertible, or $\boldsymbol{P}^{-1}$ exists. Then, $\boldsymbol{x}(t) = \boldsymbol{P}^{-1}\boldsymbol{z}(t)$ and substituting it into (7-29) yields

(7-30) $$\begin{cases} \boldsymbol{P}^{-1}\dot{\boldsymbol{z}}(t) = \boldsymbol{A}\boldsymbol{P}^{-1}\boldsymbol{z}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{P}^{-1}\boldsymbol{z}(t_0) = \boldsymbol{x}_0 \\ y(t) = \boldsymbol{c}\boldsymbol{P}^{-1}\boldsymbol{z}(t) \end{cases}$$

or

(7-31) $$\begin{cases} \dot{\boldsymbol{z}}(t) = \boldsymbol{A}_N\boldsymbol{z}(t) + \boldsymbol{b}_N u(t), \quad \boldsymbol{z}(t_0) = \boldsymbol{z}_0 \\ y(t) = \boldsymbol{c}_N\boldsymbol{z}(t) \end{cases}$$

where $\boldsymbol{A}_N = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}$, $\boldsymbol{b}_N = \boldsymbol{P}\boldsymbol{b}$, $\boldsymbol{c}_N = \boldsymbol{c}P^{-1}$ and $\boldsymbol{z}_0 = \boldsymbol{P}\boldsymbol{x}_0$. Since $|\boldsymbol{P}||\boldsymbol{P}^{-1}| = 1$, it can be obtained that

(7-32) $$|\lambda\boldsymbol{I} - \boldsymbol{A}_N| = |\lambda\boldsymbol{P}\boldsymbol{I}\boldsymbol{P}^{-1} - \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}| = |\boldsymbol{P}||\lambda\boldsymbol{I} - \boldsymbol{A}||\boldsymbol{P}^{-1}| = |\lambda\boldsymbol{I} - \boldsymbol{A}|$$
$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

i.e., both (7-29) and (7-31) have the same characteristic equation. Besides,

(7-33) $$y(t) = \boldsymbol{c}_N\boldsymbol{z}(t) = \boldsymbol{c}\boldsymbol{x}(t)$$

(7-34) $$\dot{y}(t) = \boldsymbol{c}_N\dot{\boldsymbol{z}}(t) = \boldsymbol{c}_N\boldsymbol{A}_N\boldsymbol{z}(t) + \boldsymbol{c}_N\boldsymbol{b}_N u(t) = \boldsymbol{c}\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{c}\boldsymbol{b}u(t)$$

(7-35) $$\ddot{y}(t) = \boldsymbol{c}_N\boldsymbol{A}_N\dot{\boldsymbol{z}}(t) + \boldsymbol{c}_N\boldsymbol{b}_N \dot{u}(t) = \boldsymbol{c}_N\boldsymbol{A}_N^2\boldsymbol{z}(t) + \boldsymbol{c}_N\boldsymbol{A}_N\boldsymbol{b}_N u(t) + \boldsymbol{c}_N\boldsymbol{b}_N \dot{u}(t)$$
$$= \boldsymbol{c}\boldsymbol{A}^2\boldsymbol{x}(t) + \boldsymbol{c}\boldsymbol{A}\boldsymbol{b}u(t) + \boldsymbol{c}\boldsymbol{b}\dot{u}(t)$$

which implies that both (7-29) and (7-31) represent the same IOD, expressed as

(7-36) $$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0 y(t)$$
$$= b_{n-1}u^{(n-1)}(t) + b_{n-2}u^{(n-2)}(t) + \cdots + b_1\dot{u}(t) + b_0 u(t)$$

In control engineering, one of the most commonly used SSD of (7-36) is called the controllable form and expressed as

(7-37)
$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ y(t) = \boldsymbol{c}\boldsymbol{x}(t) \end{cases}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix}$$

8. State-feedback Control of LTI Systems

➢ State-feedback Control

From this topic, we will focus on the state-feedback control applied to an LTI system in state-space description, which is expressed as

(8-1) $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{b}u(t), \quad \boldsymbol{x}(t_0)=\boldsymbol{x}_0$

(8-2) $y(t)=\boldsymbol{c}\boldsymbol{x}(t)$

where $u(t)$ is the input, $y(t)$ is the output and $\boldsymbol{x}(t)=\left[x_1(t)\ x_2(t)\ \cdots\ x_n(t)\right]^T$ is the system state with initial condition $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$ at the initial time $t=t_0$.

First, let's consider the stability problem. It is well-known that if the system satisfies the following condition

(8-3) $rank\left[\boldsymbol{b}\ \ \boldsymbol{A}\boldsymbol{b}\ \ \boldsymbol{A}^2\boldsymbol{b}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{b}\right]=n$

then the system is controllable by the use of state feedback

(8-4) $u(t)=-\boldsymbol{k}\boldsymbol{x}(t)$

such that the system state $\boldsymbol{x}(t)$ will approach the origin $\boldsymbol{0}$ from any initial state $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$.

It is noticed that the use of $\boldsymbol{x}(t)$ in the state-feedback control (8-4) implies all the state variables must be exactly obtainable or measurable, otherwise the state-feedback control (8-4) is not available. Next, let's show how to determine the control gain $\boldsymbol{k}$.

➢ Pole-placement Method

Commonly, the so-called pole- placement method is adopted. With the use of (8-4), the state equation is changed into

(8-5) $\dot{\boldsymbol{x}}(t)=(\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k})\boldsymbol{x}(t)$

which implies that the system can be stabilized when the eigenvalues of $\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}$ are located in the left-half complex plane, i.e., each eigenvalue possesses a negative real part. To apply the pole-placement method, first we have to choose n desired eigenvalues, such as λ_i, i=1,2,…,n. Then, determine $\boldsymbol{k}$ from the following characteristic polynomial

(8-6) $$\left|\lambda \boldsymbol{I}-(\boldsymbol{A}-\boldsymbol{bk})\right|=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

Note that the gain $\boldsymbol{k}$ is unique for single-input system, but not for multiple inputs system. As a result, $x_k(t)=\sum_{i=1}^{n}a_i e^{\lambda_i t}$, k=1,2,…,n. Clearly, the system is stabilized since all the state variables are driven to approach 0, i.e., $x_k(t)\big|_{t\to\infty}\to 0$, under state-feedback control.

