Ch1 Vector Spaces

1-2 Vector Spaces

Def. Let V be a set, F be a field

$+$: 向量加向量

$\cdot$: scalar multiplication

V is a vector space (over a field F) if:

(VS-1) $x+y \in V$, whenever $x, y \in V$ (加法有封閉性)

(VS0) $\alpha x \in V$ whenever $\alpha \in F$, $x \in V$ (純量乘法有封閉性)

(VS1) $x+y=y+x$, $\forall x, y \in V$ (可交換)

(VS2) $(x+y)+z = x+(y+z)$, $\forall x, y, z \in V$ (可結合)

(VS3) $\exists 0 \in V$ s.t. $0+x=x$, $\forall x \in V$ (存在加法單位元素)

(VS4) $\exists y \in V$ s.t. $x+y=0$, $\forall x \in V$ (存在加法反元素)

(VS5) $\exists 1 \in F$ s.t. $1x=x$, $\forall x \in V$ (存在 scalar multiplication 的單位元素)

(VS6) $a(bx)=(ab)x$, $\forall x \in V$, $a, b \in F$.

(VS7) $a(x+y)=ax+ay$, $\forall x, y \in V$, $a \in F$

(VS8) $(a+b)x=ax+bx$, $\forall x \in V$, $a, b \in F$.

ex. $V=\{$平面的向量$\}=\{(a,b) : a, b \in \mathbb{R}\}$, $F=\mathbb{R}$

加法: $x=(x_1, y_1)$, $y=(x_2, y_2)$, $x+y=(x_1+x_2, y_1+y_2)$

係數積: $\alpha x=(\alpha x_1, \alpha x_2)$

Q：把ex1中的F改成$\mathbb{C}$，is V a vector space over F?

A：不会，$\because$ VS0中，$\alpha x \notin V$ $\therefore$ V is not a vector space over F.

Q：那改成 $\mathbb{Q}$ 呢

A：Yes

ex. $V=\{(a,b): a,b\in\mathbb{R}\}$, $F=\mathbb{R}$

加法：$x+y=(x_1+y_1, x_2-y_2)$

係数積：$\alpha x=(\alpha x_1, \alpha x_2)$

$\Rightarrow$ 不满足VS1

$\Rightarrow$ V is not a vector space over F.

ex. $V=\left\{\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}: a_i\in\mathbb{R}, 1\le i\le n\right\}$

$F=\mathbb{R}$

$+: a,b\in V$, $a+b=\begin{pmatrix}a_1+b_1\\ \vdots\\ a_n+b_n\end{pmatrix}$, where $b=\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$.

$\cdot: \alpha\in\mathbb{R}, a\in V$, $\alpha a=\begin{pmatrix}\alpha a_1\\ \vdots\\ \alpha a_n\end{pmatrix}$

Is V a vector space over $\mathbb{R}$?

-sol-. 一个一个代$\Rightarrow$是对的

Notation. $V\triangleq F^n$

ex. 是非题：

I am OKAY.

$\mathbb{C}^n$ is a vector space over $\mathbb{C}$ (O)
$\mathbb{R}^n$ $\mathbb{R}$ (O)
$\mathbb{Q}^n$ $\mathbb{Q}$ (O)
$\mathbb{R}^n$ $\mathbb{C}$ (X)
$\mathbb{R}^n$ $\mathbb{Q}$ (O)

ex. $V=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} : a_{ij}\in F,\ 1\le i,j\le n\right\}$

$+$: $A=(a_{ij})$, $B=(b_{ij})\in V$. $A+B=(a_{ij}+b_{ij})$
$\cdot$: $\alpha A=(\alpha a_{ij})$
$\Rightarrow$ V is a vector space over F.

ex. $V=\{f : f: S\to F\}$, S is a non-empty set, $F=F$
$+$: $f+g=h$, $h(s)=f(s)+g(s)$, $\forall s\in S$
$\cdot$: $\alpha\in F, f\in V$, $(\alpha f)(s)=\alpha(f(s))$
$\Rightarrow$ $V=F(S,F)$ (由S打到F的函数集合)
is a vector space

ex. $V=\{(a_1,a_2) : a_1, a_2\in\mathbb{R}\}$
$F=\mathbb{R}$
$+$: $(a_1,a_2)+(b_1,b_2)=(a_1+b_1,0)$
$\cdot$: $\alpha(a_1,a_2)=(\alpha a_1,0)$
$\because$ $\forall\alpha\in F$, $\alpha(a_1,a_2)\neq(\alpha a_1,0)$ for $\alpha_2\neq 0$
$\therefore$ VS5 is not satisfied
$\Rightarrow$ V is not a vector space over F.

ex. V, F : 同上假設,

$+ : (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

$\cdot : \alpha(a_1, a_2) = \begin{cases} (0,0) & \text{if } \alpha = 0 \\ (\alpha a_1, \frac{a_2}{\alpha}) & \text{if } \alpha \neq 0 \end{cases}$

$\because$ VS8 is not satisfied

$\therefore$ V is not a vector space.

Thm 1.1 Let V be a vector space over a field F. Let $x, y, z \in V$, if $x+y = y+z$, then $x = y$ (消去律成立)

pf. $\because$ by VS3, VS4, given $z \in V$, $\exists\, v \in V$ s.t. $z + v = 0$

$\therefore$ $x = x + 0 = x + (z + v) \overset{VS2}{=} (x + z) + v$

$= (y + z) + v$, (由已知)

$= y + (z + v)$ (by VS2)

$= y + 0$

$= y$ (by VS3)

Q.E.D.

Cor. (1) 0 is unique

(2) 加法反元素 is unique

Remark. Write such element as $-x$

pf. (1) Let 0 and $0'$ be s.t.

$\begin{cases} x + 0 = x \\ x + 0' = x \end{cases}$

I am OKAY.

for any $x \in V$

$\Rightarrow$ '$x+v = x+0$'

$\Rightarrow$ by Thm 1.1, $v = 0'$

(2). Suppose y and y' are s.t.

$$\begin{cases} x+y=v \\ x+y'=v \end{cases}$$

$\Rightarrow x+y = x+y'$

$\Rightarrow$ by Thm 1.1, $y = y'$

Q.E.D.

Thm 1.2. Let V be a vector space, then

(1) $0x = 0$, $\forall x \in V$

(2) $-(ax) = (-a)x = a(-x)$

(3) $a0 = 0$, $\forall a \in F$, $0 \in V$

pf. (1) $\because 0x = (0+0)x = 0x+0x$ by VS8

and $0x = 0+0x$ by VS3

$\therefore$ by Thm 1.1, $0x = 0$.

(2) $\because ax+(-a)x = (a+(-a))x$ by VS8

$= 0 \cdot x$

$= 0$ by (1)

$\therefore (-a)x = -(ax)$

$\because (-1)x = -x$

$\therefore a(-x) = a((-1)x) = (a(-1))x$ by VS6

$= (-a)x$

$\Rightarrow -(ax) = (-a)x = a(-x)$

(3) The proof is similar to (1)

Q.E.D.

I am OKAY.

§1-3 Subspaces

Def. Let V be a vector space over a field F and $W \subset V$. Then W is a subspaces of V if W is a vector space over F with the same properties

(1) $x+y \in W$, $\forall x, y \in W$

(2) $\alpha x \in W$, $\forall x \in W$, $\alpha \in F$

(3) $0 \in W$

(4) $\forall x \in W$, $(-x) \in W$

Thm 1.3 Let V be a vector space, W be a subset of V. Then W is a subspace

$\Longleftrightarrow$ (1) $0 \in W$

(2) $x+y \in W$, $\forall x, y \in W$

(3) $\alpha x \in W$, $\forall \alpha \in F$, $x \in W$

Remark. In fact, $W \neq \emptyset$, then (1) is not even needed.

pf. $\Rightarrow$) If (1)(2)(3) hold, then (4) holds, trivial

$\Leftarrow$) We want $\forall x \in W \Rightarrow \exists y \in W$ s.t. $x+y=0$

$\forall x \in W \subset V$, $\exists -x \in V$ with $x+(-x)=0$

$\Rightarrow$ by (3), $-x=(-1)x \in W$

Q.E.D.

ex. $V = P(F) \triangleq \{a_n x^n + \cdots + a_1 x + a_0 : n \in \mathbb{N}, a_i \in F, i = 0, 1, \ldots, n\}$

$W = P_n(F) = \{a_n x^n + \cdots + a_0 : n \text{ is fixed}, a_i \in F, i = 0, 1, \ldots, n\}$

$\Rightarrow$. W is a subspace of V.

ex. $M_{n\times m}(F) \triangleq \{(a_{ij})_{n\times m} : a_{ij} \in F\}$

$V = M_{n\times n}(F)$

$W = \{(a_{ij})_{n\times n} : a_{ii} = 0, i = 1, 2, \ldots, n\}$

$\Rightarrow$ W is a subspace of V.

ex. $W = \{(a_{ij})_{n\times n} : a_{ij} = 0, i \neq j\}$ = The set of diagonal matrices

$\Rightarrow$ W is a subspace of $V = M_{n\times n}(F)$

ex. $W = \{(a_{ij})_{n\times n} : a_{ij} \geq 0, \text{ for all } i, j\}$

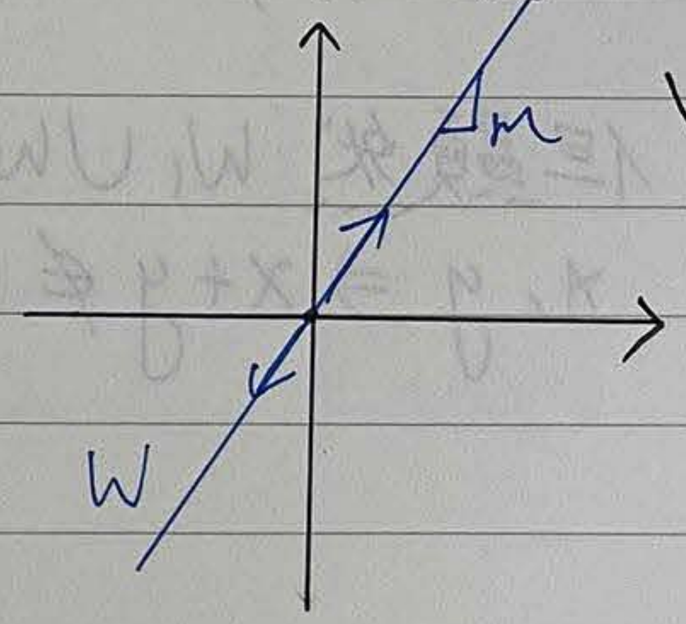
$\left(\begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix} \notin W\right)$

$\Rightarrow$ W is not a subspace of V since

$(-1)\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix} \notin W$.

ex. $V = \mathbb{R}^2$, $W = \{(x, mx) : x \in \mathbb{R}, m \text{ is fixed}\}$

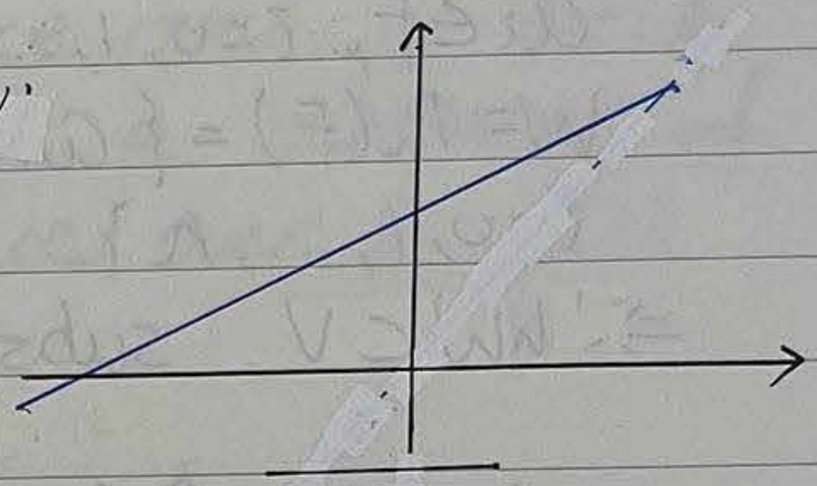
$\Rightarrow$ W is a subspace of $\mathbb{R}^2$



I am OKAY.

ex. $V=\mathbb{R}^2$, $W=\{(x, mx+b) : x\in\mathbb{R}, m \text{ is fixed}, b\neq 0\}$

$\Rightarrow$ W is not a subspace of V



Thm 1.4. W_α: Subspaces of V, $\alpha\in\Lambda$, V: a vector space $\Rightarrow \bigcap_{\alpha\in\Lambda} W_\alpha$ is a subspace of V.

pf. Let $x, y \in \bigcap_{\alpha\in\Lambda} W_\alpha$, then $x, y \in W_\alpha$ for each $\alpha\in\Lambda$.

$\Rightarrow x+y \in W_\alpha$, $\forall \alpha\in\Lambda$.

$\Rightarrow x+y \in \bigcap_{\alpha\in\Lambda} W_\alpha$.

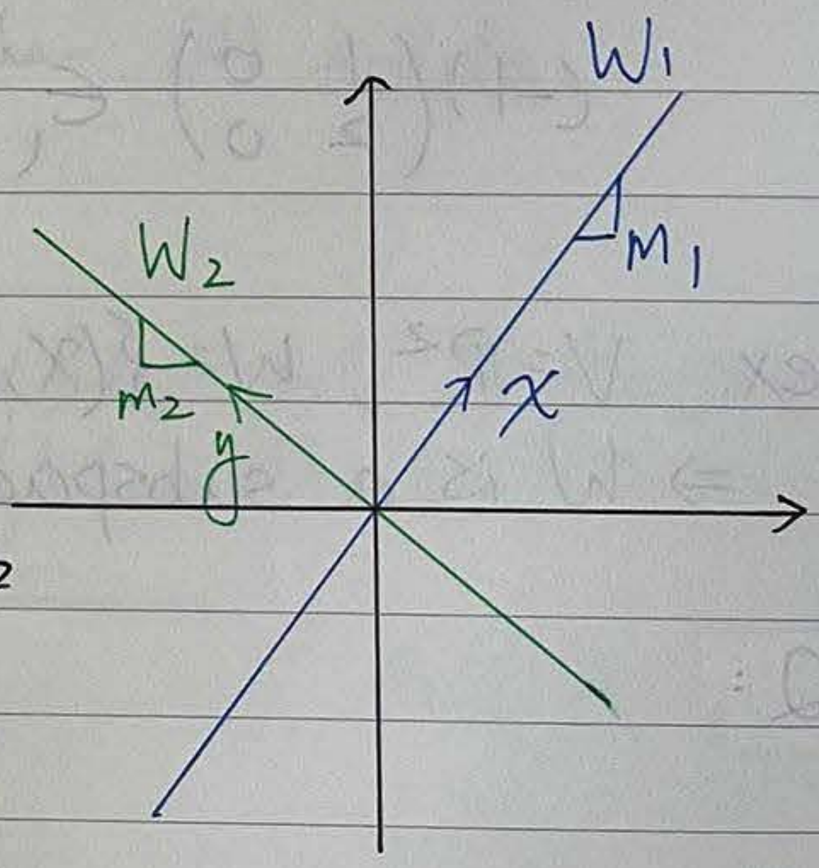
Q.E.D.

Q: 那联集会是 subspace 吗?

ex. $V=\mathbb{R}^2$

$\{(0,0)\} = W_1\cap W_2$. 称 $\{0\}$ 为 trivial subspace of any vector space

但显然 $W_1\cup W_2$ 非子空間, 如 $x, y \Rightarrow x+y \notin W_1\cup W_2$



Remark. $W_1 \cup W_2$ is not necessarily a subspace of V.

Notation. $S_1 + S_2 \triangleq \{x+y : x \in S_1, y \in S_2\}$.

ex. 承上例, $W_1 + W_2 = \mathbb{R}^2$. 事实上2子空間相加会是一个子空間

Fact: Let W_1, W_2 be subspaces of a vector space V

(1) $W_1 + W_2$ is a subspace of V

(2) $W_1 \cap W_2$ "

(3) $W_1 \cup W_2$ is not necessarily a subspaces of V

(4) $W_1 + W_2$ is the "smallest" subspace containing W_1 and W_2
(i.e., if $W \supset W_1 \cup W_2$, and W is a subspace of V, then $W \supset W_1 + W_2$)

(5) $W_1 \cup W_2$ is a subspace of V
$\Leftrightarrow$ $W_1 \subset W_2$ or $W_2 \subset W_1$

I am OKAY.

§1-4 Linear Combinations and Systems

Goal: Let V be a vector space and $S \subset V$, $S \neq \emptyset$. We want to find the smallest subspace containing S

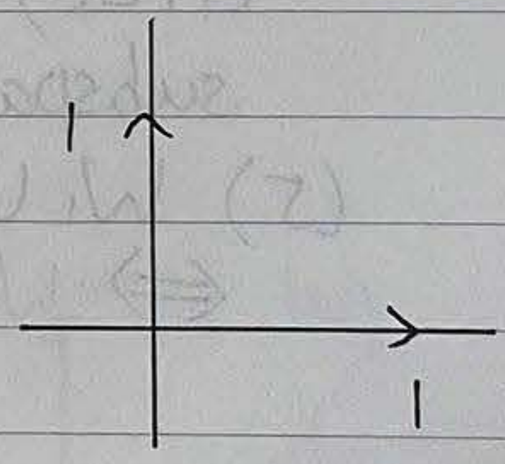
1. 介紹 linear combination, span(S)
2. 給一个向量，此向量是否能由其它向量線性組合出來 ⇒ 解联立方程式

Def. Given a vector $v \in V$, if $\exists\, u_i \in S$, $a_i \in F$, $i = 1, 2, \ldots, n$ s.t. $v = \sum_{i=1}^{n} a_i u_i$, then v is a linear combination of vectors $u_1, \ldots, u_n$ and a_i are called coefficients of such linear combination.
(or we say that v is a l.c. of vectors of S)

ex. $V = \mathbb{R}^2$, $F = \mathbb{R}$, $S = \{(1,0), (0,1)\}$

$(0,0) = 0(1,0) + 0(0,1)$

$(-1,3) = (-1)(1,0) + 3(0,1)$



任意給定但有限(n个)

Def. $\mathrm{Span}(S) = \left\{ \boxed{\sum_{i=1}^{n} a_i u_i} : a_i \in F,\ u_i \in S,\ i = 1, 2, \ldots, n \right\}$

= the set that are l.c. of vectors of S.

ex. $\mathrm{span}\{(1,0), (0,1)\} = \mathbb{R}^2$

of Linear Equatims.