➢ Regulation problem

Next, let's consider the so-called regulation problem: Drive the output $y(t)$ to the fixed destination y_d, i.e., $y(t)=y_d$. To deal with such problem, define the error vector as

(8-7) $$e(t)=y(t)-y_d=\boldsymbol{cx}(t)-y_d$$

and genetate a new state by the use of integrator as below:

(8-8) $$z(t)=\int_{t_0}^{t}e(\tau)d\tau=\int_{t_0}^{t}\left(\boldsymbol{cx}(t)-y_d\right)d\tau$$

which implies

(8-9) $$\dot{z}(t)=e(t)=\boldsymbol{cx}(t)-y_d, \qquad z(t_0)=0$$

Combined with (8-1), it leads to the following augmented system

(8-10) $$\underbrace{\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{z}(t)\end{bmatrix}}_{\dot{\boldsymbol{x}}_{aug}(t)}=\underbrace{\begin{bmatrix}\boldsymbol{A} & 0\\ \boldsymbol{c} & 0\end{bmatrix}}_{\boldsymbol{A}_{aug}}\cdot\underbrace{\begin{bmatrix}\boldsymbol{x}(t)\\ z(t)\end{bmatrix}}_{\boldsymbol{x}_{aug}(t)}+\underbrace{\begin{bmatrix}\boldsymbol{b}\\ 0\end{bmatrix}}_{\boldsymbol{b}_{aug}}u(t)+\begin{bmatrix}\boldsymbol{0}\\ -1\end{bmatrix}y_d, \qquad \underbrace{\begin{bmatrix}\boldsymbol{x}(t_0)\\ z(t_0)\end{bmatrix}}_{\boldsymbol{x}_{aug}(t_0)}=\begin{bmatrix}\boldsymbol{x}_0\\ 0\end{bmatrix}$$

It has been proved that

(8-11) $$rank\left[\boldsymbol{b}_{aug}\ \ \boldsymbol{A}_{aug}\boldsymbol{b}_{aug}\ \ \boldsymbol{A}_{aug}^2\boldsymbol{b}_{aug}\ \ \cdots\ \ \boldsymbol{A}_{aug}^n\boldsymbol{b}_{aug}\right]=n+1$$

i.e., the augmented system (8-10) is also controllable. Hence, with the use of state feedback

(8-12) $$u=-\boldsymbol{k}_{aug}\boldsymbol{x}_{aug}(t)$$

the system state $x_{aug}(t)$ will approach the origin $\boldsymbol{0}$ from the initial state

$\boldsymbol{x}_{aug}(t_0)=\begin{bmatrix}\boldsymbol{x}_0\\0\end{bmatrix}$.

Based on the pole- placement method, the augmented state equation is changed into

(8-13) $\dot{\boldsymbol{x}}_{aug}(t)=\left(\boldsymbol{A}_{aug}-\boldsymbol{b}_{aug}\boldsymbol{k}_{aug}\right)\boldsymbol{x}_{aug}(t)$

where the eigenvalues of $\boldsymbol{A}_{aug}-\boldsymbol{b}_{aug}\boldsymbol{k}_{aug}$ are assigned as η_i, i=1,2,…,n+1, all located in the left-half complex plane. Then, determine $\boldsymbol{k}_{aug}$ by the pole-placement method from the following characteristic polynomial

(8-14) $\left|\lambda\boldsymbol{I}-\left(\boldsymbol{A}_{aug}-\boldsymbol{b}_{aug}\boldsymbol{k}_{aug}\right)\right|=(\lambda-\eta_1)(\lambda-\eta_2)\cdots(\lambda-\eta_{n+1})$ (17)

As a result, $\dot{\boldsymbol{x}}_{aug}(t)\Big|_{t\to\infty}=\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\\dot{z}(t)\end{bmatrix}_{t\to\infty}\to\boldsymbol{0}$. Clearly, $\dot{z}(t)\Big|_{t\to\infty}=y(t)\Big|_{t\to\infty}-y_d$, i.e., the control goal $y(t)\Big|_{t\to\infty}=y_d$ is achieved under state-feedback control.

➢ **Observer**

If only the output $y(t)$ is measurable, not the system state $\boldsymbol{x}(t)$, then it is not available to use the state feedback $u(t)=-\boldsymbol{kx}(t)$. To deal with such problem, the so-called linear Luenberger observer is often employed to estimate $\boldsymbol{x}(t)$.

Based on $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in the original system (8-1) and (8-2), the linear Luenberger observer is constructed as below:

(8-15) $\dot{\hat{\boldsymbol{x}}}(t)=\boldsymbol{A}\hat{\boldsymbol{x}}(t)+\boldsymbol{b}u(t)+\boldsymbol{L}\left(y(t)-\hat{y}(t)\right)$, $\hat{\boldsymbol{x}}(0)=\boldsymbol{0}$

(8-16) $\hat{y}(t)=\boldsymbol{c}\hat{\boldsymbol{x}}(t)$

where $\hat{\boldsymbol{x}}(t)$ is the estimated state, $\hat{y}(t)$ is the estimated output, and the gain $\boldsymbol{L}$ is purposely introduced as a key design element. Define the estimation error as

(8-17) $\tilde{\boldsymbol{x}}(t)=\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)$

then from (8-1) and (8-15) we have

(8-18) $\dot{\tilde{\boldsymbol{x}}}(t)=\boldsymbol{A}\tilde{\boldsymbol{x}}(t)-\boldsymbol{Lc}\tilde{\boldsymbol{x}}(t)=(\boldsymbol{A}-\boldsymbol{Lc})\tilde{\boldsymbol{x}}(t)$, $\tilde{\boldsymbol{x}}(0)=\boldsymbol{x}_0$

Clearly, if the eigenvalues of $\boldsymbol{A}-\boldsymbol{Lc}$ are all located in the left-half complex plane, then the estimation error will approach zero as $t\to\infty$, i.e., (8-19) $\tilde{\boldsymbol{x}}(t)\big|_{t\to\infty} = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\big|_{t\to\infty} \to \boldsymbol{0}$

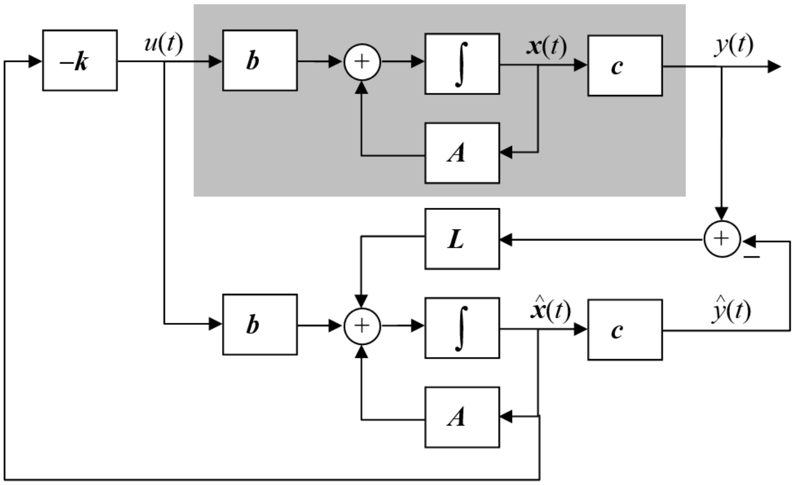
or

(8-20) $\hat{\boldsymbol{x}}(t)\big|_{t\to\infty} \to \boldsymbol{x}(t)\big|_{t\to\infty}$

In other words, the estimated state is gradually approximated to the actual state $\boldsymbol{x}(t)$ as time increases and thus, the state feedback in (8-4) can be replaced by the following control

(8-21) $u(t) = -\boldsymbol{k}\hat{\boldsymbol{x}}(t)$

which will drive the system state $\boldsymbol{x}(t)$ from $\boldsymbol{x}_0$ to $\boldsymbol{0}$. The whole system block diagram is shown in the figure.



Now, let's focus on the design of the observer gain $\boldsymbol{L}$. It is known that the eigenvalues of $\boldsymbol{A}-\boldsymbol{L}\boldsymbol{c}$ can be obtained by solving the following polynomial:

(8-22) $\left|\lambda\boldsymbol{I} - (\boldsymbol{A}-\boldsymbol{L}\boldsymbol{c})\right| = 0$

Since the determinant of a matrix $\boldsymbol{H}$ is the same as the determinant of its transpose, i.e., $|\boldsymbol{H}| = |\boldsymbol{H}^T|$, the polynomial (8-22) can be rewritten as

(8-23) $\left|\lambda\boldsymbol{I} - (\boldsymbol{A}-\boldsymbol{L}\boldsymbol{c})\right| = \left|\lambda\boldsymbol{I} - (\boldsymbol{A}^T - \boldsymbol{c}^T\boldsymbol{L}^T)\right| = 0$

Based on the pole-placement method, the existence of $\boldsymbol{L}$ requires that the system must satisfies the following condition

(8-24) $rank\left[\boldsymbol{A}\boldsymbol{c}^T \;\; \boldsymbol{A}^T\boldsymbol{c}^T \;\; \left(\boldsymbol{A}^T\right)^2\boldsymbol{c}^T \;\cdots\; \left(\boldsymbol{A}^T\right)^{n-1}\boldsymbol{c}^T\right] = n$

Due to the fact that $rank(\boldsymbol{H}) = rank(\boldsymbol{H}^T)$, (8-24) can be rewritten as

(8-25) $$rank\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{c}A \\ \boldsymbol{c}A^2 \\ \vdots \\ \boldsymbol{c}A^{n-1} \end{bmatrix} = n$$

which is the condition for a system to be observable.

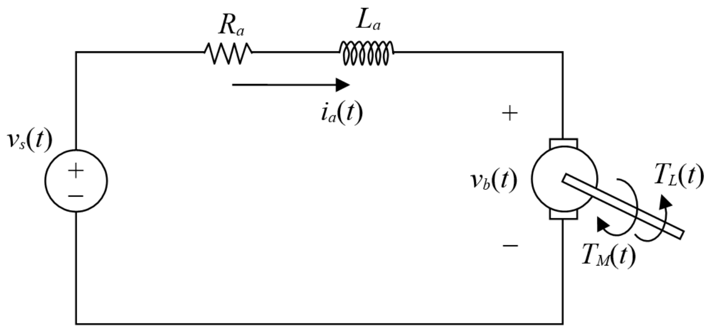
Hence, we can assign n poles $\{p_1, p_2, \cdots, p_n\}$ and then solve $\boldsymbol{L}$ from the following equation:

(8-26) $$\left|\lambda\boldsymbol{I} - \left(\boldsymbol{A}^T - \boldsymbol{c}^T\boldsymbol{L}^T\right)\right| = (\lambda - p_1)(\lambda - p_2)\cdots(\lambda - p_n)$$

which is similar to the way in determining the control gain $\boldsymbol{k}$.

9. Velocity and Position Control of a DC Motor

➢ Modeling of DC motor



The system structure of a DC motor is depicted in the figure, including the armature resistance R_a and winding leakage inductance L_a. According to the Kirchhoff's voltage law, the electrical equation of the DC motor is described as

(9-1) $$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) = v_s(t)$$

where $i_a(t)$ is the armature current, $v_b(t)$ is the back emf voltage and $v_s(t)$ is the voltage source. The back emf voltage $v_b(t)$ is proportional to the angular velocity $\omega(t)$ of the rotor in the motor, expressed as

(9-2) $$v_b(t) = k_b \omega(t)$$

where k_b is the back emf constant. In addition, the motor generates a torque T_M proportional to the armature current, given as

(9-3) $$T_M(t) = k_T i_a(t)$$

where k_T is the torque constant.

If the input voltage $v_s(t) = V_s$ is a constant, the resulted armature current $i_a(t) = I_a$, angular velocity $\omega(t) = \Omega$ and torque $T_M(t) = T$ are also constant in the steady state. From (9-1) to (9-3), we have

(9-4) $$R_a I_a + k_b \Omega = V_s$$

(9-5) $$T = k_T I_a$$

Under the conservation of power, we know that the input power $I_a V_s$ is equal to the sum of external power $T\Omega$ and the power $R_a I_a^2$ consumed in the resistance, i.e.,

(9-6) $$V_s I_a = T\Omega + R_a I_a^2$$

From (9-4), we have $R_a I_a^2 + k_b \Omega I_a = V_s I_a$, then comparing it with (9-6) yields

(9-7) $$T = k_b I_a$$

From (9-5) and (9-7), we know that both k_T and k_b are the same. Hence, we can rewrite (9-1) and (9-3) as

(9-8) $$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + k\omega(t) = v_s(t)$$

(9-9) $$T_M(t) = k i_a(t)$$

where $k = k_T = k_b$. Besides, if the DC motor is used to drive an external torque T_L(t) of payload then its mechanical behavior can be described as

(9-10) $$J_M \frac{d\omega(t)}{dt}(t) + B_M \omega(t) = T_M(t) - T_L(t)$$

where J_M is the rotor moment of inertia, B_M is the frictional coefficient and the torque $T_L(t)$ reacted from the payload.

For simplicity, we assume an external payload is directly connected to the motor, i.e., the torque is $T_L(t) = J_L \dot{\omega}(t)$. Hence, based on (9-8), (9-9) and (9-10), the dynamic equation of the DC motor can be expressed as

(9-11) $$L_a \frac{di_a(t)}{dt} + R_a i_a(t) + k\omega(t) = v_s(t)$$

(9-12) $$J \frac{d\omega(t)}{dt}(t) + B_M \omega(t) - k i_a(t) = 0$$

where $J = J_M + J_L$. Now, let's discuss the model of a DC motor in state-space description (SSD) and input-output description (IOD).