Thm 1.5 (1) Span(S) is a subspace of V.

(2) Span(S) is the smallest subspace containing S. that is, if $S \subset W$ and W is a subspace of V, then $span(S) \subset W$

For convenience, we define $span(\phi) = \{0\}$

pf. (1) Let $x, y \in span(S)$, then $\exists\, a_i, u_i$, $i=1,\dots,n$, $\exists\, b_i, v_i$, $i=1,\dots,m$ with

$$x = \sum_{i=1}^{n} a_i u_i, \quad y = \sum_{i=1}^{n} b_i u_i$$

$$\Rightarrow x+y = \sum_{i=1}^{n} (a_i+b_i) u_i$$

$\because a_i+b_i \in F$

$\therefore x+y \in span(S)$.

Likewise, $\alpha x \in span(S)$, done.

(2) We want to prove that if $x \in span(S)$, then $x \in W$, for any subspace W of V.

$\because x \in span(S)$

$\therefore x = \sum_{i=1}^{n} a_i u_i$

$\because u_i \in S$

$\therefore u_i \in W$

$\therefore W$ is a subspace

$\therefore x \in W$, $\forall x \in span(S)$, done

Q.E.D.

I am OKAY.

ex. $u_1=(1,2,1)$, $u_2=(-2,-4,-2)$, $u_3=(0,2,3)$,
$u_4=(2,0,-3)$, $u_5=(-3,8,16)$
$v=(2,6,8)$
if $x_1u_1+x_2u_2+x_3u_3+x_4u_4+x_5u_5=v$, find $x_1,\dots,x_5$?

-sol-. $\begin{cases} x_1-2x_2+0x_3+2x_4-3x_5=2 \\ \vdots \end{cases}$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2 \\ 2 & -4 & 2 & 0 & 8 & 6 \\ 1 & -2 & 3 & -3 & 16 & 8 \end{array}\right] \Rightarrow \cdots$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1=2x_2-x_5-4 \\ x_3=-3x_5+7 \\ x_4=2x_5+3 \end{cases}, \quad x_2, x_5 \text{ 為任意實數}$$

ex2. $u_1=x^3-2x^2-5x-3$
$u_2=3x^3-5x^2-4x-9$
$\begin{cases} v_1=2x^3-2x^2+12x-6 \\ v_2=3x^3-2x^2+7x+8. \end{cases}$, find l.c. of u_1 and u_2 which is v_1, v_2.
au_1+bu_2

-sol-.
$$\left[\begin{array}{cc|cc} 1 & 3 & 2 & 3 \\ -2 & -5 & -2 & -2 \\ -5 & -4 & 12 & 7 \\ -3 & -9 & -6 & 8 \end{array}\right].$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 2 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 11 & 22 & 22 \\ 0 & 0 & 0 & 17 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 2 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 17 \end{array}\right] \Rightarrow \begin{cases} v_1 \text{ 可用 } u_1, u_2 \text{ 之 l.c. 表示} \\ v_2 \text{ 無法用“} \end{cases}$$

$$\Rightarrow \begin{cases} a+3b=2 \\ b=2 \end{cases} \Rightarrow \begin{cases} a=-4 \\ b=2 \end{cases} \Rightarrow v_1 = -4u_1 + 2u_2.$$

ex.(1) $\text{span}\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}\right\} \overset{?}{=} \mathbb{R}^{2\times 2}$

(2) $S = \text{span}\left\{\begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}\right\} \overset{?}{=} \mathbb{R}^{2\times 2}$

(3) $S = \text{span}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}\right\} \overset{?}{=} \mathbb{R}^{2\times 2}$

-sol- (1) Yes (2) No (3) No 如前述方法可求之.
不过，觀察一下，(2)(3) 对角线都是1，因此形如 $\begin{pmatrix}1&0\\0&2\end{pmatrix}$ 一定組不出來. 因此不必真的算

I am OKAY.

§1.5 Linear Independence and Linear

Goal: Let V be a vector space $W \subset V$, W is a subspace of V. Find the smallest set S s.t. $\text{span}(S) = W$.

Def. (1) A subset $S \subset V$ is called linear dependent if $\exists$ distinct vectors $u_i \in S$ and not all zero coeff. $a_i \in F$ s.t. $\sum a_i u_i = 0$

(2) If S is not linear dependent, then it is called linear independent.
(That is, if $\sum_{i=0}^{n} a_i u_i = 0$, then $a_i = 0$ for all $i = 1, 2, \dots, n$. Here u_i, $i = 1, \dots, n$ are distinct vectors of S).

ex. $V = \mathbb{R}^2$, $S = \{(0,0)\}$
$\because 2(0,0) = (0,0)$
$\therefore S$ is linear dependent.

ex. $S = \{(0,0), (1,0), (0,1)\}$
$\Rightarrow$ In fact, any subset containing the zero vector is linear dependent.

ex. $S = \{(1,0), (0,1)\}$
$\because a(1,0) + b(0,1) = 0$ implies
$\begin{cases} a = 0 \\ b = 0 \end{cases}$
$\therefore S$ is linear independent.

Dependence.

Thm 1.6. Let $S_1 \subseteq S_2 \subset V$,

(1) If S_2 is linear independent, then S_1 is linear independent

(2) If S_1 is linear dependent, then S_2 is linear dependent.

ex. $P_2(\mathbb{R}) = V$, $S = \{1, x\}$

$\because a_0 \cdot 1 + a_1 \cdot x = 0$, $\forall x \in \mathbb{R}$ implies $a_0 = a_1 = 0$

$\therefore$ S is linear independent

ex. $P_2(\mathbb{R}) = V$, $S_1 = \{x, 2x\}$

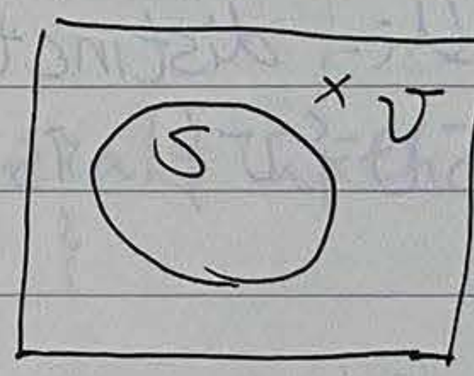
$a_1 x + a_2(2x) = 0$, we can pick $a_1 = 2$, $a_2 = -1$

$\Rightarrow S_1$ is linear dependent

Thm 1.7 $S \subset V$: vector space S is linear independent, $v \in V - S$, then $S \cup \{v\}$ is linear dependent

$\Leftrightarrow$ $v \in \text{span}(S)$

(or equivalently, $v \notin \text{span}(S) \Leftrightarrow S \cup \{v\}$ is linear independent)



ex. $V = \mathbb{R}^2$, $S = \{(1,0)\}$

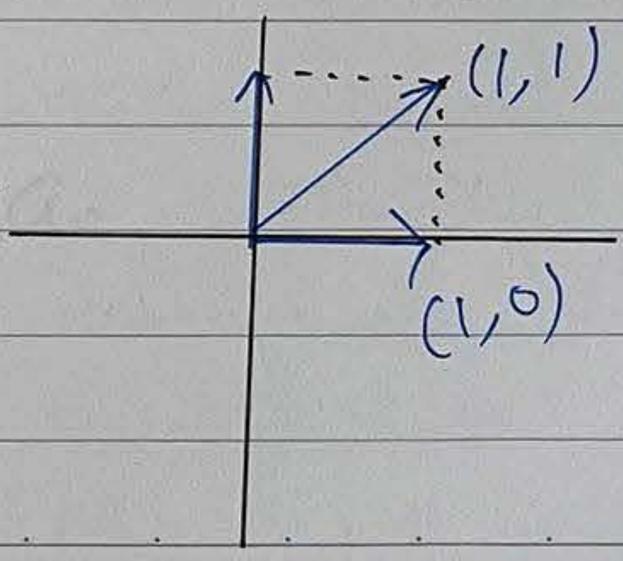
$S \cup \{(1,1)\} = \{(1,0), (1,1)\}$

$\Rightarrow$ linear independent.

$v_1 = (2,0) \in \text{span}(S)$

$S \cup \{(2,0)\} = \{(1,0), (2,0)\}$

$-2(1,0) + (2,0) = (0,0)$



I am OKAY.

pf. ⇒) 已知: $S\cup\{v\}$ is linear dependent

求證: $v\in span(S)$

由已知, $\exists$ distinct $u_i\in S\cup\{v\}$ $i=1,2,\dots,n$,
and a_i, not all zero s.t. $\sum_{i=1}^{n} a_i u_i = 0$ —(*)

$\because u_i\in S\cup\{v\} \Rightarrow \exists\, u_i = v$, for some i.
Otherwise, S is linear dependent ($\rightarrow\leftarrow$)
$\Rightarrow$ W.L.O.G., let such $u_i = u_n = v$.
Furthermore, $a_n \neq 0$. Otherwise $\sum_{i=1}^{n-1} a_i u_i = 0$,

$\Rightarrow a_1 u_1 + \dots + a_n u_n = 0$

$\Rightarrow v = u_n = -\frac{1}{a_n}(a_1 u_1 + \dots + a_{n-1} u_{n-1}) \in Span(S)$.

Note that $u_1, \dots, u_{n-1} \in S$, Done!

⇐) 已知 $v\in span(S)$, 求證 $S\cup\{v\}$ is linear dependent.

$\because$ 由已知, $\exists\, a_i\in F$, $u_i\in S$, $i=1,2,\dots,n$.
s.t. $v=\sum_{i=1}^{n} a_i v_i \Rightarrow 0 = (-1)v + \sum_{i=1}^{n} a_i u_i$

$\therefore$ W.L.O.G. suppose u_i all distinct
$\Rightarrow S\cup\{v\}$ is linear dependent

Q.E.D.

§1.6 Bases and Dimension.

* 基底：以「不多」也「不少」的向量代表 V.
  維度：$\dim V$ = 基底元素的數

Def. Let $\beta \subset V$ where V is a vector space.
Then β is called a basis of V if the following holds
(1) β is a linear independent set (不多)
(2) $\text{span}(\beta) = V$ (不少)

ex. $V = \mathbb{R}^2$, $\beta = \{(1,0),(0,1)\}$, $\beta' = \{(1,0),(1,1)\}$

ex2. $V = M_{2\times 3}(\mathbb{R})$, $\beta = \left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}$

ex3. $V = \{0\}$, $\beta = \{\emptyset\} \Rightarrow \{0\}$ 之維度為 0!

ex4. $P_2(\mathbb{R}) = V$, $\beta = \{1, x, x^2\}$ or $\{1, 1+x, 2x^2\}$

ex5. $P(\mathbb{R})$ = the set of polynomials with real coefficient. $\beta = \{1, x, x^2, \dots\}$

pf. $\because \forall f \in P(\mathbb{R})$
$\therefore f(x) = a_n x^n + \dots + a_1 x + a_0$
$\Rightarrow f \in \text{span}(\beta)$.

Claim: β is l.i.

I am OKAY.

pf of claim: pick any finite number vectors in β. ∃最大次數 K s.t. any linear combination of such finite vectors is a polynomial with degree at most k.
(例: x^2, x^{100}, x^{200} ⇒ $k=200$)
∴ K次多項式 $\equiv 0$ implies all their coefficients $=0$
∴ β is l.i.
⇒ β is a basis of $P(\mathbb{R})$

Q.E.D.

Thm 1.8 β is a basis of $V \iff \forall v \in V$,
$\exists!\ a_i,\ i=1,\dots,r$ s.t. $v=a_1u_1+\cdots+a_nu_n$

ex. $V=\mathbb{R}^2$, $\beta=\{(1,0),(0,1)\}$, $v=(1,2)$
$\Rightarrow v=1\cdot(1,0)+2\cdot(0,1)$
$a_1\ u_1\ a_2\ u_2$ → 沒有其它 a_1, a_2 之組合

pf. ⇒) Suppose $\exists\ a_i, b_i,\ i=1,\dots,n$ s.t.
$v=a_1u_1+\cdots+a_nu_n=b_1u_1+\cdots+b_nu_n$
$\Rightarrow (a_1-b_1)u_1+\cdots+(a_n-b_n)u_n=0$.
$\Rightarrow a_1=b_1,\ a_2=b_2,\ \cdots,\ a_n=b_n$

⇐) $\mathrm{Span}(\beta)=V$ is obvious. We want β is linear independent.
Let $a_1u_1+\cdots+a_nu_n=0$
∵ $a_1,\dots,a_n$ is unique and $a_1=\cdots=a_n=0$ 為一可行的 $(a_1,\dots,a_n)$
∴ $u_1,\dots,u_n$ is linear independent

$\Rightarrow \{u_1, \dots, u_n\}$ is a basis of V

Q.E.D.

Remark. V has a finite basis β. Then by Thm 1.8, $\exists!\ (a_1, \dots, a_n)$ s.t. $v = a_1u_1 + \dots + a_nu_n$.
$\Rightarrow (a_1, \dots, a_n)$ 就好如同坐標
$\Rightarrow V \cong F^n$

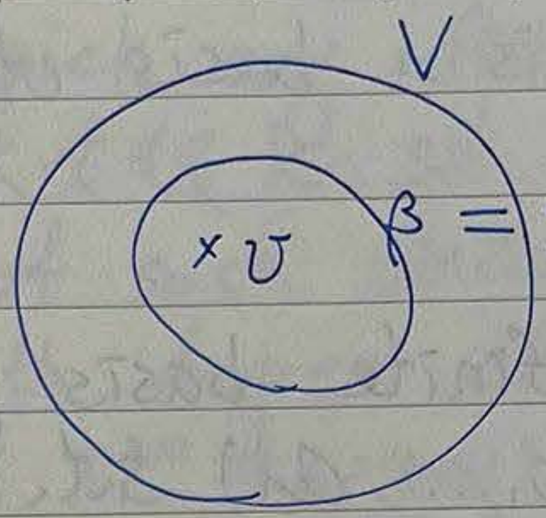
Thm 1.9. Let $V = \mathrm{Span}(G)$, $|G| < \infty$
$\Rightarrow V$ has a finite basis.

pf. (1) $V = \{0\}$, $\beta = \varnothing$
(2) $V \neq \{0\}$:
$\because V \neq \varnothing$, $|G| < \infty$ → $|G| < \infty$ 總會走到生效的
$\therefore \exists\ u_1$: nonzero vector $\in \beta$, and $\{u_i\}$ is linear independent
$\Rightarrow \exists\ \{u_1, \dots, u_n\} = \beta$ s.t. β is linear independent and $\forall v \in V$, $\beta \cup \{v\}$ is linear dependent.

思考
$G_1 = \{(1,0,0), (0,0,1), (0,1,0)\}$, $\mathrm{span}(G_1) = \mathbb{R}^3$
$u_1 = (1,0,0)$, $u_2 = (0,0,1)$, $u_3 = (0,1,0)$, u_4 選誰都會重複.

Claim: β is a basis. We prove by $\mathrm{span}(\beta) = V$.
pf of claim: by Thm 1.7, if $v \in V$, then we have 2 cases:

I am OKAY.



$\begin{cases} ① v \in V-S \Rightarrow v \in \text{span}(S) \text{ by Thm 1.7} \\ ② v \in S \Rightarrow v \in \text{span}(S), \text{ obviously} \end{cases}$

$\because v$ is arbitrary

$\therefore \beta$ is a basis of V

Q.E.D.

Thm 1.10 (Replacement Thm)

Let $V = \text{span}(G)$, $|G| = n$. Let L be a linear independent set in V with $|L| = m$. Then

(1) $n \geq |L| = m$ → G可用另外n个向量取代，其中m个为 l.i

(2) $\exists H \subset V$ s.t. $|H| = n-m$ and $\text{span}(L \cup H) = V$

ex. $V = \mathbb{R}^3$, $G = \{(1,0,0), (0,1,0), (0,0,1)\} \Rightarrow n = 3$

任何 l.i. 之 L 元素个数不会比 3 大！

(在 $\mathbb{R}^3$ 中找不到 4 个线性独立之向量！)

pf. (1) Prove by induction on m.

If $m = 0$, then $L = \emptyset$.

$\Rightarrow n \geq 0$ and $H = G$

$\Rightarrow$ when $m = 0$, (1) and (2) holds.

Suppose $|L| = m$, let $L = \{l_1, l_2, \ldots, l_{m+1}\}$ be linear independent in V. We want

① $n \geq m+1$

② $\exists H \subset V, |H| = n-m-1$ s.t. $\text{span}(L \cup H) = V$

Let $L' = \{l_1, \dots, l_m\}$, then L' is linear independent in V by Thm 1.6.

Suppose $\exists H' = \{h_1, \dots, h_{n-m}\} \subset V$ s.t.

$\begin{cases} \text{span}(L' \cup H') = V \quad -(*) \\ m \leq n. \end{cases}$

$\because$ If $n = m$, then $H' = \emptyset \therefore \text{span}(L') = V$

$\Rightarrow l_{m+1} \in \text{span}(L')$

$\Rightarrow L = \{l_1, \dots, l_{m+1}\}$ linear dependent $\rightarrow\leftarrow$

$\Rightarrow n > m$

$\because$ by $(*)$, $l_{m+1} = a_1 l_1 + \cdots + a_m l_m + b_1 h_1 + \cdots + b_{n-m} h_{n-m}$

Claim: $\exists\, b_i \neq 0$.

pf of claim: $\because$ If not, then $l_{m+1} = a_1 l_1 + \cdots + a_m l_m$

$\therefore \quad 0 = a_1 l_1 + \cdots + a_m l_m + (-1) l_{m+1}$

$\Rightarrow L$ is linear dependent, $\rightarrow\leftarrow$

$\Rightarrow \exists\, b_i \neq 0$.