First, consider the case which requires the DC motor to move in a constant speed. Then, the angular velocity is selected as the output, expressed as

(9-13) $$y_\omega(t) = \omega(t)$$

From (9-11) and (9-12) and choosing the state variables as $x_1(t) = i_a(t)$ and $x_2(t) = \omega(t)$, we have

(9-14) $R_a x_1(t) + L_a \dot{x}_1(t) + k x_2(t) = v_s(t)$

(9-15) $J\dot{x}_2(t) + B_M x_2(t) - k x_1(t) = 0$

which can be rewritten as the state equation

(9-16) $\dot{x}_1(t) = -\dfrac{R_a}{L_a} x_1(t) - \dfrac{k}{L_a} x_2(t) + \dfrac{1}{L_a} v_s(t)$

(9-17) $\dot{x}_2(t) = \dfrac{k}{J} x_1(t) - \dfrac{B_M}{J} x_2(t)$

and the output equation

(9-18) $y_\omega(t) = x_2(t)$

Hence, the SSD is shown as below:

(9-21) $\dot{\boldsymbol{x}}_\omega(t) = \boldsymbol{A}_\omega \boldsymbol{x}_\omega(t) + \boldsymbol{b}_\omega u(t)$

(9-22) $y_\omega(t) = \boldsymbol{c}_\omega \boldsymbol{x}_\omega(t)$

where the state vector is $\boldsymbol{x}_\omega(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, the input vector is $u(t) = v_s(t)$, and the system matrices are $\boldsymbol{A}_\omega = \begin{bmatrix} -\dfrac{R_a}{L_a} & -\dfrac{k}{L_a} \\ \dfrac{k}{J} & -\dfrac{B_M}{J} \end{bmatrix}$, $\boldsymbol{b}_\omega = \begin{bmatrix} \dfrac{1}{L_a} \\ 0 \end{bmatrix}$ and $\boldsymbol{c}_\omega = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

If the control goal is to drive the DC motor to a desired position, not velocity, then the output is chosen as the angular position $y_\theta(t) = \theta(t) = \int_0^t \omega(\tau) d\tau$. To include the angular position as a system state variable, we often change the integral form $\theta(t) = \int_0^t \omega(\tau) d\tau$ into the differential form as below:

(9-23) $\dot{\theta}(t) = \omega(t)$

and choose $x_3(t) = \theta(t)$ as a new state variable. Therefore, the resulted state equations are

(9-24) $\dot{x}_1(t) = -\dfrac{R_a}{L_a} x_1(t) - \dfrac{k}{L_a} x_2(t) + \dfrac{1}{L_a} v_s(t)$

(9-25) $\dot{x}_2(t) = \dfrac{k}{J} x_1(t) - \dfrac{B_M}{J} x_2(t)$

(8-26) $\dot{x}_3(t) = x_2(t)$

and the output equation is

(9-27) $y_\theta(t) = x_3(t)$

In matrix form, we have

(9-28) $\dot{\boldsymbol{x}}_\theta(t) = \boldsymbol{A}_\theta \boldsymbol{x}_\theta(t) + \boldsymbol{b}_\theta u(t)$

(9-29) $y_\theta(t) = \boldsymbol{c}_\theta \boldsymbol{x}_\theta(t)$

where $\boldsymbol{x}_\theta(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}^T$, $\boldsymbol{A}_\theta = \begin{bmatrix} -R_a/L_a & -k/L_a & 0 \\ k/J & -B_M/J & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{b}_\theta = \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{c}_\theta = [0 \ 0 \ 1]$.

- **Velocity Control**

For the velosity control, we employ the modeling in (9-21) and (9-22), shown as

(9-30) $\dot{\boldsymbol{x}}_\omega(t) = \boldsymbol{A}_\omega \boldsymbol{x}_\omega(t) + \boldsymbol{b}_\omega u(t)$

(9-31) $y_\omega(t) = \boldsymbol{c}_\omega \boldsymbol{x}_\omega(t)$

where $\boldsymbol{x}_\omega(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i_a(t) \\ \omega(t) \end{bmatrix}$, $\boldsymbol{A}_\omega = \begin{bmatrix} -\dfrac{R_a}{L_a} & -\dfrac{k}{L_a} \\ \dfrac{k}{J} & -\dfrac{B_M}{J} \end{bmatrix}$, $\boldsymbol{b}_\omega = \begin{bmatrix} \dfrac{1}{L_a} \\ 0 \end{bmatrix}$ and $\boldsymbol{c}_\omega = [0 \ 1]$.

It is easy to check that

(9-32) $rank[\boldsymbol{b}_\omega \ \ \boldsymbol{A}_\omega \boldsymbol{b}_\omega] = rank\begin{bmatrix} L_a^{-1} & -R_a L_a^{-2} \\ 0 & kJ^{-1}L_a^{-1} \end{bmatrix} = 2$

(9-33) $rank\begin{bmatrix} \boldsymbol{c}_\omega \\ \boldsymbol{c}_\omega \boldsymbol{A}_\omega \end{bmatrix} = rank\begin{bmatrix} 0 & 1 \\ kJ^{-1} & -B_M J^{-1} \end{bmatrix} = 2$

Therefore, the DC motor is controllable and observable.

Besides, its IOD can be obtained starting from the characteristic equation is

(9-34) $|\lambda \boldsymbol{I} - \boldsymbol{A}_\omega| = \begin{vmatrix} \lambda + \dfrac{R_a}{L_a} & \dfrac{k}{L_a} \\ -\dfrac{k}{J} & \lambda + \dfrac{B_M}{J} \end{vmatrix} = \lambda^2 + \underbrace{\left(\dfrac{R_a}{L_a} + \dfrac{B_M}{J}\right)}_{a_1}\lambda + \underbrace{\dfrac{R_a B_M + k^2}{JL_a}}_{a_0} = 0$

According to Cayley-Hamilton theorem, we have $\boldsymbol{A}_\omega^2 + a_1 \boldsymbol{A}_\omega + a_0 \boldsymbol{I} = \boldsymbol{0}$. Then, calculate $\ddot{y}_\omega(t)+a_1\dot{y}_\omega(t)+a_0y_\omega(t)$ as below:

(9-35) $$\begin{aligned}&\ddot{y}_\omega(t)+a_1\dot{y}_\omega(t)+a_0y_\omega(t)\\&=\boldsymbol{c}_\omega\left(\boldsymbol{A}_\omega\dot{\boldsymbol{x}}_\omega(t)+\boldsymbol{b}_\omega\dot{u}(t)\right)+a_1\boldsymbol{c}_\omega\left(\boldsymbol{A}_\omega\boldsymbol{x}_\omega(t)+\boldsymbol{b}_\omega u(t)\right)+a_0\boldsymbol{c}_\omega\boldsymbol{x}_\omega(t)\\&=\boldsymbol{c}_\omega\boldsymbol{A}_\omega^2\boldsymbol{x}_\omega(t)+\boldsymbol{c}_\omega\boldsymbol{A}_\omega\boldsymbol{b}_\omega u(t)+\underbrace{\boldsymbol{c}_\omega\boldsymbol{b}_\omega}_{=0}\dot{u}(t)\\&\quad+a_1\boldsymbol{c}_\omega\boldsymbol{A}_\omega\boldsymbol{x}_\omega(t)+a_1\underbrace{\boldsymbol{c}_\omega\boldsymbol{b}_\omega}_{=0}u(t)+a_0\boldsymbol{c}_\omega\boldsymbol{x}_\omega(t)\\&=\boldsymbol{c}_\omega\underbrace{\left(\boldsymbol{A}_\omega^2+a_1\boldsymbol{A}_\omega+a_0\boldsymbol{I}\right)}_{\boldsymbol{0}}\boldsymbol{x}(t)+\boldsymbol{c}_\omega\boldsymbol{A}_\omega\boldsymbol{b}_\omega u(t)=\underbrace{kJ^{-1}L_a^{-1}}_{b_0}u(t)\end{aligned}$$

which results in the IOD as

(9-36) $$\ddot{y}_\omega(t)+a_1\dot{y}_\omega(t)+a_0y_\omega(t)=b_0u(t)$$

Most importantly, a different SSD to (9-21) and (9-22) can be directly obtained from (9-36) and expressed as