W.L.O.G, let $b_i = b_1$

$\Rightarrow h_1 = -\frac{a_1}{b_1} l_1 \cdots - \frac{a_m}{b_1} l_m - \frac{b_2}{b_1} h_2 + \cdots - \frac{b_{n-m}}{b_1} h_{n-m}$

$\Rightarrow H = \{h_2, \dots, h_{n-m}\}$

$\Rightarrow \text{span}(L \cup H) = V$

$\Rightarrow |H| = n-m-1 = n-(m+1)$

Q.E.D.

I am OKAY.

Def. Dimension (維度)

(1) Let β be a basis for V and $|\beta| < \infty$. Then $\dim(V) = |\beta|$
In which case V is said to be finite dimensional.

(2) If V is not finite dimension, then V is infinite dimension

ex. $V = \mathbb{R}^2$

$\begin{cases} \beta = \{(1,0), (0,1)\} \\ \beta = \{(1,1), (-2,1)\} \end{cases}$ 都是 basis.

Q: Is such def. well-defined?

Cor 1 of Thm 1.10:

(1) Let $V = \text{span}(G)$, $|G| = n$. If L is a linear independent subset of V, then $|L| < n \Rightarrow |L|$ is finite.

(2) Let β and γ be basis of V, then $|\beta| = |\gamma|$.

pf: (1) If $|L| = \infty$, then $\exists\, \overline{L} \subset L$ s.t. $|\overline{L}| = n+1$

$\Rightarrow$ by Thm 1.10 (1), $n \geq n+1$ $\rightarrow\leftarrow$

$\Rightarrow |L|$ must be finite.

$\Rightarrow$ by Thm 1.10 (1), $n \geq |L|$.

(2): Let $G = \beta$. $V = \text{span}(\beta)$, then $|\gamma| \leq |\beta|$.

" γ " γ " $|\beta| \leq |\gamma|$

$\therefore |\beta| = |\gamma|$

Q.E.D.

Cor2. If $\dim(V) = n < \infty$

(1) If $\text{span}(G) = V$, then $|G| \geq n$. Moreover, If $|G| = n$, then G is a basis for V (基底是「最小」的 spanning set)

(2) If L is a linear independent subset of V, then $|L| \leq n$. Moreover, if $|L| = n$, then L is a basis for V. (基底是「最大」的 independent set).

pf. (1) $\because \dim(V) = n < \infty, \therefore \exists$ a basis β of V s.t. $|\beta| = n$.

$\because V = \text{span}(G)$,

$\therefore$ If $\begin{cases} |G| = \infty \Rightarrow \infty \geq n \\ |G| < \infty \Rightarrow |G| \geq n \end{cases}$

If $|G| = n$, then we claim G is l.i.:

pf of claim: Suppose G is linear dependent by Thm 1.10(1), Cor1(1), $\exists \bar{G} \subset G$ s.t. $\text{span}(\bar{G}) = \text{span}(G) = V$ and $\bar{G} \leq n-1$, $\rightarrow\leftarrow$.

$\Rightarrow \exists \beta$ be a basis for V and $|\beta| = n$

$\Rightarrow \beta$ is linear-indepedent.

$\Rightarrow n-1 \geq n$ $\rightarrow\leftarrow$

$\Rightarrow G$ is linear dependent.

$\Rightarrow G$ is a basis for V.

(2) $\because \dim(V) = n$

$\therefore \exists \beta$: a basis for V with $|\beta| = n$

I am OKAY.

and $\text{span}(\beta)=V$.
$\Rightarrow$ by Cor 1.(1), $n \geq |L|$

this follows
Now we want $\text{span}(L)=V$. by Thm 1.10(2),
$\beta = G$, $L = L$, $V = \text{span}(L)$.
where
Q.E.D.

Thm 1.11. If $\dim(V)<\infty$, If W is a subspace of V, then:
(1) $\dim(W) \leq \dim(V)$
(2) If $\dim(W)=\dim(V)$, then $W=V$

pf. (1) If γ is a basis of W, β is a basis of V then in Thm 1.10(2), take
$\beta = G$
$\gamma = L$
$\begin{cases} \beta = G \\ \gamma = L \end{cases}$
$\Rightarrow \text{span}(\beta) = G$.
$\Rightarrow |\beta| = \dim V$ and $|\beta| \leq |\gamma|$
$\Rightarrow \dim(W) \leq \dim(V)$
$\sqcup$

(2). by Thm 1.10(2), $|\gamma| = n$, $\exists H \subset V$ s.t.
$|H| = 0$ and $\text{span}(\gamma \cup H) = V = W$.
$L = \gamma$
Q.E.D.

ex. $V=\{0\}$, $\dim V = |\beta| = |\emptyset| = 0$

ex. $V=\mathbb{R}^3$, $\dim V = |\beta| = |\{(1,0,0),(0,1,0),(0,0,1)\}|$
$=3$.

ex. $V = M_{2\times 3}(\mathbb{R})$

$\beta = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$\Rightarrow \dim V = 6$.

ex. $P_n(\mathbb{R})$, $\beta = \{1, x, \cdots, x^n\}$

$\Rightarrow \dim(P_n(\mathbb{R})) = n+1$

ex. $P(\mathbb{R})$, $\beta = \{1, x, x^2, \cdots, x^n, \cdots\}$

$\Rightarrow P(\mathbb{R})$ is infinite dimensional (why?)

ex. $V = \mathbb{C}$, $F = \mathbb{C} \Rightarrow \beta = \{1\}$ or $\{i\} \Rightarrow \dim V = 1$.

ex. $\beta = \{u, v\}$ is a basis for V. Prove that $\{u+v, u-v\}$ is also a basis for V.

pf. $\dim(V) = 2$, Need only to show that $\{u+v, u-v\}$ is l.i.

If $a(u+v) + b(u-v) = 0$

$\Rightarrow (a+b)u + (a-b)v = 0$

$\because \{u, v\}$ is l.i.

$\therefore \begin{cases} a+b=0 \\ a-b=0 \end{cases}$

$\Rightarrow a = b = 0$

$\Rightarrow \{u+v, u-v\}$ is l.i. #

I am OKAY.

Def. In $P_n(\mathbb{R})$, if $c_0, c_1, \dots, c_n$ are distinct numbers, define the Lagrange polynomial

$$f_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x-c_j)}{(c_i-c_j)}$$

ex. $$f_0(x) = \frac{(x-c_1)(x-c_2)\cdots(x-c_n)}{(c_0-c_1)(c_0-c_2)\cdots(c_0-c_n)}$$

§1.7 Maximal Linearly Independent Subsets.

Q: Does every vector spaces (no matter finite/infinite dimension) have a basis?

Def. Let F be a family of sets. A member M of F is called maximal (w.r.t. set inclusion) if M is contained in no member of F other than M itself.

Def. Let F be the family of all subsets of a nonempty set S, then F is called a power set of S.

ex. Let S, T be disjoint sets, F = union of their power sets $\Rightarrow S, T$ are both maximal in F.

ex. Let F be the family of all finite subsets of an infinite set S.
$\because$ For any member M in F and $s \in S$, $s \notin M$, $M \cup \{s\}$ is a member of F and contains M
$\therefore$ F has no maximal elements.

Def. A collection of sets C is called a chain if $\forall A, B$ in C, either $A \subseteq B$ or $B \subseteq A$

ex. If $n \in \mathbb{N}$, $A_n = \{1, \dots, n\}$, then $C = \{A_n \mid n \in \mathbb{N}\}$ is a chain.

I am OKAY.

Axiom. (Maximal Principle)
Let F be a family of sets. If for each chain $C \subseteq F$, $\exists$ a member of F that contains each member of C, then F contains a maximal member.

Remark. Maximal Principle is equivalent to the Axiom of Choice

Def. Let S be a subset of a vector space V. A maximal linearly independent subset of S is a subset $B \subseteq S$ s.t.:
(1) B is linearly independent
(2) The only linearly independent subset of S that contains B is B itself.

Remark. It follows that A basis β for V is a maximal linearly independent subset of V.

現在可以來證一些東西了.

Thm 1.12 Let V be a vector space and S a subset that generates V. If β is a maximal linearly independent subset of S, then β is a basis for V.

pf. Let β be the m.l.i.s of S, then $S \subseteq \text{span}(\beta)$ (Otherwise, $\exists v \in S$

s.t. $v \notin \text{span}(\beta)$, then by Thm 1.7 $\beta \cup \{v\}$ is l.i., contradict with the maximality of β)

$\because \text{span}(S) = V$

$\therefore$ by Thm 1.5, $\text{span}(\beta) = V$.

Q.E.D.

Thm 1.13 Let S be a linearly independent subset of a vector space V. There exists a maximal linearly independent subset of V that contains S.

pf. Let F denote the family of all linearly independent subsets of V that contain S. Let U be the union of the members of chain C. We want to prove that U is the desired set in maximal principle.

Let C be a chain in F.

$\because$ each member of C is a subset of V containing S

$\therefore S \subseteq U \subseteq V$

Let $u_1, u_2, \dots, u_n \in U$, $a_1, a_2, \dots, a_n \in F$ s.t. $a_1u_1 + \cdots + a_nu_n = 0$

$\because u_i \in U$, for $i = 1, \dots, n$

$\therefore \exists$ a set A_i in C s.t. $u_i \in A_i$

$\because C$ is a chain

$\therefore \exists A_k$ contain all the other A_i's.

$\Rightarrow u_i \in A_k$, $\forall i = 1, \dots, n$

$\because A_k$ is a linearly independent set

$\therefore a_1u_1 + \cdots + a_nu_n = 0$ implies $a_1 = \cdots = a_n = 0$.

I am OKAY.

$\Rightarrow$ U is linearly independent

$\Rightarrow$ $U \in F$

$\because$ U contains each member of C

$\therefore$ by maximal principle, F has a maximal element.

$\Rightarrow$ This element is a m.l.i.s. subset of V that contains S.

Q.E.D.

Cor. Every vector space has a basis.

Ch2. Linear Transformation and Matrices

2-1. Linear Transformations, Null spaces, and Ranges

Def. T is a linear transformation from a vectorspace V to another vector space W if

$\begin{cases} (1)\ T(x+y) = T(x)+T(y),\ \forall x,y \in V \\ (2)\ T(cx) = cT(x),\ \forall c \in F,\ x \in V. \end{cases}$

Facts: (1) $T(0)=0$

(2) $T(x-y) = T(x)-T(y)$

(3) T is linear $\Leftrightarrow T(cx+y) = cT(x)+T(y)$, $\forall c \in F,\ x,y \in V$.

pf. (1) $T(0) = T(0+0) = T(0)+T(0)$

$\Rightarrow 0 + T(0) = T(0) + T(0)$

$\Rightarrow 0 = T(0)$

(2) $T(x-y) = T(x+(-1)y) \overset{(1)}{=} T(x) + T((-1)y)$

$\overset{(2)}{=} T(x) + (-1)T(y) = T(x) - T(y)$

(3). $\Rightarrow$): $\because T(cx+y) = T(cx)+T(y) = cT(x)+T(y)$, $\forall x,y \in V$ and $c \in F$

$\therefore T$ is linear.

$\Leftarrow$) Let $c=1$, then $T(x+y) = T(x)+T(y)$.

" 0, " $T(cx+0) = cT(x)+T(0)$

$= cT(x)+0 = cT(x)$

Q.E.D.

ex.① $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f((a_1,a_2)) = (a_1+a_2, 1, a_1)$. Is f a linear transform?

↗ 常簡寫為 $f(a_1,a_2)$

② $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1,a_2) = (a_1+a_2, a_1)$. Is T a L.T?

-sol- ① No, since $T(0,0)=(0,1,0) \neq 0$

② Let $x=(a_1,a_2)$, $y=(b_1,b_2)$. then

$\because cx+y = (ca_1,ca_2)+(b_1,b_2) = (ca_1+b_1, ca_2+b_2)$

$\therefore T(cx+y) = T(ca_1+b_1, ca_2+b_2)$

$= (ca_1+b_1+ca_2+b_2,\ ca_1+b_1)$

$= c(a_1+a_2, a_1) + (b_1+b_2, b_1)$

$= cT(x) + T(y)$

ex. $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$. $T(p(x)) = p'(x)$

Claim: 微分是線性運算

Wanted: $(cp(x)+g(x))' = cp'(x)+g'(x)$ by calculus

the result holds

ex. $T(a_1,a_2) = (2a_1+a_2, 3a_2)$ ○

$T(a_1,a_2) = (a_1, 0)$ ○

$T(a_1,a_2) = (|a_1|, a_2)$ ×

$T(a_1,a_2) = (\sin a_1, a_2)$ ×

⇒ 坐標式的寫法就看各分量是不是①一次式 且②沒有常數項就好.

Fact: ① ex38, 39 ⇒ In Def, (1), (2) 兩條件互相獨立.

② $F=\mathbb{Q}$ ⇒ (1) 可推得 (2)

I am OKAY.

pf. (2) Let $c=\frac{m}{n}$, $n,m\in\mathbb{Z}, n\neq 0$.

(a) Claim: $T(mx)=mT(x)$, $m\geq 0, m\in\mathbb{Z}$

pf of claim: $T(\underbrace{x+\cdots+x}_{m})=T(x)+T(\underbrace{x+\cdots+x}_{m-1})$

$=\cdots=mT(x)$.

(b) Claim: $T(-x)=-T(x)$

pf of claim: $T(x)=T(2x+(-x))=T(2x)+T(-x)$

$=2T(x)+T(-x)$

$\Rightarrow -T(x)=T(-x)$

(c) Claim: $T(mx)=mT(x)$, $m<0$, $m\in\mathbb{Z}$.

pf of claim: $T(mx)=T((-m)(-x)) \underset{(a)}{=} (-m)T(-x)$

$\underset{(b)}{=} mT(x)$

(d) Claim: $T(\frac{x}{n})=\frac{1}{n}T(x)$

pf of claim: $T(\frac{x}{n})=T\left(x+\left(-\frac{n-1}{n}x\right)\right)$

$=T(x)+T((-n+1)\frac{x}{n})$

$=T(x)+(-n+1)T(\frac{x}{n})$

$\Rightarrow 0=T(x)-nT(\frac{x}{n})$

$\Rightarrow nT(\frac{x}{n})=T(x)$

$\Rightarrow T(\frac{x}{n})=\frac{1}{n}T(x)$.

(e) Hence, $T(\frac{m}{n}x)=\frac{m}{n}T(x)$.

Q.E.D.

Def: $T: V \longrightarrow W$ is a linear transformation

(1) Null space of $T = N(T)$ (or $\ker(T)$)

$\triangleq \{x \in V : T(x) = 0\}$

(2) Range of $T = R(T) \triangleq \{T(x) : x \in V\}$

$= \{y \in W : \exists x \text{ st. } T(x) = y\}$

ex. $T(a_1, a_2) = (2a_1 + a_2, 3a_2)$. $V = \mathbb{R}^2 = W$. Find $N(T)$ and $R(T)$?

-sol-. (1) To find $N(T)$,

$$\begin{cases} 2a_1 + a_2 = 0 \\ 3a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}$$

$\Rightarrow N(T) = \{(0,0)\}$ → is a subspace of $\mathbb{R}^2$

$\dim N(T) = 0$

(2) $R(T) = \{(2a_1 + a_2, 3a_2) : a_1, a_2 \in \mathbb{R}\}$

$= \{(x, y) : x, y \in \mathbb{R}\}$

$= \mathbb{R}^2$ → is a subspace of $\mathbb{R}^2$

$\dim R(T) = 2$.

ex. $T(a_1, a_2) = (a_1, 0)$. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is called a projection.

$N(T) = \{(0, a_2) : a_2 \in \mathbb{R}\}$ = the y-axis, is a subspace of $\mathbb{R}^2$. $\beta = \{(0,1)\}$

$\Rightarrow \dim(N(T)) = 1$.

$R(T) = \{(a_1, 0) : a_1 \in \mathbb{R}\}$ = the x-axis

$\Rightarrow \dim R(T) = 1$

$\Rightarrow \dim N(T) + \dim R(T) = 2 = \dim \mathbb{R}^2$

I am OKAY.

ex. $T: \mathbb{R}^3 \to \mathbb{R}^2$ $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = (0, 0)$

$N(T) = \{(a_1, a_1, 0) : a_1 \in \mathbb{R}\}$

$R(T) = \{(a_1 - a_2, 2a_3) : a_1, a_2, a_3 \in \mathbb{R}\}$

$= \{(x, y) : x, y \in \mathbb{R}\}$

$= \mathbb{R}^2$

$\because$ basis of $N(T) = \beta = \{(1, 1, 0)\}$

$\therefore \dim N(T) = 1$

$\because R(T) = \mathbb{R}^2$

$\therefore \dim R(T) = 2$

$\Rightarrow \dim N(T) + \dim R(T) = 3 = \dim \mathbb{R}^3$

ex. $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

If $f(x) = x - 1 \Rightarrow T(f(x)) = \begin{pmatrix} 0-1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$f(x) = x^2 + x + 1$, $T(f(x)) = \begin{pmatrix} 3-7 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$

① $T(f(x)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$

$$\Rightarrow \begin{cases} f(1) - f(2) = 0 \\ f(0) = 0 \end{cases}$$

$\because f(0) = 0$,

$\therefore$ If $f(x) = ax^2 + bx + c$, then $c = 0$

$\because f(1) - f(2) = 0$

$\therefore a + b + \cancel{c} = 4a + 2b + \cancel{c}$

$\Rightarrow b = -3a$

$\Rightarrow N(T) = \{ax^2 - 3ax : a \in \mathbb{R}\}$

$\because \beta = \{x^2 - 3x\}$ is a basis of $N(T)$

$\therefore \dim N(T) = 1.$

② $R(T) = \left\{ \begin{pmatrix} f(1)-f(0) & 0 \\ 0 & f(0) \end{pmatrix} : f \in P_2(\mathbb{R}) \right\}$

取幾筆數多項式即可！ $= \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}$

= The set of diagonal matrices of size 2×2.