(9-37) $$\dot{\boldsymbol{z}}(t)=\boldsymbol{A}\boldsymbol{z}(t)+\boldsymbol{b}u(t)$$

(9-38) $$y(t)=\boldsymbol{c}\boldsymbol{z}(t)$$

where $\boldsymbol{z}(t)=\begin{bmatrix}z_1(t)\\z_2(t)\end{bmatrix}$, $\boldsymbol{A}=\begin{bmatrix}0&1\\-a_0&-a_1\end{bmatrix}$, $\boldsymbol{b}=\begin{bmatrix}0\\1\end{bmatrix}$ and $\boldsymbol{c}=\begin{bmatrix}b_0&0\end{bmatrix}$.

The velocity control is a regulation problem, which is required to drive $y_\omega(t)=\omega(t)=x_2(t)$ to a desired velocity $y_{\omega d}=\omega_d$. Hence, we have to define the error function

(9-39) $$e_\omega(t)=y_\omega(t)-y_{\omega d}=\boldsymbol{c}_\omega\boldsymbol{x}_\omega(t)-y_{\omega d}$$

and genetate a new state variable

(9-40) $$z_\omega(t)=\int_{t_0}^{t}e_\omega(\tau)d\tau=\int_{t_0}^{t}\left(\boldsymbol{c}_\omega\boldsymbol{x}_\omega(t)-y_{\omega d}\right)d\tau$$

which is equivalent to a new state equation shown as

(9-41) $$\dot{z}_\omega(t)=e_\omega(t)=\boldsymbol{c}_\omega\boldsymbol{x}_\omega(t)-y_{\omega d}$$

Now, an augmented system can be formed by (9-30) and (9-41), given as

(9-42) $$\underbrace{\begin{bmatrix}\dot{\boldsymbol{x}}_\omega(t)\\\dot{z}_\omega(t)\end{bmatrix}}_{\dot{\boldsymbol{x}}_{aug}(t)}=\underbrace{\begin{bmatrix}\boldsymbol{A}_\omega&0\\\boldsymbol{c}_\omega&0\end{bmatrix}}_{\boldsymbol{A}_{aug}}\cdot\underbrace{\begin{bmatrix}\boldsymbol{x}_\omega(t)\\z_\omega(t)\end{bmatrix}}_{\boldsymbol{x}_{aug}(t)}+\underbrace{\begin{bmatrix}\boldsymbol{b}_\omega\\0\end{bmatrix}}_{\boldsymbol{b}_{aug}}u(t)+\begin{bmatrix}\boldsymbol{0}\\-1\end{bmatrix}y_{\omega d}$$

It can be proved that

(9-43) $rank\begin{bmatrix} \boldsymbol{b}_{aug} & \boldsymbol{A}_{aug}\boldsymbol{b}_{aug} & \boldsymbol{A}_{aug}^2\boldsymbol{b}_{aug} \end{bmatrix} = 3$

i.e., the augmented system is also controllable. Next, two cases are discussed, depending on all the state variables are measurable or not.

Case-I: $x_1(t) = i_a(t)$ and $x_2(t) = \omega(t)$ measurable

In this case, we generate a new state $z_\omega(t)$ shown in (9-40) and employ the augmented system in (9-42). From (9-43), we know that it is a controllable system and thus, we can use the state feedback control

(9-44) $u = -\boldsymbol{k}_{aug}\boldsymbol{x}_{aug}(t) = -\boldsymbol{k}_{aug}\begin{bmatrix} \boldsymbol{x}_\omega(t) \\ z_\omega(t) \end{bmatrix}$

and change the augmented system into

(9-45) $\dot{\boldsymbol{x}}_{aug}(t) = \left(\boldsymbol{A}_{aug} - \boldsymbol{b}_{aug}\boldsymbol{k}_{aug}\right)\boldsymbol{x}_{aug}(t)$

Based on the pole- placement method, the eigenvalues of $\boldsymbol{A}_{aug} - \boldsymbol{b}_{aug}\boldsymbol{k}_{aug}$ are assigned as η_i, i=1,2,3, all located in the left-half complex plane, i.e., $Re(\eta_i) < 0$, to stabilize the system. Besides, $\boldsymbol{k}_{aug}$ can be uniquely determined from the following characteristic polynomial

(9-46) $\left|\lambda\boldsymbol{I} - \left(\boldsymbol{A}_{aug} - \boldsymbol{b}_{aug}\boldsymbol{k}_{aug}\right)\right| = (\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \eta_3)$

As a result, $\dot{\boldsymbol{x}}_{aug}(t)\Big|_{t\to\infty} = \begin{bmatrix} \dot{\boldsymbol{x}}_\omega(t) \\ \dot{z}_\omega(t) \end{bmatrix}_{t\to\infty} \to \boldsymbol{0}$, which guarantees that

(9-47) $\dot{z}_\omega(t)\Big|_{t\to\infty} = y_\omega(t)\Big|_{t\to\infty} - y_{\omega d} = 0$

Clearly, the control goal $y_\omega(t)\Big|_{t\to\infty} = y_{\omega d}$ is achieved.

Case-II: only $y_\omega(t) = \omega(t) = x_2(t)$ measurable

In this case, state variables are not all measurable, so we have to create an observer for (9-30) and (9-31), shown as below:

(9-48) $\dot{\hat{\boldsymbol{x}}}_\omega(t) = \boldsymbol{A}_\omega\hat{\boldsymbol{x}}_\omega(t) + \boldsymbol{b}_\omega u(t) + \boldsymbol{L}\left(y_\omega(t) - \hat{y}_\omega(t)\right)$

(9-49) $\hat{y}_\omega(t) = \boldsymbol{c}_\omega\hat{\boldsymbol{x}}_\omega(t)$

where $\hat{\boldsymbol{x}}_\omega(t)$ is the estimated state of $\boldsymbol{x}_\omega(t)$ and $\hat{y}_\omega(t)$ is the estimated output. Further, define the estimation error as