$\Rightarrow \beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $R(T)$

$\Rightarrow \dim(R(T)) = 2$

$\Rightarrow \dim N(T) + \dim R(T) = 3 = \dim P_2(\mathbb{R})$

Thm 2.1. T: linear transform $V \to W$.

$\Rightarrow \begin{cases} N(T) \text{ is a subspace of } V \\ R(T) \quad " \quad W \end{cases}$

pf. 是否是子空間另需檢查封閉性 ($T(0)=0$ ⇒ $N(T)$ and $R(T)$ are nonempty.)

Let $x, y \in N(T) \Rightarrow T(x) = T(y) = 0$

$\Rightarrow 0 = T(x) + T(y) = T(x+y)$

$\Rightarrow x+y \in N(T)$.

$\because x \& N(T)$

$\therefore T(x) = 0$

$\Rightarrow cT(x) = c \cdot 0 = 0 = T(cx)$

$\Rightarrow cx \in N(T), \forall c \in F.$

I am OKAY.

Similarly, $R(T)$ can be proved to be a subspace of V.

Q.E.D.

Thm 2.2. Let $\beta = \{v_1, \dots, v_n\}$ be a basis of V.
Then $R(T) = \text{span}(T(\beta)) = \text{span}\{T(v_1), \dots, T(v_n)\}$

$\| \quad \| \quad \| \quad \

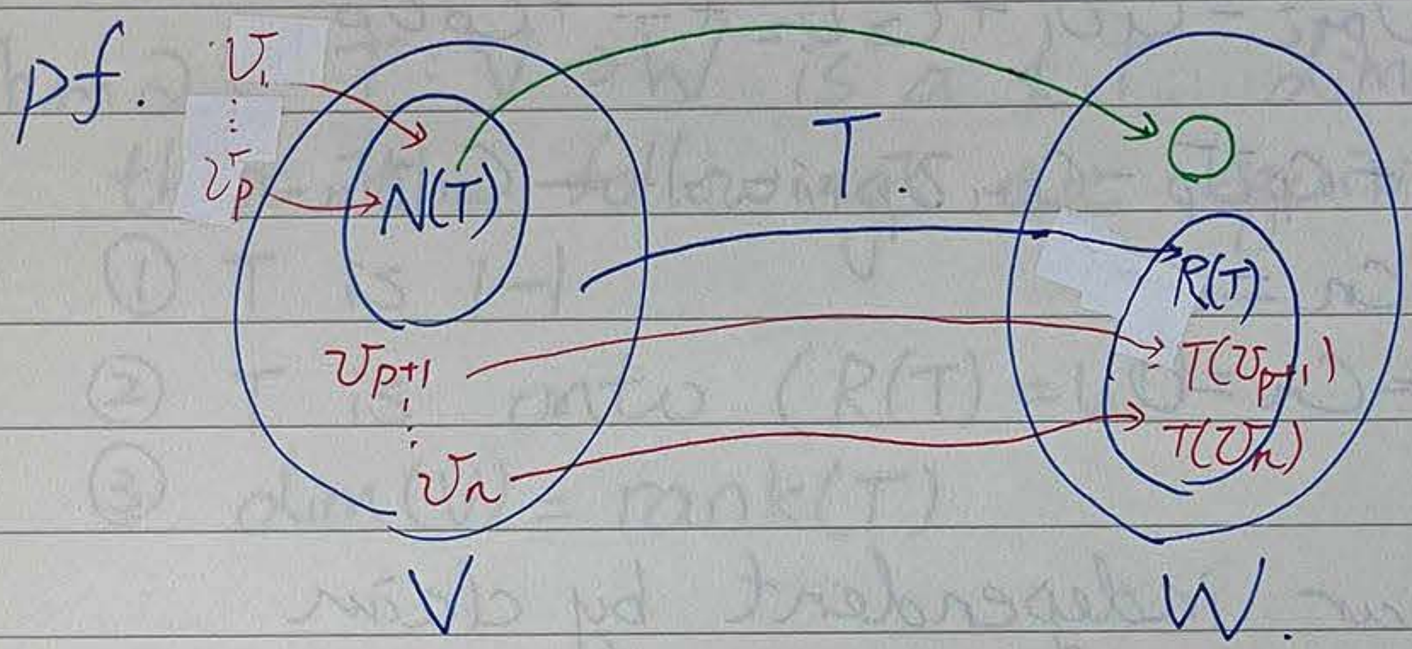
Thm 2.3 (Dimension Thm)

Let V, W be vector spaces, $T: V \to W$ is a linear transform, $\dim(V) < \infty$. Then

$$\dim(N(T)) + \dim(R(T)) = \dim(V)$$

$\dim(N(T))$ = Nullity(T) ; $\dim(R(T))$ = Rank(T)

pf.



Let $\dim(N(T)) = p$ and $\beta = \{v_1, \dots, v_p\}$ be a basis of $N(T)$ and $\gamma = \{v_1, \dots, v_p, v_{p+1}, \dots, v_n\}$ be a basis of V.

$\Rightarrow$ We want $\dim R(T) = n-p$ $\iff X \triangleq \{T(v_{p+1}), \dots, T(v_n)\}$ is a basis of $R(T)$ $\iff$

$\begin{cases} \text{① } \text{Span}(X) = R(T) \\ \text{② } X \text{ is l.i.} \end{cases}$

① by Thm 2.2,

$$R(T) = \text{span}\{T(v_1), \dots, T(v_p), T(v_{p+1}), \dots, T(v_n)\}$$
$$= \text{span}\{0, \dots, 0, T(v_{p+1}), \dots, T(v_n)\}$$
$$= \text{span}\{T(v_{p+1}), \dots, T(v_n)\}$$

② Suppose $\sum_{i=1}^{n-p} c_{p+i} T(v_{p+i}) = 0$

I am OKAY.

Claim: $c_{p+1} = c_{p+2} = \cdots = c_n = 0$

pf of claim: $\because$ T is linear

$\therefore T\left(\sum_{i=1}^{n-p} c_{p+i} v_{p+i}\right) = 0$

$\Rightarrow \sum_{i=1}^{n-p} c_{p+i} v_{p+i} \in N(T)$

$\Rightarrow \sum_{i=1}^{n-p} c_{p+i} v_{p+i} = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$

$\Rightarrow c_1 v_1 + \cdots + c_p v_p - c_{p+1} v_{p+1} - \cdots - c_n v_n = 0$

$\Rightarrow c_1 = \cdots = c_n = 0$

$\Rightarrow c_{p+1} = \cdots = c_n = 0$

$\Rightarrow X$ is linear independent by claim

$\Rightarrow X$ is a basis of $R(T)$

$\Rightarrow \dim R(T) = n-p$

$\Rightarrow \dim N(T) + \dim R(T) = \dim V = n.$

Q.E.D.

Thm 2.4. $T: V \to W$ is a L.T. then

T is 1-1 $\iff N(T) = \{0\}$

pf. $\Rightarrow$) $\because T(0) = 0$ and T is 1-1

$\therefore N(T) = \{0\}$

$\Leftarrow$) If $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$

$\Rightarrow x - y \in N(T) = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x=y$

$\Rightarrow T$ is 1-1

Q.E.D.

Remark. The Thm is true for V and W being infinite-dimensional.

Thm 2.5 $T: V \to W$ is a L.T. $\dim(V) = \dim(W) < \infty$, then the followings are equivalent (TFAE):

① T is 1-1

② T is onto ($R(T) = W$)

③ $\dim(V) = \text{rank}(T)$

Remarks: (1) T is nonlinear $\Rightarrow$ the conclusion of the Thm fail.

(2). If V, W are not finite dimentional $\Rightarrow$ The Thm fails, too.

pf. ① $\Rightarrow$ ②: $\because$ T is 1-1

$\therefore$ by Thm 2.4, $N(T) = \{0\}$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow$ by Thm 2.3, $\dim(R(T)) = \text{rank}(T) = \dim(V)$ (已知)

$\because R(T)$ is a subspace of W

$\therefore R(T) = W$

$\Rightarrow T$ is onto.

② $\Rightarrow$ ③: $\because R(T) = W$, $\dim(W) = \dim V$

$\therefore \dim R(T) = \dim V$

$\Rightarrow$ Nullity$(T) = 0$ by Thm 2.3

I am OKAY.

$\Rightarrow$ by Thm 2.3, $\dim V = \operatorname{rank} T$.

③ $\Rightarrow$ ①: $\because$ by Thm 2.3, $\text{Nullity}(T) = 0$

$\therefore N(T) = \{0\}$

$\Rightarrow$ by Thm 2.4, T is 1-1

Q.E.D.

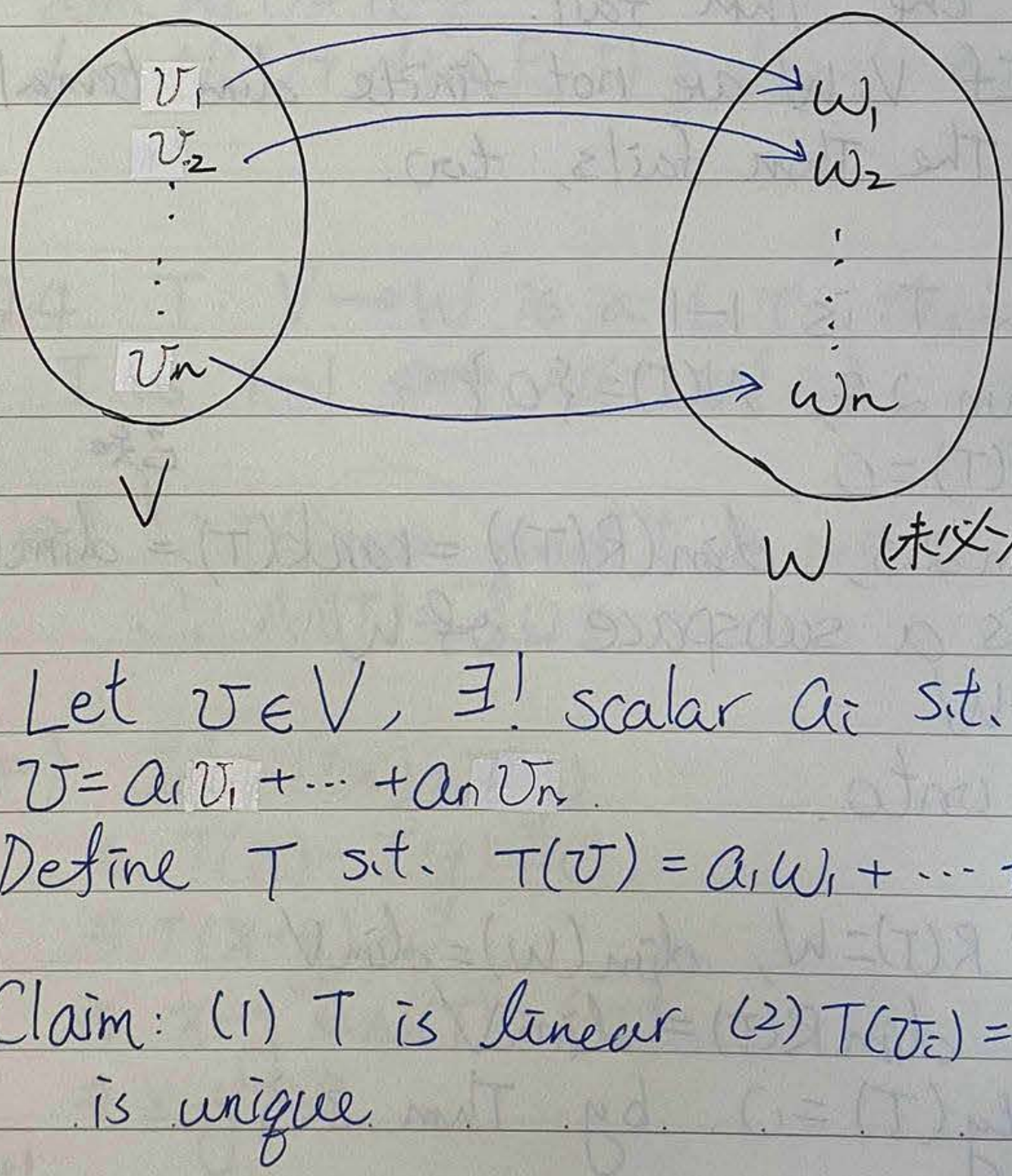
Thm 2.6. Let $\beta = \{v_1, \dots, v_n\}$ be a basis for V and $w_i \in W$, $i = 1, \dots, n$. Then $\exists$ linear transformation T s.t.

(1) T is linear

(2) $T(v_i) = w_i$

(3) T is unique.

(拿推基底的行李，拿推全世界)



pf. Let $v \in V$, $\exists!$ scalar a_i s.t.

$v = a_1 v_1 + \cdots + a_n v_n$.

Define T s.t. $T(v) = a_1 w_1 + \cdots + a_n w_n$

Claim: (1) T is linear (2) $T(v_i) = w_i$ (3) T is unique

pf of claim:

(1) Let $u, v \in V$ and $d \in F$

$\Rightarrow u = \sum_{i=1}^{n} b_i v_i$ and $v = \sum_{i=1}^{n} c_i v_i$, for

some $b_1, \dots, b_n, c_1, \dots, c_n \in F$

$\Rightarrow du + v = \sum_{i=1}^{n} d(b_i + c_i) v_i$

$\Rightarrow T(du+v) = \sum_{i=1}^{n} (db_i + c_i) w_i = d \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i$

$= dT(u) + T(v)$

(2). $T(v_i) = T(0v_1 + \dots + v_i + \dots + 0v_n)$

$= 0 \cdot T(v_1) + \dots + T(v_i) + \dots + 0T(v_n)$

$= w_i$

(3). Let $U: V \to W$ be another L.T. and

$U(v_i) = w_i$, $\forall i = 1, \dots, n$, then $\forall x \in V$,

$\therefore x = \sum_{i=1}^{n} a_i v_i$, for some $a_1, \dots, a_n \in F$

$\therefore U(x) = \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i = T(x)$

$\Rightarrow U = T$

Q.E.D.

I am OKAY.

§ 2.2 Matrix Representation of a Linear

Def. An ordered basis for a finite dimensional basis for V is a basis for V endowed with a specific order

Remark. That is, $\beta=\{u_1, u_2, \dots, u_n\}$, $\beta'=\{u_2, u_1, \dots, u_n\}$, then $\beta \neq \beta'$ as ordered basis

ex. $\beta=\{e_1, e_2\}$, $\beta'=\{e_2, e_1\}$ in $\mathbb{R}^2$
$\Rightarrow \beta' \neq \beta$.

Def. Let $\beta=\{u_1, \dots, u_n\}$ be an ordered basis for V, let $x \in V$ and $x=a_1u_1+\cdots+a_nu_n$ then $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is called the coordinate vector of x w.r.t. β. we shall denote by $[x]_\beta$ $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

ex. $V=P_2(\mathbb{R})$, ① $\beta=\{1, x, x^2\}$
Let $v=-2+3x+x^2 \in P_2(\mathbb{R})$
$=-2(1)+3(x)+1(x^2)$
$a_1\ u_1\ \ a_2\ u_2\ \ a_3\ u_3$
$\Rightarrow$ coordinate vector w.r.t. β is $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$

② w.r.t. $\beta'=\{1, x^2, x\} \Rightarrow \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\Rightarrow [v]_\beta=\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, $[v]_{\beta'}=\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

Transformation

Def. Let $T: V \to W$ be a L.T. let $\beta = \{u_1, \dots, u_n\}$ and $\gamma = \{v_1, \dots, v_m\}$ be ordered bases for V and W, respectively. Let

$$T(u_j) = \sum_{i=1}^{m} a_{ij} v_i, \quad j = 1, \dots, n.$$

Let $A = (a_{ij})$ $1 \le i \le m$, $1 \le j \le n$.
Then A, of dimension $m \times n$, is called the matrix representation of T w.r.t. β and γ. Write $A = [T]_\beta^\gamma$

ex. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (a_1 + a_2, a_2 - a_1, a_1 + 3a_2)$
$\beta = \{(1,0), (0,1)\}$
$\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$
Find $[T]_\beta^\gamma$?

-sol-. $\because T(1,0) = (1,-1,1) = 1(1,0,0) - 1(0,1,0) + 1(0,0,1)$
$T(0,1) = (1,1,3) = 1(1,0,0) + 1(0,1,0) + 3(0,0,1)$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 3 \end{pmatrix}_{3\times 2} \quad \#$$

ex2. $T: P_2(\mathbb{R}) \to \mathbb{R}$ and $T(f(x)) = f(2)$.
$\beta = \{1, x, x^2\}$, $\gamma = \{1\}$

$$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}_{1\times 3}$$

since $\begin{cases} T(1) = 1 \cdot 1 \\ T(x) = 2 \cdot 1 \\ T(x^2) = 4 \cdot 1 \end{cases}$

I am OKAY.

ex. $T: P_2(\mathbb{R}) \to M_{2\times 2}(F)$ by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$$

$\beta = \{1, x, x^2\}$

$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Find $[T]_\beta^\gamma$?

-sol-. $\because T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$$= (0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (0)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

#

蒐集所有 V 到 W 的線性函數的集合

Def. (1) $\mathcal{L}(V, W) = \{T \mid T: V \to W$ is a L.T., V and W are vector spaces$\}$

(2) If $V = W$, then we write $\mathcal{L}(V, V) = \mathcal{L}(V)$

Thm 2.7: $\mathcal{L}(V, W)$ is a vector space. (both over F)

We define $(U+T)(x) = U(x) + T(x)$ ($\in W$)

Let $\alpha \in F$, $T \in \mathcal{L}(V, W)$, $(\alpha T)(x) = \alpha(T(x))$ ($\in W$)

pf. (V-0) $(U+T): V \to W$ (of course!)