(9-50) $\tilde{\boldsymbol{x}}_\omega(t)=\boldsymbol{x}_\omega(t)-\hat{\boldsymbol{x}}_\omega(t)$

then from (9-30) and (9-48) we have

(9-51) $\dot{\tilde{\boldsymbol{x}}}_\omega(t)=\boldsymbol{A}_\omega\tilde{\boldsymbol{x}}_\omega(t)-\boldsymbol{L}\boldsymbol{c}_\omega\tilde{\boldsymbol{x}}_\omega(t)=\left(\boldsymbol{A}_\omega-\boldsymbol{L}\boldsymbol{c}_\omega\right)\tilde{\boldsymbol{x}}_\omega(t)$

Again, based on pole-placement method the gain $\boldsymbol{L}$ is determined by choosing the eigenvalues of $\boldsymbol{A}_\omega-\boldsymbol{L}\boldsymbol{c}_\omega$ all located in the left-half complex plane. As a result, the estimation error will approach zero as $t\to\infty$, i.e.,

(9-52) $\tilde{\boldsymbol{x}}_\omega(t)\big|_{t\to\infty}=\boldsymbol{x}_\omega(t)-\hat{\boldsymbol{x}}_\omega(t)\big|_{t\to\infty}\to\boldsymbol{0}$

In other words, the estimated state $\hat{\boldsymbol{x}}_\omega(t)$ is gradually approximated to the actual state $\boldsymbol{x}_\omega(t)$ as time increases. Therefore, the augmented system in (9-42) should be replaced by

(9-53) $$\underbrace{\begin{bmatrix}\dot{\hat{\boldsymbol{x}}}_\omega(t)\\ \dot{z}_\omega(t)\end{bmatrix}}_{\dot{\hat{\boldsymbol{x}}}_{aug}(t)}=\underbrace{\begin{bmatrix}\boldsymbol{A}_\omega & 0\\ \boldsymbol{c}_\omega & 0\end{bmatrix}}_{\boldsymbol{A}_{aug}}\cdot\underbrace{\begin{bmatrix}\hat{\boldsymbol{x}}_\omega(t)\\ z_\omega(t)\end{bmatrix}}_{\hat{\boldsymbol{x}}_{aug}(t)}+\underset{\boldsymbol{b}_{aug}}{\begin{bmatrix}\boldsymbol{b}_\omega\\ 0\end{bmatrix}}u(t)+\begin{bmatrix}\boldsymbol{0}\\ -1\end{bmatrix}y_{\omega d}$$

and the state feedback control (9-44) is changed into

(9-54) $$u=-\boldsymbol{k}_{aug}\hat{\boldsymbol{x}}_{aug}(t)=-\boldsymbol{k}_{aug}\begin{bmatrix}\hat{\boldsymbol{x}}_\omega(t)\\ z_\omega(t)\end{bmatrix}$$

which indubitably achieve the control goal $y_\omega(t)\big|_{t\to\infty}=y_{\omega d}$.

In fact, we also can create an observer for (9-37) and (9-38), which is derived from (9-36), shown as below:

(9-55) $\dot{\hat{\boldsymbol{z}}}(t)=\boldsymbol{A}\hat{\boldsymbol{z}}(t)+\boldsymbol{b}u(t)+\boldsymbol{L}\left(y(t)-\hat{y}(t)\right)$

(9-56) $\hat{y}(t)=\boldsymbol{c}\hat{\boldsymbol{z}}(t)$

where $\hat{\boldsymbol{z}}(t)$ is the estimated state of $\boldsymbol{z}(t)$ and $\hat{y}(t)$ is the estimated output. Obviously, the state feedback control (9-44) is changed into

(9-57) $$u=-\boldsymbol{k}_{aug}\begin{bmatrix}\hat{\boldsymbol{z}}(t)\\ z_\omega(t)\end{bmatrix}$$

which still can achieve the control goal $y_\omega(t)\big|_{t\to\infty}=y_{\omega d}$.

➢ Position Control

For the position control, we employ the modeling in (9-28) and (9-29), shown as

(9-58) $\dot{\boldsymbol{x}}_\theta(t) = \boldsymbol{A}_\theta \boldsymbol{x}_\theta(t) + \boldsymbol{b}_\theta u(t)$

(9-59) $y_\theta(t) = \boldsymbol{c}_\theta \boldsymbol{x}_\theta(t)$

where $\boldsymbol{x}_\theta(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}^T$, $\boldsymbol{A}_\theta = \begin{bmatrix} -R_a/L_a & -k/L_a & 0 \\ k/J & -B_M/J & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{b}_\theta = \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{c}_\theta = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

It is easy to check that

(9-60) $rank\begin{bmatrix} \boldsymbol{b}_\theta & \boldsymbol{A}_\theta \boldsymbol{b}_\theta & \boldsymbol{A}_\theta^2 \boldsymbol{b}_\theta \end{bmatrix} = rank\begin{bmatrix} L_a^{-1} & -R_a L_a^{-2} & R_a^2 L_a^{-3} - k^2 J^{-1} L_a^{-2} \\ 0 & kJ^{-1}L_a^{-1} & -kR_a J^{-1} L_a^{-2} - B_M k J^{-2} L_a^{-1} \\ 0 & 0 & kJ^{-1}L_a^{-1} \end{bmatrix} = 3$

i.e., (9-58) is controllable.

The goal of position control is to drive $y_\theta(t) = \theta(t)$ to a desired velocity $y_{\theta d} = \theta_d$.

Clearly, this is also a regulation problem and we have to define the error function as

(9-61) $e_\theta(t) = y_\theta(t) - y_{\theta d} = \boldsymbol{c}_\theta \boldsymbol{x}_\theta(t) - y_{\theta d}$

and genetate a new state as below:

(9-62) $z_\theta(t) = \int_{t_0}^{t} e_\theta(\tau)d\tau = \int_{t_0}^{t} \left(\boldsymbol{c}_\theta \boldsymbol{x}_\theta(t) - y_{\theta d}\right)d\tau$

which implies

(9-63) $\dot{z}_\theta(t) = e_\theta(t) = \boldsymbol{c}_\theta \boldsymbol{x}_\theta(t) - y_{\theta d}$

Combining with (9-58) yields the following augmented system

(9-64) $\underbrace{\begin{bmatrix} \dot{\boldsymbol{x}}_\theta(t) \\ \dot{z}_\theta(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}_{aug}(t)} = \underbrace{\begin{bmatrix} \boldsymbol{A}_\theta & 0 \\ \boldsymbol{c}_\theta & 0 \end{bmatrix}}_{\boldsymbol{A}_{aug}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{x}_\theta(t) \\ z_\theta(t) \end{bmatrix}}_{\boldsymbol{x}_{aug}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{b}_\theta \\ 0 \end{bmatrix}}_{\boldsymbol{b}_{aug}} u(t) + \begin{bmatrix} \boldsymbol{0} \\ -1 \end{bmatrix} y_{\theta d}$

It can be proved that

(9-65) $rank\begin{bmatrix} \boldsymbol{b}_{aug} & \boldsymbol{A}_{aug}\boldsymbol{b}_{aug} & \boldsymbol{A}_{aug}^2\boldsymbol{b}_{aug} & \boldsymbol{A}_{aug}^3\boldsymbol{b}_{aug} \end{bmatrix} = 4$

i.e., the augmented system is still controllable. Here, we only consider all the state variables are measurable and thus, we can use the state feedback