欲證：$U+T$ 仍為 linear.

① $\forall \alpha \in F$, $\forall x, y \in V$, we want

$$Q(\alpha x+y) = \alpha Q(x) + Q(y)$$

$\because$ 左 $= U(\alpha x+y) + T(\alpha x+y) = \alpha U(x) + U(y) + \alpha T(x) + T(y)$

$= \alpha[U(x)+T(x)] + [U(y)+T(y)]$

$=$ 右

$\therefore (U+T)(\alpha x+y) = \alpha(U+T)(x) + (U+T)(y)$

② we want αT is linear $\iff (\alpha T)(cx+y)$

$= c(\alpha T)(x) + (\alpha T)(y)$, $\forall c \in F$, $x, y \in V$

This can be done similarly to ①.

(V-1) $(U+T)(x) \stackrel{?}{=} (T+U)(x)$, $\forall x \in V$.

$\because (U+T)(x) = U(x) + T(x)$

$= T(x) + U(x)$ (since W is a vector space)

$= (T+U)(x)$

$\therefore U+T = T+U$

the others are similar.

Q.E.D.

Thm 2.8. $T, U : V \xrightarrow{\text{linear}} W$

(1) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(2) $[\alpha T]_\beta^\gamma = \alpha [T]_\beta^\gamma$

where β is a basis of V, γ is a basis for W.

I am OKAY.

ex. $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ 2a_1 - 4a_2)$

$U(a_1, a_2) = (a_1 - a_2,\ 2a_1,\ 3a_1 + 2a_2)$

$\mathbb{R}^2$ basis: $\beta = \{(1,0), (0,1)\}$

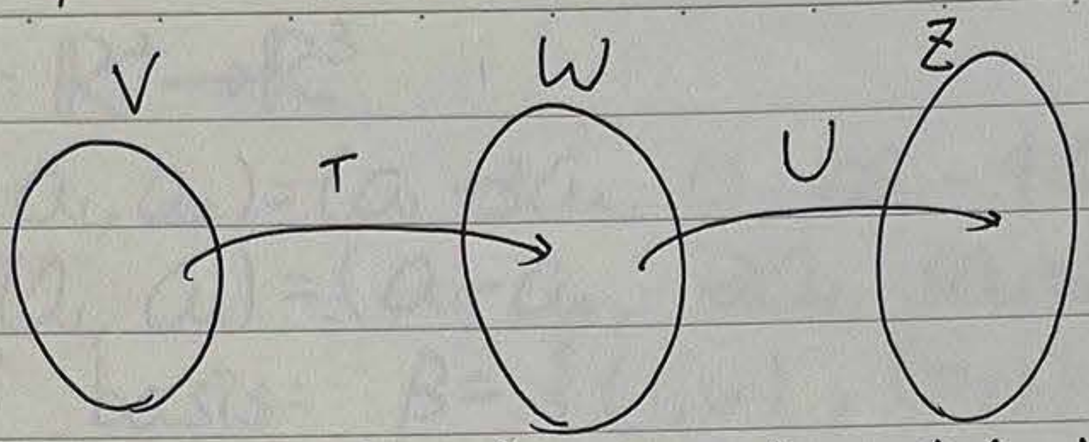
$\mathbb{R}^3$ basis: $\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$

$(T+U)(a_1, a_2) = (2a_1 + 2a_2,\ 2a_1,\ 5a_1 - 2a_2)$

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}, \quad [U]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$$

$$[T] + [U] = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix}$$

2-3 Compositin of Linear Transformatins and



$UT \triangleq U \circ T$, $(UT)(x) = U(T(x))$

Thm 2.9. Let V, W and Z be vector spaces over the same field F. Let
$\begin{cases} T: V \to W \text{ be linear transforms,} \\ U: W \to Z \end{cases}$
then $UT \triangleq U \circ T$ is linear.

pf. Needed: $(UT)(\alpha x+y) = \alpha(UT)(x) + (UT)(y)$,
$\forall x, y \in V,\ \alpha \in F$.

$\because (UT)(\alpha x+y) = U(T(\alpha x+y))$
$= U(\alpha T(x) + T(y))$ (since T is linear)
$= \alpha U(T(x)) + U(T(y))$ (since U ")
$= \alpha (UT)(x) + (UT)(y)$
$\therefore UT$ is linear

Q.E.D.

Thm 2.10. Let V be a vector space and $U, T, U_i, T_i \in \mathcal{L}(V)$, then
(1) $(T_1 + T_2)U = T_1U + T_2U$
(2) $T(U_1+U_2) = TU_1 + TU_2$
(3) $(TU_1)U_2 = T(U_1U_2)$
(4) $IT = TI = T$ (I: identical mapping)

只要此些運算有定義就成立，未必都要在 $\mathcal{L}(V)$

Matrix Multiplication

pf. (1) $(T_1+T_2)U = T_1U + T_2U$ → We want

$((T_1+T_2)U)(x) = (T_1U+T_2U)(x)$, $\forall x \in V$

$\because ((T_1+T_2)U)(x) = ((T_1+T_2))(U(x))$

$= (T_1+T_2)(U(x))$

$= T_1(U(x)) + T_2(U(x))$

$= (T_1U)(x) + (T_2U)(x)$

$= (T_1U+T_2U)(x)$, $\forall x \in V$.

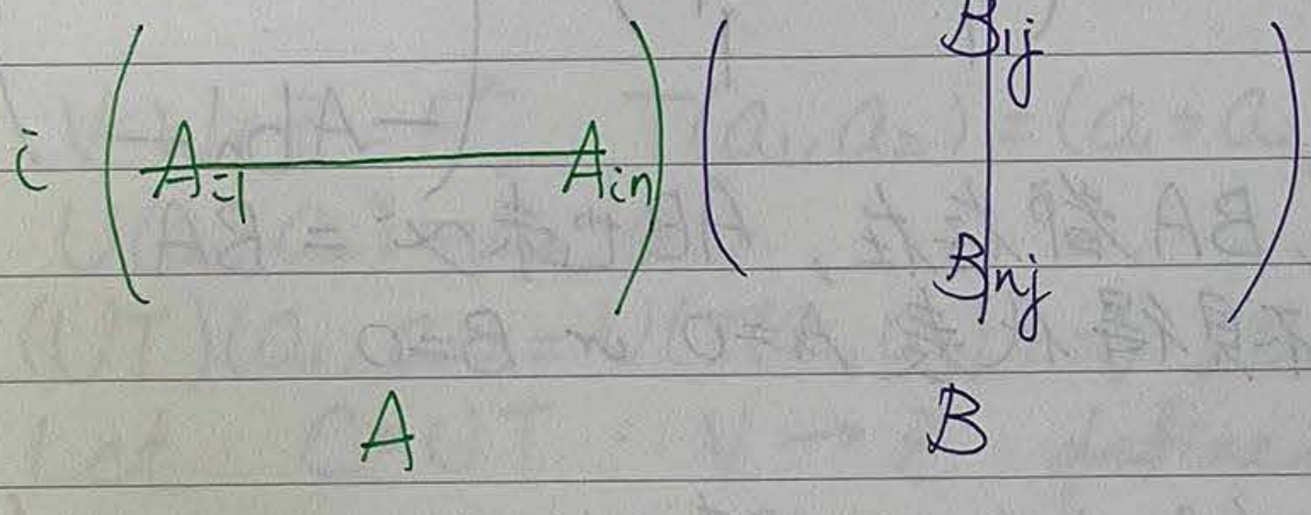
$\therefore$ (1) holds

(2), (3), (4) can be proved similarly.

Q.E.D.

Def. Let $A=(A_{ij})_{m\times n}$ and $B=(B_{ij})_{n\times p}$ be matrices. Define $AB=C=(C_{ij})_{m\times p}$ as

$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ (A第i列布B第j行內積)



ex. $A=\begin{pmatrix}1&1&1\\2&2&3\end{pmatrix}_{2\times3}$, $B=\begin{pmatrix}1&1&1\\2&2&2\\3&3&3\end{pmatrix}_{3\times3}$

$\Rightarrow AB=\begin{pmatrix}6&6&6\\15&15&18\end{pmatrix}_{2\times3}$, 但BA不可乘.

I am OKAY.

Thm. Let A, A_1, A_2 be $m\times n$ matrices, B, B_1, B_2 be $n\times p$ matrices, $\alpha\in F$, then:

(1) $(AB)^t = B^tA^t$

(2) $A(B_1+B_2) = AB_1 + AB_2$

(3) $(A_1+A_2)B = A_1B + A_2B$

(4) $a(AB) = A(aB) = (aA)B$

(5) $I_m A_{m\times n} = A_{m\times n} I_n = A_{m\times n}$, where

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

(6) Write

$$B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{pmatrix}$$

$$\text{then } AB = A\begin{pmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{pmatrix} = \begin{pmatrix} -Ab_1- \\ \vdots \\ -Ab_n- \end{pmatrix}$$

(7) 即使 AB, BA 都存在, AB 也未必 = BA

(8) $AB=0$ 不見得代表 $A=0$ or $B=0$

pf. (1) $\because A^t = (A_{ji})_{n\times m}$, $B^t = (B_{ji})_{p\times n}$, and

$$C_{ij} \triangleq \sum_{k=1}^{n} A_{ik}B_{kj}$$

$$\therefore C_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{jk}$$

$$= \sum_{k=1}^{n} (B^t)_{ik}(A^t)_{kj}.$$

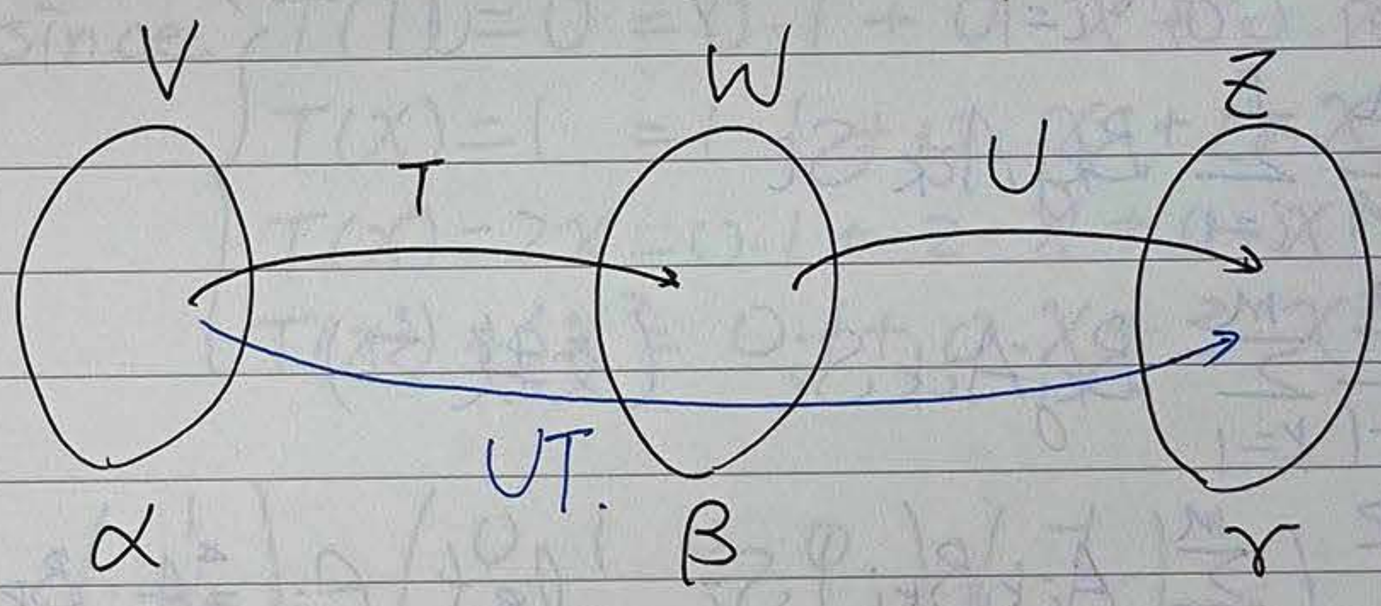
$\Rightarrow (AB)^t = B^t A^t$

(2) ~ (8) 等之，這些不是高中教的嗎？

Q.E.D.

Thm 2.11 Let V, W, Z be finite dimensional vector spaces with ordered bases α, β and γ.
Let $T: V \to W$ and $U: W \to Z$ be linear transformations. Then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

(線性函數的合成 $\leftrightarrow$ 矩陣相乘)



ex. $V = W = Z = F^2$, $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2)$
$U(a_1, a_2) = (a_2, a_1)$.
$(UT)(a_1, a_2) = U(T(a_1, a_2)) = (a_1 + a_2, a_1 - a_2)$
Let $C = UT : V \to Z$ defined by.
$(a_1, a_2) \mapsto (a_1 + a_2, a_1 - a_2)$
Take $\alpha = \beta = \gamma = \{(1,0), (0,1)\}$

$$\Rightarrow \begin{cases} C(1,0) = (1,1) = 1(1,0) + 1(0,1) \\ C(0,1) = (-1,1) = -1(1,0) + 1(0,1) \end{cases}$$

$$\Rightarrow \begin{cases} [T]_\alpha = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and } [C]_\alpha = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ [U]_\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}$$

I am OKAY.

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$[U]_\alpha \quad [T]_\alpha \quad [C]_\alpha$

pf. $\because C(u_j) = (UT)(u_j) = U(T(u_j))$

$$= U\left(\sum_{k=1}^{m} B_{kj} v_k\right) \quad (\text{by def})$$

$$= \sum_{k=1}^{m} B_{kj} U(v_k) \quad (\text{linear})$$

$$= \sum_{k=1}^{m} B_{kj} \left(\sum_{i=1}^{p} A_{ik} s_i\right) \quad (\text{by def})$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{p} B_{kj} A_{ik} s_i$$

$$= \sum_{i=1}^{p} \sum_{k=1}^{m} B_{kj} A_{ik} s_i \quad (\text{交換 } \Sigma)$$

$$= \sum_{i=1}^{p} \left(\sum_{k=1}^{m} A_{ik} B_{kj}\right) s_i, \text{ let } C_{ij} = A_{ij} B_{kj}$$

$$= \sum_{i=1}^{p} C_{ij} s_i$$

$\therefore (C_{ij}) = [UT]_\beta^\gamma$

Q.E.D.

Thm 2.14 Let V and W be vector spaces. $T: V \to W$ is a linear transform. Let β be an ordered basis of V and γ a basis of W, and $\dim(W) < \infty$. Then $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$, $\forall v \in V$

V W
$\|$ $\|$

ex. $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ is defined by $T(f(x)) = f'(x)$. Let $\beta = \{1, x, x^2, x^3\}$, $\gamma = \{1, x, x^2\}$

Let $v = x^3 - 2x^2 + x - 1$, $T(v) = 3x^2 - 4x + 1$

$= 1 \cdot 1 + (-4)x + (3)x^2$

$\Rightarrow [T(v)]_\gamma = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $[v]_\beta = \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}$

$\Rightarrow [T]_\beta^\gamma = (A_{ij})_{3\times 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

since $\begin{cases} T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ T(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 \end{cases}$

$\Rightarrow \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}$

pf. (Tricky!)

<法 I> (不 tricky 的方法) (Try yourself)

<法 II> (tricky)

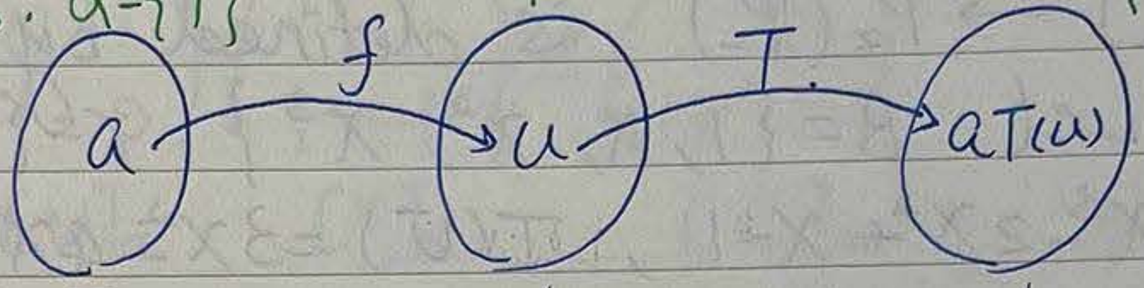
For fix $u \in V$, define $f: F \to V$ and $g: F \to W$ by $f(a) = au$, $g(a) = aT(u)$, where $a \in F$.

$\Rightarrow g = T \circ f$ since $Tf(a) = T(f(a)) = T(au)$

$\Rightarrow$ $\boxed{aT(u) = g(a)}$ //

I am OKAY.

ordered
basis: $\alpha=\{1\}$ $\quad \beta \quad \gamma=\{u_1,\cdots,u_m\}$



dim $\quad F\ 1 \quad V\ n \quad W\ m$

m維向量 → viewed as m×1 矩陣

$$\Rightarrow [T(v)]_\gamma = [\underbrace{g(1)}_{\in W}]_\gamma = [g]_\alpha^\gamma$$

(If $g(1)=b_1u_1+\cdots+b_mu_m$, then

$$[g]_\alpha^\gamma=\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = [g(1)]$$)

$$= [T]_\beta^\gamma [f]_\alpha^\beta \quad \text{by Thm 2.71}$$

$$= [T]_\beta^\gamma [f(1)]_\beta$$

$$= [T]_\beta^\gamma [v]_\beta$$

Q.E.D.

Thm 2.15. Let A be a m×n matrix. $L_A: F^n \to F^m$ is a linear transformation, where $L_A(x)=Ax$.
Let β, γ be standard ordered basis for F^n and F^m, then:

(a) $[L_A]_\beta^\gamma = A$

(b) $L_A = L_B \Rightarrow A=B$

(c) $L_{A+B}=L_A+L_B$ and L

(d) If $T: F^n \to F^m$ is linear, then $\exists!$ m×n matrix C s.t. $T=L_C$. In fact,

$C = [T]_\beta^\gamma$.

We say that $L_A(x)$ is a left-multiplication transformation. Note that

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

(e_1, e_2, e_n, n 個)

(e) $L_{AE} = L_A L_E$ (合成) ($m \times n$, $n \times p$) → identity mapping on F^n

(f) $m = n \Rightarrow L_{I_n} = I_{F^n}$

pf. (a) Let $\beta = \{e_1, \dots, e_n\}$, $\gamma = \{\bar{e}_1, \dots, \bar{e}_m\}$

$\because L_A(e_j) = Ae_j$

$$= \begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix}$$ → 矩陣 A 的第 j 行

$$= A_{1j}\bar{e}_1 + \cdots + A_{mj}\bar{e}_m$$

$\therefore [L_A]_\beta^\gamma = A$.

(b)(c) follows from Thm 2.10.

(d) $\because$ By Thm 2.14, $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$, and $[x]_\beta = x$ (since $x \in F^n$), $[T(x)]_\gamma = T(x)$

(ex. $x \in F^3$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$)

$\therefore T(x) = [T]_\beta^\gamma x$

$\Rightarrow$ Let $C = [T]_\beta^\gamma$, and the uniqueness

I am OKAY.

of C follows from (b).

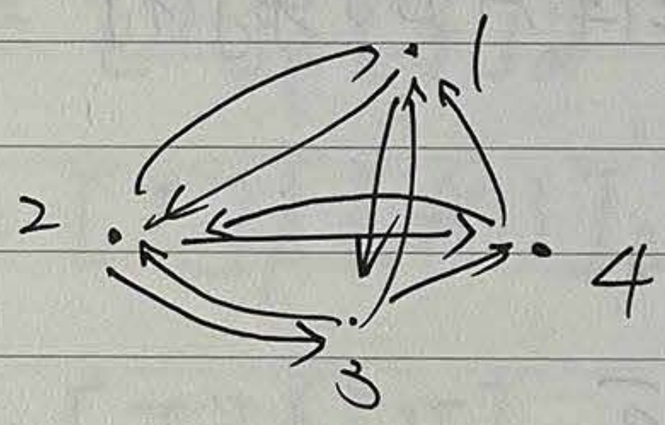
(e). $\because L_{AE}(x) = AE(x) = A(Ex) = L_A(Ex)$
$= L_A(L_E(x)) = (L_A \circ L_E)(x)$, $\forall x \in F^p$
$\therefore L_{AE} = L_A L_E$

(f) is obvious

Q.E.D.

△ Application to graph.

1. directed graph:



2. Complete graph: 任兩點都互通

問題: clique: A (maximum) complete subgraph in a graph
ex. {1,2,3}, {2,3,4} are clique, but {1,2,4}, {1,3,4} are not.

Hard Problems:
(1) Finding a clique
(2) Listing all clique

Graph → Incidence Matrix

$$\text{出}\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{pmatrix}\triangleq A \qquad \text{終}\ 1\ 2\ 3\ 4$$

(對角線放 0)

可回答的問題:

(1) 從 i 到 j 兩步內可到的走法總數

$$(A^2)_{13}=\sum_{k=1}^{4}A_{1k}A_{k3}=A_{11}A_{13}+A_{12}A_{23}+A_{13}A_{33}+A_{14}A_{43}$$

$\Rightarrow$ 總數 $=(A+A^2)_{13}$

$\Rightarrow$ 從 i 到 j 2步內可到的走法總數 $=(A+A^2)_{ij}$

$\Rightarrow$ 推論: 從 i 到 j, m 步內可到的走法

$=(A+A^2+\cdots+A^m)_{ij}$

(2) 給定一了 vertex i, 如何判斷 i 在一了 clique 中?

Let $B=\begin{cases}1, & \text{if } A_{ij}=A_{ji}=1\\ 0, & \text{otherwise}\end{cases}$

左圖之 $B=\begin{pmatrix}0&1&1&0\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{pmatrix}$

$i\in$ a clique $\Leftrightarrow (B^3)_{ii}>0$. (exercise 19)

I am OKAY.

3. Dominate Graph

$\forall i, j$, $A_{ij} = 1 \iff A_{ji} = 0$.

ex.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Facts: (1) $\exists i$ s.t. i 在2步内能reach其它点

(2) $\exists j$ s.t. j 被其它点找到

(3) $\exists ?$ i s.t. $A+A^2$ 的第 i 列的元素都 >0 (除了对角线) (exercise 21)

(4) $\exists j$ 全 j 行

2-4 Invertibility and Isomorphisms

Q: 若向量空間 V, W 之維度相同, 且 $\dim(V)=\dim(W)<\infty$, 則 V and W 「可看成」一樣的嗎?

A: 「可看成一樣」較字上指 $\exists$ L.T. $T: V\to W$ that is invertible

Def. Let $T: V\to W$ be a linear transform. We say T is invertible if $\exists\, U: W\to V$ s.t. $UT=I_V$, where I_V is the identity mapping in V, and $TU=I_W$.

Prop. Let U be an inverse of T, then U is unique $\Rightarrow$ write U as T^{-1}.

pf. Let U_1, U_2 be two inverse of T.
We want $U_1(w)=U_2(w)$, $\forall w\in W$.

$\because U_1(w)=(U_2T)(U_1(w))$
$=U_2((TU_1)(w))$
$=U_2(I_W(w))$
$=U_2(w)$, $\forall w\in W$

$\therefore$ U is unique

Q.E.D.

Thm. (1) A L.T. $T: V\to W$ is invertible $\Leftrightarrow$ T is 1-1 and onto!

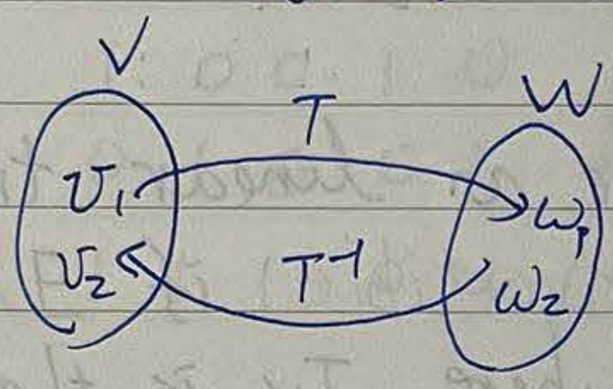
(2) If T_1, T_2 are invertible L.T., then Thm 2.17 $\Leftarrow$ T_1T_2 is well-defined and $(T_1T_2)^{-1}=T_2^{-1}T_1^{-1}$.

(3). If T is invertible, then so are T^{-1}.

I am OKAY.

pf. (2). (outlined:)
step 1: Show that $T_1 T_2$ is invertible
step 2: $(T_1 T_2)^{-1} = T_2^{-1} T_1^{-1}$.

(3). Claim: $T^{-1}: W \to V$ is linear



pf of claim: Let $w_1, w_2 \in W$ and $a \in F$.
We want $T^{-1}(aw_1 + w_2) = aT^{-1}(w_1) + T^{-1}(w_2)$.
$\because \exists\, v_1, v_2 \in V$ s.t. $T(v_1) = w_1$, $T(v_2) = w_2$
$\therefore T^{-1}(aw_1 + w_2) = T^{-1}(aT(v_1) + T(v_2))$
$= T^{-1}(T(av_1 + v_2))$
$= (T^{-1}T)(av_1 + v_2)$
$= Iv(av_1 + v_2)$
$= av_1 + v_2$
$= aT^{-1}(w_1) + T^{-1}(w_2)$
$\Rightarrow T^{-1}$ is linear.
$\Rightarrow$ by a similar argument, change $T^{-1} = S$ and $T = S^{-1}$, we can prove that T^{-1} is invertible.

Q.E.D.

Def. A: an $n \times n$ matrix.
We say that A is invertible if $\exists\, B$: $n \times n$ matrix s.t. $AB = BA = I_{n \times n}$

Remark. B is unique. We denote $B = A^{-1}$.

Q: $\dim(F^4)=4$, $\dim M_{2\times2}(F)=4$, $\dim P_3(F)=4$
這些空間「一樣」嗎？

lemma. Let $T: V\to W$ be an invertible L.T.
then $\dim(V)<\infty \iff \dim(W)<\infty$

pf. $\Rightarrow$) $\because \dim(V)<\infty$, let $\beta=\{v_1,\dots,v_n\}$ be an ordered basis for V
$\therefore$ by Thm 2.2, $R(T)=\text{span}\{T(v_1),\dots,T(v_n)\}$
$\because T$ is invertible
$\therefore T$ is onto
$\Rightarrow R(T)=W$.
$\Rightarrow \dim(W)<\infty$ by Thm.

$\Leftarrow$) 將 $\Rightarrow$) 中的 T 以 T^{-1} 取代即得.

Q.E.D.

Thm 2.18. Let V, W be vector spaces, $\dim(V)$, $\dim(W)<\infty$, β, γ are ordered basis for V and W respectively. Then a linear transform $T: V\to W$ is invertible $\iff$ $[T]_\beta^\gamma$ is invertible
In this case, $[T^{-1}]_\gamma^\beta=([T]_\beta^\gamma)^{-1}$

pf. $\Rightarrow$) $\because T$ is invertible
$\therefore$ by Thm 2.17, T is 1-1 and onto
$\Rightarrow \dim(V) = \text{Rank}(T) = \dim(W)$
1-1, dim Thm. onto

I am OKAY.

$\because \exists\, U: W \to V$ s.t. $UT = I_V$ and $TU = I_W$
(since T is invertible)

$\therefore \begin{cases} [UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = [I_V]_\beta = I_n \\ [TU]_\beta = [T]_\beta^\gamma [U]_\gamma^\beta = [I_W]_\gamma = I_n \end{cases}$

$\Rightarrow [T]_\beta^\gamma$ has an inverse $[U]_\gamma^\beta = [T^{-1}]_\gamma^\beta$

$\Rightarrow ([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$

$\Leftarrow$) Let $[T]_\beta^\gamma = A$ and $A^{-1} = B$. Let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$

(思考 | 1. 如何由 B 找 U?
→ U 的行為可由基底的行為來決定 (線性)
→ 以 v_i 之組合表達 w_j, 可得 U 之矩陣表達式)

$\because \exists!\, U: W \to V$, a linear transform, s.t.
$U(w_j) = \sum_{i=1}^{n} B_{ij} v_i$ by Thm 2.6

$\therefore [U]_\gamma^\beta = B$

$\because [UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$

$\therefore UT = I_V$

$\Rightarrow$ by an analogous argument, $TU = I_W$

$\Rightarrow T$ is invertible

Q.E.D.

Def. Two vector spaces V and W are isomorphic if $\exists$ an invertible L.T. $T: V \to W$ such T is called an isomorphism.
We denote $V \sim W$

ex. $V = P_3(F)$, $W = M_{2\times 2}(F)$.

Define $T(a+bx+cx^2+dx^3) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow$ T is linear

$\because N(T) = \{0\}$

$\therefore$ T is 1-1

$\Rightarrow$ T is invertible.

Thm. If $\dim(V), \dim(W) < \infty$, then $V \sim W$
$\Leftrightarrow \dim V = \dim W$

pf. $\Rightarrow$) $\because V \sim W$

$\therefore \exists\, T: V \to W$ which is linear and invertible

$\Rightarrow$ by dim. Thm, $\dim(V) = \text{Rank}(T) \overset{\text{onto}}{=} \dim(W)$

$\Leftarrow$) Let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_n\}$ be basis of V and W

$\Rightarrow$ by Thm 2.6, $\exists!\, T: V \to W$, a L.T, s.t. $T(v_j) = w_j$

* Needed next: T invertible?

(思考 | 1. $\text{Range}(T) = \text{span}\{T(v_1), \dots, T(v_n)\}$
$= \text{span}\{w_1, \dots, w_n\}$
$= W$.
$\Rightarrow$ T is onto)

I am OKAY.

$\because$ T is onto
$\therefore$ T is 1-1
$\Rightarrow$ T is invertible
$\Rightarrow V \sim W$

Q.E.D.

ex. $\because \dim P_3(F) = \dim M_{2\times 2}(F) = \dim F^4$
$\therefore P_3(F) \sim M_{2\times 2}(F) \sim F^4$

ex. $L(V,W) = \{T \mid T: V \to W \text{ is linear}\}$
If $\dim(V) = n$, $\dim(W) = m$, what is $\dim L(V,W)$?

Thm 2.20. Let V, W be vector spaces, $\dim V = n$, $\dim W = m$. Then $L(V,W) \sim M_{m\times n}(F)$, $\dim L(V,W) = mn$. " V " Z

pf. We want to find a L.T. $\Phi: U \to Z$ which is invertible.
Let $\Phi: L(V,W) \to M_{m\times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$, $\forall T \in L(V,W)$. β, γ are ordered basis of V, W.

$\because \Phi(aT_1 + T_2) = a\Phi(T_1) + \Phi(T_2)$ (trivial)
$\therefore \Phi$ is linear

To show that Φ is invertible, we show Φ is onto, that is, $\forall A \in M_{m\times n}(F)$, $\exists T \in L(V,W)$ s.t. $\Phi(T) = A$.

Let $A=(A_{ij})_{m\times n}\in M_{m\times n}$, then by Thm 2.6, $\exists!\ T: V\to W$ in $L(V,W)$ s.t.

$$T(v_j)=\sum_{i=1}^{m} A_{ij}w_i\ ,\ 1\le j\le n.$$

$\Rightarrow [T]_\beta^\gamma = A = \Phi(T)$, $\forall A\in M_{m\times n}$. $\Rightarrow \Phi$ is onto.

$\because Null(\Phi)=\{T: \Phi(T)=0\}=\{T: T(v_j)=0, \forall v_j\in\beta\}=\{0\}$ 零函数

$\therefore \Phi$ is 1-1

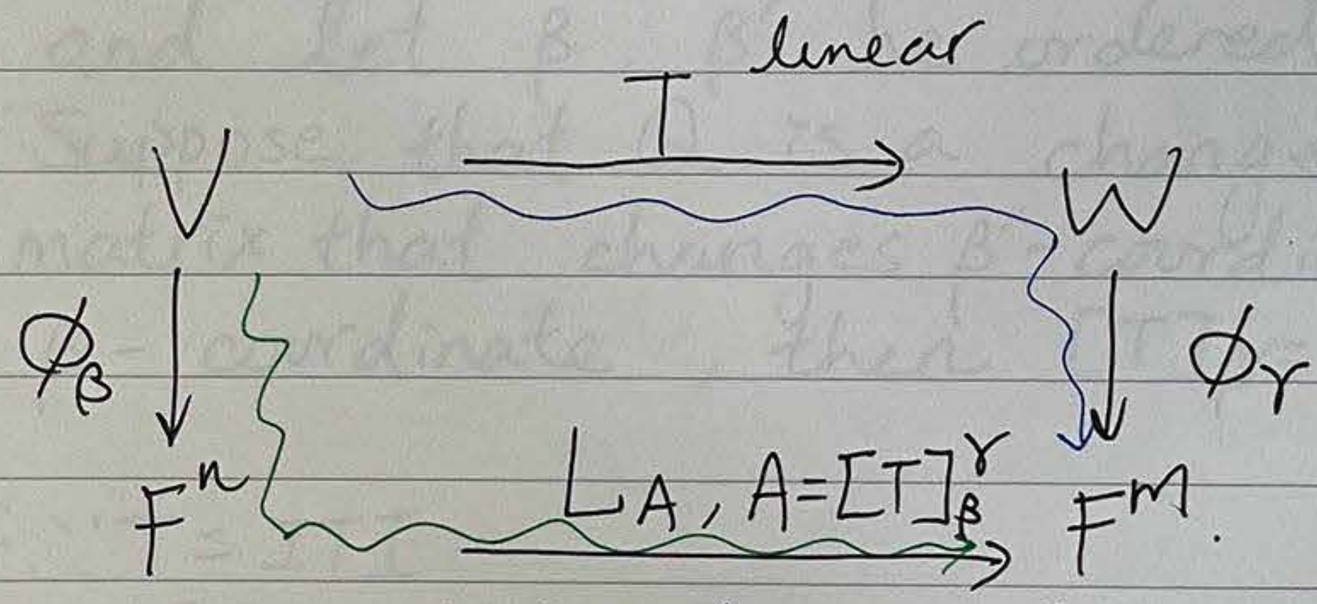
$\Rightarrow \Phi$ is invertible

Q.E.D.

Def: Let V be a vector space, β be an ordered basis for V, $\dim(V)=n$. Define $\phi_\beta: V\to F^n$ by $\phi_\beta(x)=[x]_\beta$, $\forall x\in V$. ϕ_β is called the standard representation of V w.r.t. an ordered basis β.

Thm 2.21. ϕ_β is an isomorphism from V to F^n

⇒ 該V往F^n看齊



$L_A(x)=Ax$, $x\in F^n$

⇒ 所有有限維向量空間上的線性變換都可以寫成矩陣乘向量！

I am OKAY.

$T: V \to W$, $L_A: F^n \to F^m$ 本質是一樣的，即：

$L_A \phi_\beta = \phi_\gamma T. \Rightarrow T = \phi_\gamma^{-1} L_A \phi_\beta$.

Let $v \in V$, $L_A \phi_\beta(v) = L_A(\phi_\beta(v)) = L_A([v]_\beta)$

$= A[v]_\beta = \underbrace{[T]_\beta^\gamma}_{\text{matrix}} \underbrace{[v]_\beta}_{\text{vector}} = \underbrace{[T(v)]_\gamma}_{\text{vector}}$

$= \phi_\gamma(T(v))$

$= (\phi_\gamma T)(v)$.