(9-66) $u = -\boldsymbol{k}_{aug}\boldsymbol{x}_{aug}(t) = -\boldsymbol{k}_{aug}\begin{bmatrix} \boldsymbol{x}_\theta(t) \\ z_\theta(t) \end{bmatrix}$

and change the augmented system into

(9-67) $\dot{\boldsymbol{x}}_{aug}(t) = \left(\boldsymbol{A}_{aug} - \boldsymbol{b}_{aug}\boldsymbol{k}_{aug}\right)\boldsymbol{x}_{aug}(t)$

Based on the pole- placement method, the eigenvalues of $\boldsymbol{A}_{aug} - \boldsymbol{b}_{aug}\boldsymbol{k}_{aug}$ are assigned as η_i, i=1,2,3,4, all located in the left-half complex plane, i.e., $Re(\eta_i) < 0$, such that the system is stabilized. Besides, $\boldsymbol{k}_{aug}$ can be uniquely determined from the following characteristic polynomial

(9-68) $\left|\lambda \boldsymbol{I} - \left(\boldsymbol{A}_{aug} - \boldsymbol{b}_{aug}\boldsymbol{k}_{aug}\right)\right| = (\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \eta_3)(\lambda - \eta_4)$

As a result, $\dot{\boldsymbol{x}}_{aug}(t)\big|_{t\to\infty} = \begin{bmatrix} \dot{\boldsymbol{x}}_\theta(t) \\ \dot{z}_\theta(t) \end{bmatrix}_{t\to\infty} \to \boldsymbol{0}$, which guarantees that

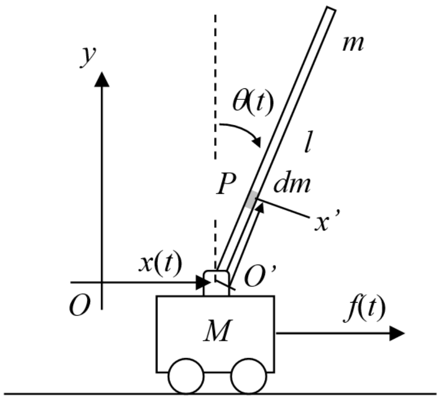
(9-69) $\dot{z}_\theta(t)\big|_{t\to\infty} = y_\theta(t)\big|_{t\to\infty} - y_{\theta d} = 0$

Clearly, the control goal $y_\theta(t)\big|_{t\to\infty} = y_{\theta d}$ is achieved.

10. Position Control of an Inverted Pendulum

➢ Modeling of Invereted pendulum

In control engineering, the inverted pendulum depicted in the figure is the most common system used to demonstrate control theories. Here, let's introduce the mathematical modeling of the inverted pendulum by the so-called Lagrange's method.



Let $x(t)$ be the position of the cart moving on the horizontal surface, $\theta(t)$ be the angular position of the pendulum deviated from the vertical angle, and $f(t)$ be the input force. Assume *m* and *l* are the mass and the length of the uniform pendulum and *M* is the mass of the cart. The use of Lagrange's first defines the lagrangian as

(10-1) $L = T - U$

where *T* is the kinetic energy and *U* is the potential energy of the system. To find the kinetic energy and potential energy, let's define the inertial coordinate (x, y) with origin *O* and the coordinate (x', θ) with origin *O'* at the end of the pendulum connected to the cart. The total kinetic energy *T* is

(10-2) $T = T_M + T_m = \frac{1}{2}M\dot{x}^2(t) + \frac{1}{2}\int_m v_p^2(t)dm$

whete $T_M = \frac{1}{2}M\dot{x}^2(t)$ is the kinetic energy of the cart and T_m is the kinetic energy of the pendulum. The position *P* is $\left(x(t) + x'sin\theta(t),\, x'cos\theta(t)\right)$ and then the velocity of dm at *P* is $v_p(t) = \sqrt{\left(\dot{x}(t) + x'cos\theta(t)\dot{\theta}(t)\right)^2 + \left(x'sin\theta(t)\dot{\theta}(t)\right)^2}$, i.e., (10-3) $v_p^2(t) = \dot{x}^2(t) + 2\dot{x}(t)x'cos\theta(t)\dot{\theta}(t) + x'^2\dot{\theta}^2(t)$

Since $dm = \frac{m}{l}dx'$, we have

(10-4) $T_m = \frac{m}{2l}\int_0^l \left(\dot{x}^2(t) + 2\dot{x}(t)x'cos\theta(t)\dot{\theta}(t) + x'^2\dot{\theta}^2(t)\right)dx'$

$$= \frac{m}{2}\dot{x}^2(t) + \frac{ml}{2}\dot{x}(t)cos\theta(t)\dot{\theta}(t) + \frac{ml^2}{6}\dot{\theta}^2(t)$$

Hence, the total kinetic energy is

(10-5) $T = \frac{1}{2}(M+m)\dot{x}^2(t) + \frac{ml}{2}cos\theta(t)\dot{x}(t)\dot{\theta}(t) + \frac{ml^2}{6}\dot{\theta}^2(t)$

Let the reference level of the potential energy be the horizontal axis of inertial coordinate, then the total potential energy is

(10-6) $U = U_M + U_m = 0 + mg\frac{l}{2}cos\theta(t)$

where the potential energy of the cart is $U_M = 0$ and the potential energy of the pendulum is $U_m = mg\frac{l}{2}cos\theta(t)$. It is obvious that the Lagrangian in (10-1) is

(10-7) $L = \frac{1}{2}(M+m)\dot{x}^2(t) + \frac{ml}{2}cos\theta(t)\dot{x}(t)\dot{\theta}(t) + \frac{ml^2}{6}\dot{\theta}^2(t) - mg\frac{l}{2}cos\theta(t)$

The Lagrange's formula is shown as

(10-8) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i(t)}\right) - \frac{\partial L}{\partial q_i(t)} = Q_i(t)$

where $q_i(t)$ is the generalized coordinate and $Q_i(t)$ is the generalized force or torque applied to the coordinate $q_i(t)$. From the Lagrangian L in (10-7), it is clear that the generalized force applied to $x(t)$ is $f(t)$ and no generalized torque applied to the coordinate $\theta(t)$. Therefore, the Lagrange's equations in (10-8) are

(10-9) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}(t)}\right) - \frac{\partial L}{\partial x(t)} = f(t)$

(10-10) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}(t)}\right) - \frac{\partial L}{\partial \theta(t)} = 0$

where

(10-11) $\frac{\partial L}{\partial x(t)} = 0$

(10-12) $\dfrac{\partial L}{\partial \dot{x}(t)} = (M+m)\dot{x}(t) + \dfrac{ml}{2} cos\theta(t)\dot{\theta}(t)$