§2-5 Change of Coordinate Matrix

Q: $T: V \to V$ 為 linear operator. 若 β, β' 為 V 的 ordered bases. 則 $[T]_\beta$ 和 $[T]_{\beta'}$ 有何关係.

Thm. $I: V \to V$, $I(v) = v$, β, β' are basis of V
by Thm 2.22, (1) $[I]^\beta_{\beta'}$ is invertible
(2) $[v]_\beta = [I]^\beta_{\beta'} [v]_{\beta'}$

pf. (1) Of course!
(2) Trivial! $\because [I(v)] = v$
$\therefore [I(v)]_\beta = [v]_\beta = [I]^\beta_{\beta'} [v]_{\beta'}$

Q.E.D.

Def. Let $Q = [I]^\beta_{\beta'}$. Then Q is called a coordinate matrix. That is, Q change β'-coordinates to β-coordinate

Thm 2.24. Let T be a linear operator on a finite-dimensional vector space V, and let β, β' be ordered bases of V. Suppose that Q is a change of coordinate matrix that changes β'-coordinates into β-coordinate, then $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

pf. $\because T = ITI$
$\therefore [T]_{\beta'} = [ITI]_{\beta'} = [I]^{\beta'}_{\beta} [T]_\beta [I]^\beta_{\beta'}$
$= Q^{-1}[T]_\beta Q$

I am OKAY.

圖示：

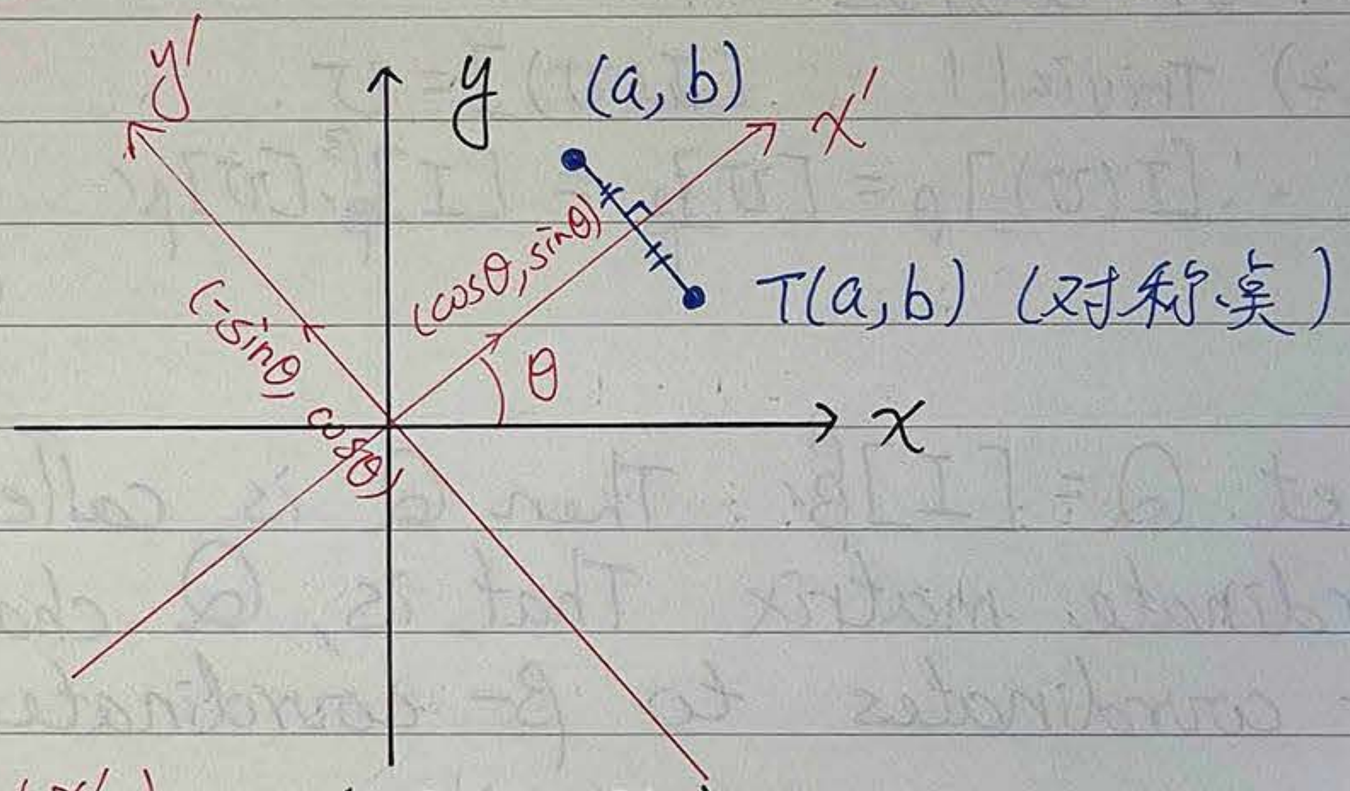
$$V_{\beta'} \xrightarrow{I} V_{\beta} \xrightarrow{I} V_{\beta} \xrightarrow{I} V_{\beta'}$$

Q.E.D.

Def. $A, B \in M_{n\times n}(F)$

A is said to be similar to B if $\exists$ Q: invertible s.t. $B = Q^{-1}AQ$

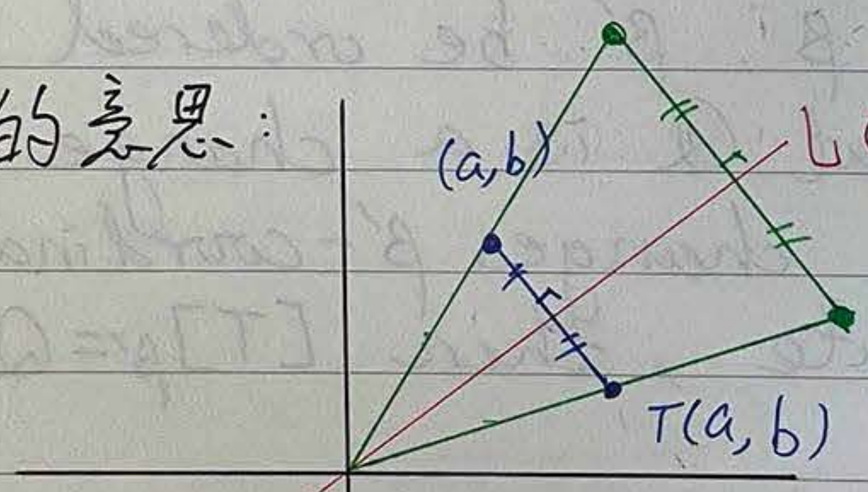
ex.



① $\begin{pmatrix} x' \\ y' \end{pmatrix} = (\ ?\ ) \begin{pmatrix} x \\ y \end{pmatrix}$

② $T(a,b) = (\ ?\ )(a,b)$

②的意思：



令T表对L对称、⇒ 可知 $T(\alpha(a,b)) = \alpha T(a,b)$

且 $T((a_1,b_1)+(a_2,b_2)) = T(a_1,b_1) + T(a_2,b_2)$

$\Rightarrow T$ 為線性變換.

-sol- ① $I(v)=v$, $I:\mathbb{R}^2\to\mathbb{R}^2$, 但是把基底從 $\beta=\{(1,0),(0,1)\}$ 改成 $\beta'=\{(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)\}$.

$$\Leftrightarrow [I]_{\beta}^{\beta'}[v]_{\beta}=[v]_{\beta'}$$

$$\Rightarrow \begin{pmatrix}x'\\y'\end{pmatrix}=[I]_{\beta}^{\beta'}\begin{pmatrix}x\\y\end{pmatrix}$$

$\because I((1,0))=a_1(\cos\theta,\sin\theta)+b_1(-\sin\theta,\cos\theta)=(1,0)$

$I((0,1))=a_2(\cos\theta,\sin\theta)+b_2(-\sin\theta,\cos\theta)=(0,1)$

$\therefore \begin{cases}a_1\cos\theta-b_1\sin\theta=1\\ a_1\sin\theta+b_1\cos\theta=0\end{cases}$, and

$\begin{cases}a_2\cos\theta-b_2\sin\theta=0\\ a_2\sin\theta+b_2\cos\theta=1\end{cases}$

$$\Rightarrow \begin{cases}a_1=\cos\theta\\ b_1=-\sin\theta\end{cases},\quad \begin{cases}a_2=\sin\theta\\ b_2=\cos\theta\end{cases}$$

$$\Rightarrow [I]_{\beta}^{\beta'}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \text{ or}$$

$$\begin{cases}x'=x\cos\theta+y\sin\theta\\ y'=-x\sin\theta+y\cos\theta\end{cases}$$

② $\mathbb{R}^2 \xrightarrow{I} \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{I} \mathbb{R}^2$

$\beta \qquad \beta' \qquad \beta' \qquad \beta$

$\because T=ITI$

$\therefore [T]_{\beta}=[I]_{\beta'}^{\beta}[T]_{\beta'}[I]_{\beta}^{\beta'}$

$$\Rightarrow \begin{cases} T(\cos\theta, \sin\theta) = 1\cdot(\cos\theta, \sin\theta) + 0(\cos\theta, \sin\theta) \\ T(-\sin\theta, \cos\theta) = -1(-\sin\theta, \cos\theta) \end{cases}$$

$$= 0(\cos\theta, \sin\theta) + (-1)(-\sin\theta, \cos\theta)$$

$$\Rightarrow [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\because \begin{cases} I(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta) \\ \qquad\qquad = \cos\theta(1,0) + \sin\theta(0,1) \\ I(-\sin\theta, \cos\theta) = (-\sin\theta, \cos\theta) \\ \qquad\qquad = -\sin\theta(1,0) + \cos\theta(0,1) \end{cases}$$

$$\therefore [I]_{\beta'}^{\beta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow [T]_\beta = [I]_{\beta'}^{\beta}\,[T]_{\beta'}\,[I]_{\beta}^{\beta'}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\Rightarrow [T(a,b)] = [T]_\beta \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

2-6 Dual Spaces.

Def. Let V be a vector space. a linear transform $f: V \to F$ is called a *functional*

Def. For a vector space V over F, we define the *dual space* of V to be the vector space $\mathcal{L}(V, F)$, denoted by V^*

Q: What is $\dim(V^*)$?

A: by Thm 2.20, $\dim(V^*) = \dim \mathcal{L}(V, F)$
$= \dim(V)\dim(F) = \dim(V)$.

Remark. We can also define V^{**}, the double dual of V.

Thm 2.24. Suppose that V is a finite dimensional vector space with the ordered basis $\beta = \{x_1, x_2, \dots, x_n\}$. Let f_i be the i^{th} coordinate function w.r.t. β, and $\beta^* = \{f_1, f_2, \dots, f_n\}$. Then β^* is an ordered basis for V^* and $\forall f \in V^*$,
$f = \sum_{i=1}^{n} f(x_i) f_i$.

Def. Let V be a finite dimensional vector space, and let $\beta = \{x_1, \dots, x_n\}$ be an ordered basis for V. If $x \in V$ and

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$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

and $a_1, \dots, a_n \in F$. Then define a function $f_i : V \to F$ by $f_i(x) = a_i$, $i = 1, \dots, n$. We say f_i is the i^{th} coordinate function w.r.t. the basis β. Note that $f_i(x_j) = \delta_{ij}$.

pf of Thm. Let $g \in V^*$, $g = \sum_{i=1}^{n} f(x_i) f_i$

$$\therefore g(x_j) = \left(\sum_{i=1}^{n} f(x_i) f_i\right)(x_j)$$

$$= \sum_{i=1}^{n} f(x_i) f(x_j)$$

$$= \sum_{i=1}^{n} f(x_i) \delta_{ij}$$

$$= f(x_j), \quad \forall j = 1, \dots, n$$

$\therefore g = f$ by Cor. of Thm 2-6

$\Rightarrow \beta^*$ generates V^*. since $\dim(V^*) = n$

$\Rightarrow \beta^*$ is a basis of V^* by Cor 2(a) of replacement Thm

Q.E.D.

Def. We call the ordered basis $\beta^* = \{f_1, \dots, f_n\}$ of V^* that satisfies $f_i(x_j) = \delta_{ij}$, $1 \le i, j \le n$, the dual basis of β.

ex. Let $\beta = \{(2,1), (3,1)\}$ be an ordered basis for $\mathbb{R}^2$. Find the dual basis of β, say $\beta^* = \{f_1, f_2\}$.

-sol-. $\because \begin{cases} 1 = f_1(2,1) \\ 0 = f_1(3,1) \end{cases}$ and that f_1, f_2 is linear

$\therefore \begin{cases} f_1(2,1) = 2f_1(e_1) + f_1(e_2) = 1 \\ f_1(3,1) = 3f_1(e_1) + f_1(e_2) = 0 \end{cases}$ by Thm 2.24.

$\Rightarrow \begin{cases} f_1(e_1) = -1 \\ f_1(e_2) = 3 \end{cases}$

Similarly, $\because \begin{cases} f_2(2,1) = 2f_2(e_1) + f_2(e_2) = 0 \\ f_2(3,1) = 3f_2(e_1) + f_2(e_2) = 1 \end{cases}$

$\therefore \begin{cases} f_2(e_1) = 1 \\ f_2(e_2) = -2 \end{cases}$

$\Rightarrow \forall (x,y) \in \mathbb{R}^2, \begin{cases} f_1(x,y) = -x + 3y \\ f_2(x,y) = x - 2y \end{cases}$ #.

Thm2.25. Let V, W be finite-dimensional vector spaces over F with ordered bases β and γ. For any L.T. $T \in \mathcal{L}(V,W)$, define the mapping $T^t : W^* \to V^*$ by $T^t(g) = gT$, $\forall g \in W^*$. Then T^t is a L.T., and $[T]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$

pf. $\because T^t(g) = gT$, $\forall g \in W^*$, and T^t is linear, (obvious)

$\therefore T^t(g) \in V^*$ and T^t maps W^* to V^*

Let $\beta = \{x_1, \cdots, x_n\}$ and $\gamma = \{y_1, \cdots, y_n\}$

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with dual basis $\beta^* = \{f_1, \dots, f_n\}$ and $\gamma^* = \{g_1, \dots, g_m\}$, and denote $A = [T]_\beta^\gamma$.

To find the j^{th} column of $[T^t]_{\gamma^*}^{\beta^*}$, we find $T^t(g_j)$ first.

$\because$ by Thm 2.24, $T^t(g_j) = g_j T = \sum_{s=1}^{n} (g_j T)(x_s) f_s$

$\therefore$ The row i, column j entry of $[T^t]_{\gamma^*}^{\beta^*}$ is

$$\begin{aligned}(g_j T)(x_i) &= g_j(T(x_i)) \\ &= g_j\left(\sum_{k=1}^{m} A_{ki} y_k\right) \\ &= \sum_{k=1}^{m} A_{ki} g_j(y_k) \\ &= \sum_{k=1}^{m} A_{ki} \delta_{jk} \\ &= A_{ji}\end{aligned}$$

$$\Rightarrow [T^t]_{\gamma^*}^{\beta^*} = A^t$$

Q.E.D.

Def. The linear transformation T^t defined in Thm2.25 is called the transpose of T. It is clear that T^t is the unique L.T., say U, s.t. $[U]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

ex. Define $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ by $T(p(x)) = (p(0), p(2))$. Let β, γ be the standard ordered bases for $P_1(\mathbb{R})$ and $\mathbb{R}^2$, respectively, we have

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. Find $[T^t]_{\gamma^*}^{\beta^*}$.

-sol-. Let $\beta^* = \{f_1, f_2\}$ and $\gamma^* = \{g_1, g_2\}$.

Suppose that $[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$T^t(g_1) = af_1 + cf_2$.

$\because T^t(g_1) = (af_1 + cf_2)(1)$

$= af_1(1) + cf_2(1)$

$= a(1) + c(0)$

$= a$, and

$T^t(g_1)(1) = g_1(T(1))$

$= g_1(1, 1)$

$= 1$

$\therefore a = 1$

$\Rightarrow$ by a similar argument, we may obtain $c = 0$, $b = 1$.

$\Rightarrow [T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = ([T]_\beta^\gamma)^t$ #.

Notation: Let V be a finite dimensional vector spaces. Define $\hat{x}: V^* \to F$ by $\hat{x}(f) = f(x)$, $\forall f \in V^*$

(1) $\hat{x}$ is a linear functional $\Rightarrow \hat{x} \in V^{**}$

(2) use the correspondence $x \longleftrightarrow \hat{x}$ to define an isomorphism between V and V^{**}

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lemma. Let V be a finite-dimensional vector space, and let $x \in V$. If $\hat{x}(f) = 0$, then $x = 0$.

pf. Suppose $x \neq 0$, choose an ordered basis $\beta = \{x_1, \dots, x_n\}$ for V s.t. $x_1 = x$
Let $\{f_1, \dots, f_n\}$ be the dual basis of β.
$\Rightarrow f_1(x_1) = 1 \neq 0$.
$\because f_1 \in V^*$, $\hat{x}(f_1) \neq 0$
$\therefore x \neq 0$ implies $\hat{x}(f_1) \neq 0$
$\Rightarrow \hat{x}(f) = 0$ implies $x = 0$

Q.E.D.

(2.26)
Thm. Let V be a finite dimensional vector space, and define $\Psi: V \to V^{**}$ by $\Psi(x) = \hat{x}$. Then Ψ is an isomorphism

pf. (1) Ψ is linear:
Let $x, y \in V$, $c \in F$. For $f \in V^*$, we have
$\because \Psi(cx+y)(f) = f(cx+y)$
$= cf(x) + f(y)$
$= c\hat{x}(f) + \hat{y}(f)$
$= (c\hat{x} + \hat{y})(f)$
$\therefore \Psi(cx+y) = c\hat{x} + \hat{y} = c\Psi(x) + \Psi(y)$.

(2) Ψ is 1-1:
Suppose $\Psi(x)$ is the zero functional on V^* for some $x \in V$, then by lemma, $\hat{x}(f) = 0$, $\forall f \in V^*$

$\Rightarrow x = 0$.