(10-13) $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}(t)}\right) = (M+m)\ddot{x}(t) + \dfrac{ml}{2} cos\theta(t)\ddot{\theta}(t) - \dfrac{ml}{2} sin\theta(t)\dot{\theta}^2(t)$

(10-14) $\dfrac{\partial L}{\partial \theta(t)} = -\dfrac{ml}{2} sin\theta(t)\dot{x}(t)\dot{\theta}(t) + mg\dfrac{l}{2} sin\theta(t)$

(10-15) $\dfrac{\partial L}{\partial \dot{\theta}(t)} = \dfrac{ml}{2} cos\theta(t)\dot{x}(t) + \dfrac{ml^2}{3}\dot{\theta}(t)$

(10-16) $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}(t)}\right) = -\dfrac{ml}{2} sin\theta(t)\dot{x}(t)\dot{\theta}(t) + \dfrac{ml}{2} cos\theta(t)\ddot{x}(t) + \dfrac{ml^2}{3}\ddot{\theta}(t)$

Hence, we have

(10-17) $(M+m)\ddot{x}(t) + \dfrac{ml}{2} cos\theta(t)\ddot{\theta}(t) - \dfrac{ml}{2} sin\theta(t)\dot{\theta}^2(t) = f(t)$

(10-18) $\dfrac{ml}{2} cos\theta(t)\ddot{x}(t) + \dfrac{ml^2}{3}\ddot{\theta}(t) - mg\dfrac{l}{2} sin\theta(t) = 0$

which is a nonlinear system. The poisition control of inverted pendulum is to keep the angular position vertically, i.e., $\theta(t)=0$, and simultaneously drive the cart from any initial position back to $x(t)=0$.

In order to use the state-feedback control, it is required to linearize the nonlinear model of inverted pendulum.

- **Linearization**

For the control purpose $\theta(t)=0$, we often put the pendulum in vertical initially and assume $\theta(t)\approx 0$ in the whole control process, which implies we can use the approximation of $cos\theta(t)\approx 1$, $sin\theta(t)\approx\theta(t)$ and $\dot{\theta}^2(t)\approx 0$. Hence, (10-17) and (10-18) can be linearized as below:

(10-19) $(M+m)\ddot{x}(t) + \dfrac{ml}{2}\ddot{\theta}(t) = f(t)$

(10-20) $\dfrac{ml}{2}\ddot{x}(t) + \dfrac{ml^2}{3}\ddot{\theta}(t) - mg\dfrac{l}{2}\theta(t) = 0$

or in the following matrix form

(10-21) $$\begin{bmatrix} M+m & \frac{ml}{2} \\ \frac{ml}{2} & \frac{ml^2}{3} \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix}=\begin{bmatrix} f(t) \\ mg\frac{l}{2}\theta(t) \end{bmatrix}$$

Since $\Delta=\frac{ml^2}{3}(M+m)-\left(\frac{ml}{2}\right)^2>0$, we know that $\begin{bmatrix} M+m & \frac{ml}{2} \\ \frac{ml}{2} & \frac{ml^2}{3} \end{bmatrix}$ is invertible and (10-21) can be written as

(10-22) $$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix}=\begin{bmatrix} M+m & \frac{ml}{2} \\ \frac{ml}{2} & \frac{ml^2}{3} \end{bmatrix}^{-1}\begin{bmatrix} f(t) \\ mg\frac{l}{2}\theta(t) \end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix} \frac{ml^2}{3} & -\frac{ml}{2} \\ -\frac{ml}{2} & M+m \end{bmatrix}\begin{bmatrix} f(t) \\ mg\frac{l}{2}\theta(t) \end{bmatrix}$$

Hence, we obtain

(10-23) $$\ddot{x}(t)=-\frac{m^2l^2g}{4\Delta}\theta(t)+\frac{ml^2}{3\Delta}f(t)$$

(10-24) $$\ddot{\theta}(t)=\frac{(M+m)mlg}{2\Delta}\theta(t)-\frac{ml}{2\Delta}f(t)$$

The truth of (10-24) implies that $\theta(t)$ can be directly controlled to vertical position by the so-called PD control $f(t)=k_1\dot{\theta}(t)+k_0\theta(t)$ without considering $x(t)$. Unfortunately, from (10-23) the use of PD control leads to $\ddot{x}(t)=0$, which means the cart will move in the way of $x(t)=at+b$, not back to $x(t)=0$.

To fulfill the poisition control of $\theta(t)=0$ and $x(t)=0$, we will derive the state equation for (10-23) and (10-24) and design the state-feedback control. Now, let's choose the input $u(t)=f(t)$ and state variables $x_1(t)=x(t)$, $x_2(t)=\theta(t)$, $x_3(t)=\dot{x}(t)$ and $x_4(t)=\dot{\theta}(t)$. Hence, the state equation can be obtained as

(10-25) $$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)}=\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m^2l^2g}{4\Delta} & 0 & 0 \\ 0 & \frac{(M+m)mlg}{2\Delta} & 0 & 0 \end{bmatrix}}_{\boldsymbol{A}}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)}+\underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{ml^2}{3\Delta} \\ -\frac{ml}{2\Delta} \end{bmatrix}}_{\boldsymbol{b}}u(t)$$

➢ State-feedback Control

It can be obtained that

(10-26) $rank\begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} & \boldsymbol{A}^2\boldsymbol{b} & \boldsymbol{A}^3\boldsymbol{b} \end{bmatrix}$

$$= rank\begin{bmatrix} 0 & \dfrac{ml^2}{3\Delta} & 0 & \dfrac{m^3l^3g}{8\Delta^2} \\ 0 & -\dfrac{ml}{2\Delta} & 0 & -\dfrac{(M+m)m^2l^2g}{4\Delta^2} \\ \dfrac{ml^2}{3\Delta} & 0 & \dfrac{m^3l^3g}{8\Delta^2} & 0 \\ -\dfrac{ml}{2\Delta} & 0 & -\dfrac{(M+m)m^2l^2g}{4\Delta^2} & 0 \end{bmatrix} = 4$$

i.e., (10-25) is controllable. Here, we assume all the state variables are measurable and thus, we can use the state feedback control

(10-27) $u = -\boldsymbol{kx}(t)$

and change the system into

(10-28) $\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{bk})\boldsymbol{x}(t)$

Based on the pole- placement method, the eigenvalues of $\boldsymbol{A} - \boldsymbol{bk}$ are assigned as η_i, i=1,2,3,4, all located in the left-half complex plane, i.e., $Re(\eta_i) < 0$, such that the system is stabilized. Besides, $\boldsymbol{k}$ can be uniquely determined from the following characteristic polynomial

(10-29) $\left|\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{bk})\right| = (\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \eta_3)(\lambda - \eta_4)$

Hence, $\boldsymbol{x}(t)\big|_{t\to\infty} \to \boldsymbol{0}$, which guarantees $x_1(t) = x(t) \to 0$ and $x_2(t) = \theta(t) \to 0$, i.e., the control goal is achieved.