(c) ψ is isomorphism

$\because \dim(V) = \dim(V^{**})$

$\therefore \psi$ is an isomorphism between V and V^{**}

Q.E.D.

Cor. Let V be a finite dimensional vector space with dual space V^*. Then every ordered basis for V^* is the dual basis of $\{x_1, \dots, x_n\}$

pf. Let $\beta^* = \{f_1, \dots, f_n\}$ be an ordered basis for V^*.

$\because$ by Thm 2.24 and 2.26, β^* has a dual basis, say $\{\hat{x}_1, \dots, \hat{x}_n\}$ in V^{**}

$\therefore \delta_{ij} = \hat{x}_i(f_j) = f_j(x_i)$, $\forall i, j$.

$\Rightarrow \{f_1, \dots, f_n\}$ is the dual basis of $\{x_1, \dots, x_n\}$

Q.E.D.

*結論: A finite dimensional vector space is isomorphic to its double dual via the map $x \mapsto \hat{x}$

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Remark. This is not true for infinite-dimensional vector spaces. In this case, no two of V, V^*, and V^{**} are isomorphic.

Ch3: Elementary Matrix Operations and

§3.1 Elementary Matrix Operations and

Def. Elementary row (column) operations
type I: 列(行)交換
type II: 列(行)乘一非0純量
type III: 列(行)乘一非0純量加到另一列(行)
abbriveatly, e.r.o. or e.c.o.

ex. $A=\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix} \xrightarrow{I} A_1=\begin{pmatrix}2&4&6\\1&3&5\end{pmatrix}$

$\underset{II}{\downarrow}$

$A_2=\begin{pmatrix}-2&-6&-10\\2&4&6\end{pmatrix}$

Def. E: elementary matrix is a matrix derive from performing an elementary operation on I_n.

ex. $I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=E$

Q: $A \longrightarrow B$ by e.r.o or e.c.o.

$\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix} \xrightarrow[I,\text{e.r.o.}]{} \begin{pmatrix}2&4&6\\1&3&5\end{pmatrix}$

$I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow[I.\text{ero}]{} \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\Rightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix}$

System of Linear Equations.

Elementary Matrices

$$(-2)C_2\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix}\xrightarrow[\text{III}]{\text{e.c.o}}\begin{pmatrix}1&1&5\\2&0&6\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow[\text{III}]{\text{eco}}\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}=E$$

$$\Rightarrow\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix}\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&1&5\\2&0&6\end{pmatrix}$$

① $A_{m\times n}=\begin{pmatrix}-a_1-\\-a_2-\\\vdots\\-a_m-\end{pmatrix}\xrightarrow[\text{type III}]{\text{ero}}\begin{pmatrix}a_1\\a_2-2a_1\\a_3\\\vdots\\a_m\end{pmatrix}=B$

$I_m=\begin{pmatrix}-e_1-\\\vdots\\-e_m-\end{pmatrix}\xrightarrow[\text{type III}]{\text{ero}}\begin{pmatrix}e_1\\e_2-2e_1\\\vdots\\e_m\end{pmatrix}$

$$\Rightarrow EA=\begin{pmatrix}e_1A\\(e_2-2e_1)A\\e_3A\\\vdots\\e_mA\end{pmatrix}=\begin{pmatrix}a_1\\a_1-2a_2\\a_3\\\vdots\\a_m\end{pmatrix}$$

② $A=\begin{pmatrix}|&&|\\a_1&\cdots&a_n\\|&&|\end{pmatrix}\xrightarrow[\text{type III}]{\text{eco}}B=\begin{pmatrix}|&|&|&&|\\a_1,&a_2-a_1,&a_3,&\cdots,&a_n\\|&|&|&&|\end{pmatrix}$

$I_n=\begin{pmatrix}|&&|\\e_1&\cdots&e_n\\|&&|\end{pmatrix}\xrightarrow[\text{type III}]{\text{eco}}E=\begin{pmatrix}|&|&|&&|\\e_1,&e_2-2e_1,&e_3&\cdots&e_n\\|&|&|&&|\end{pmatrix}$

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$$\Rightarrow AE = \begin{pmatrix} | & | & | & & | \\ Ae_1 & A(e_2-2e_1) & A_3 & \cdots & Ae_n \\ | & | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | & & | \\ a_1 & a_2-2a_1 & a_3 & \cdots & a_n \\ | & | & | & & | \end{pmatrix}$$

Thm 3.1 Let $A \in M_{n\times n}(F)$, and
$\begin{cases} A \xrightarrow{ero} B \\ A \xrightarrow{eco} B_1 \end{cases}$, then $\exists$ elementary matrices E and E_1 s.t. $B=EA$ and $B=AE_1$.
In fact, E & E_1 的獲得由 $A \to B$ 是一樣.

Thm 3.2 Let E be an elementary matrix, then: (1) E is invertible
(2) E^{-1} is of the same type of that of E

§3-2 The rank of a Matrix and Matrix Inverses:

Def. The rank of a matrix $A = \text{Rank}(A) \triangleq \text{rank}(L_A)$ where $L_A : F^n \to F^m$ and $L_A(x) = Ax$.

Thm 3.3. Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces, and let β and γ be ordered bases for V and W. Then $\text{rank}(T) = \text{rank}([T]_\beta^\gamma)$

Recall.

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \text{1-1 onto linear } \phi_\beta \downarrow & & \downarrow \phi_\gamma \\ F^n & \xrightarrow[L_A]{} & F^m \end{array}$$

"commute"

$L_A \phi_\beta = \phi_\gamma T$

$\Rightarrow T = \phi_\gamma L_A \phi_\beta^{-1}$

where $L_A : F^n \to F^m$ and $L_A(x) = Ax$

pf. $\because T = \phi_\gamma^{-1} L_A \phi_\beta$

$\therefore \text{Rank}(T) = \dim(T(V))$, where $T(V) = \{T(v) \mid v \in V\}$

$\hookrightarrow = \dim(\phi_\gamma^{-1} L_A \phi_\beta)(V)$

$= \dim(\phi_\gamma^{-1} L_A(F^n))$

$\overset{?}{=} \dim(L_A(F^n))$ —(*).

$= \text{rank}(L_A)$

$= \text{rank}(A)$

$= \text{rank}([T]_\beta^\gamma)$

Where (*) holds by exercise 2.4.17. (The exercise states that isomorphism is ranking-preserving)

Q.E.D.

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Thm. Let $A \in M_{m\times n}(F)$, $P \in M_{n\times n}(F)$, $Q \in M_{m\times m}(F)$
Suppose P, Q are invertible, then:

(1) $\text{rank}(AP) = \text{rank}(A)$

(2) $\text{rank}(QA) = \text{rank}(A)$

(3) $\text{rank}(QAP) = \text{rank}(A)$

pf. (1) $\text{rank}(AP) = \text{rank}(L_{AP}) = \text{rank}(L_A L_P)$

$= \dim(L_A L_P(F^n))$.

L_P is onto $= \dim(L_A(L_P(F^n)))$

$= \text{rank}(L_A(F^n))$

$= \text{rank}(L_A)$

$= \text{rank}(A)$

(2) $\text{rank}(QA) = \text{rank}(L_{QA})$

$= \text{rank}(L_Q L_A)$

$= \dim(L_Q L_A(F^n))$ → V'

rank preserving exercise 2.4.17 $= \dim(L_Q(L_A(F^n)))$

$= \dim(L_A(F^n))$

$= \text{Rank}(L_A)$

$= \text{Rank}(A)$

(3). This follows directly from (1)(2).

Q.E.D.

Cor. e.r.o and e.c.o are rank-preserving operations.

pf. Suppose $A \xrightarrow{\text{e.c.o}} B$, then by Thm 3.1, $\exists$ E: elementary matrix s.t. $EA = B$.

∵ by Thm 3.2, E is invertible
∴ by Thm 3.4(ii) $\operatorname{rank}(EA) = \operatorname{rank}(A)$
$\Rightarrow \operatorname{rank}(B) = \operatorname{rank}(A)$

Q.E.D.

Thm. Let $A \in M_{m\times n}(F)$,

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

then $\operatorname{Rank}(A) = \dim \operatorname{span}\{a_1, \ldots, a_n\}$
= the maximum number of linearly indep. vectors $a_1, \ldots, a_n$.

ex. $$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}$$

$\Rightarrow \operatorname{rank}(I) = 3$

ex. $$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}$$

$\Rightarrow \operatorname{rank}(A) = 2$

pf. $\operatorname{rank}(A) = \operatorname{rank}(L_A) = \dim \operatorname{range}(L_A)$
$= \dim L_A(F^n)$, where $L_A : F^n \to F^m$
$= \dim \operatorname{span}\{Ae_1, \ldots, Ae_n\}$, where $\{e_1, \ldots, e_n\}$ is ordered basis for F^n.
$= \dim \operatorname{span}\{a_1, \ldots, a_n\}$

Q.E.D.

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Thm 3.6. Suppose $\operatorname{rank}(A)=r$, where A is an $m\times n$ matrix, $r\le m, r\le n$.
then by a finite number of e.c.o and e.r.o, A can be transformed into

$$A=\begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where O_1 is a $r\times(n-r)$ zero matrix
O_2 " $(m-r)\times r$ "
O_3 " $(m-r)\times(n-r)$ "
and $r\le \min\{m,n\}$

pf. May assume $A\neq 0$. Prove by induction on m.
When $m=1$, $A=(a_1,\dots,a_n)$
$\because A\neq 0$
$\therefore a_i\neq 0$ for some i
$\Rightarrow$ change a_i, a_1, then $A=(a_i, a_2,\dots, a_1,\dots,a_n)$
$\Rightarrow$ use type II e.c.o., $A=(1, \frac{a_2}{a_i},\dots,\frac{a_1}{a_i},\dots,\frac{a_n}{a_i})$
$\Rightarrow$ by a series of type III e.c.o.
$A=(1,0,\dots,0)$

When A is a $m\times n$ matrix, $\exists A_{ij}\neq 0$.

$$A=\begin{pmatrix} & \vdots & \\ \cdots & A_{ij} & \cdots \\ & \vdots & \end{pmatrix}_{m\times n} \begin{matrix} \\ i \\ \\ \end{matrix}$$
$$\qquad\qquad\quad j$$

$\Rightarrow$ by a type I e.c.o and e.r.o.

$$A=\left(\begin{array}{c|c} A_{ij} & \\ \hline & A_{11} \end{array}\right) \begin{matrix} i \\ \\ j \end{matrix}$$

$\Rightarrow$ by type II e.r.o.

$$A=\left(\begin{array}{c|c} 1 & ? \\ \hline ? & ? \end{array}\right)$$

$\Rightarrow$ by type III e.c.o. and e.r.o.

$$A=\left(\begin{array}{c|c} 1 & O_{1\times(n-1)} \\ \hline O_{(m-1)\times 1} & B_{(m-1)\times(n-1)} \end{array}\right)$$

$\Rightarrow$ by induction hypothesis, B can be transformed into

$$B=\begin{pmatrix} I_{r-1} & O_1' \\ O_2' & O_3' \end{pmatrix}$$

$\Rightarrow$ A can be transformed into

$$A \longrightarrow D=\begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

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and $r \leq \min\{n, m\}$

Q.E.D.

Cor1. Let A be an $m \times n$ matrix, $\text{rank}(A) = r$
$\Rightarrow \exists$ invertible matrix $B: m \times n$, $C: n \times n$
s.t.

$$BAC = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix} = D$$

pf. $\because$ by Thm 3.2, $\exists E_1', \ldots, E_k', E_1, \ldots, E_p$ (elementary matrices) s.t.
$E_k' \cdots E_1' A E_1, \cdots E_p = D$
$\therefore$ We may let $E_k' \cdots E_1' = B$, $E_1 \cdots E_p = C$

Q.E.D.

Cor2. Let A be an $m \times n$ matrix, then:
(1) $\text{rank}(A) = \text{rank}(A^t)$
(2) $\text{rank}(A)$ = maximum number of linearly indep. column (or row) vectors of A.

pf. $\because D = BAC$ by Cor 1
$\therefore D^t = C^t A^t B^t$

$$\Rightarrow D^t = \begin{pmatrix} I_r^t & O_2^t \\ O_1^t & O_3^t \end{pmatrix} = \begin{pmatrix} I_r & O_2^t \\ O_1^t & O_3^t \end{pmatrix}$$

$\Rightarrow \text{rank}(A) = \text{rank}(D) = \text{rank}(D^t) = \text{rank}(A^t)$

Q.E.D.

Cor 3. Let A be an $n \times n$ & invertible matrix. then $\exists$ elementary matrices $E_1, \dots, E_p$ s.t. $A = E_1 \cdots E_p$ for some $p \in \mathbb{N}$.

pf. $\because$ A is invertible

$\therefore$ L_A is invertible and thus onto

$\Rightarrow \operatorname{rank}(L_A) = \operatorname{rank}(A) = n$.

$\Rightarrow$ by Cor 2, $\exists\, E_1, \dots, E_p, E_1', \dots, E_k'$, s.t. $E_k' \cdots E_1' A E_1 \cdots E_p = I_n$, (elementary matrices)

$\Rightarrow A = (E_k)^{-1} \cdots (E_1')^{-1} (E_p)^{-1} \cdots (E_1)^{-1}$. We can rename the matrices by $E_1, \dots, E_p$.

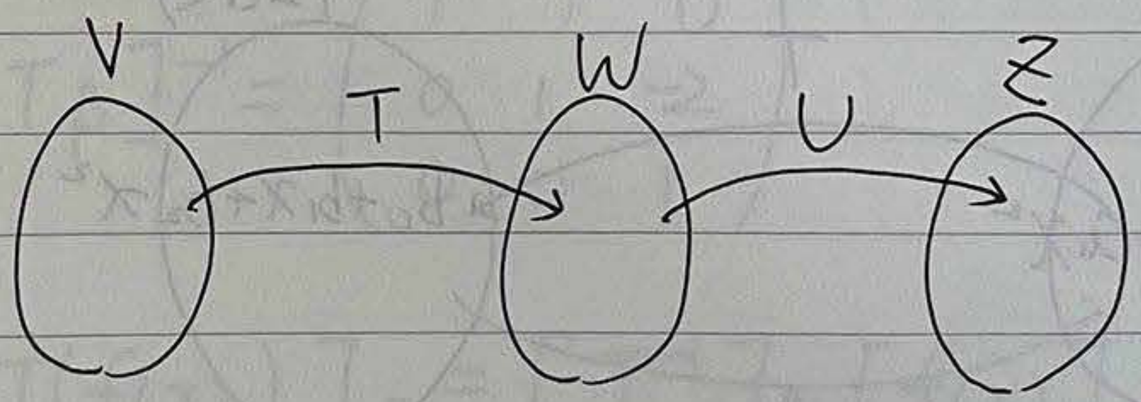
Q.E.D.

Thm 3.17. Let $T: V \to W$ and $U: W \to Z$ be linear transformations on finite dimensional vector spaces V, W and Z, and let A and B be matrices s.t. AB is defined. then: (1) $\operatorname{rank}(UT) \leq \operatorname{rank}(U)$

(2) $\operatorname{rank}(UT) \leq \operatorname{rank}(T)$

(3) $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$

(4) $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$



I am OKAY.

△ The inverse of matrix

Def. Augmented Matrix.
Let $A \in M_{m\times n}(F)$, $B \in M_{m\times p}(F)$,
$(A|B)_{m\times(n+p)}$

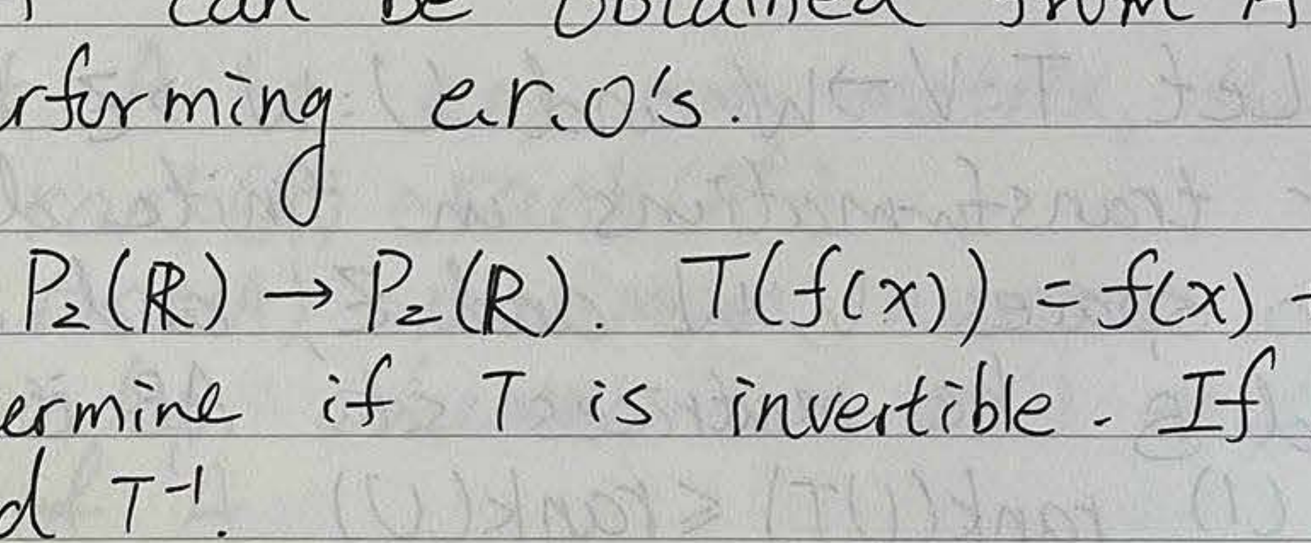
$\star$ $(A|I) \xrightarrow[\text{Thm 3.6}]{} (E_p \cdots E_1 | I)$

$\Rightarrow (E_p'(E_p \cdots E_1) | I)$
$\Rightarrow (E_{p-1} \cdots E_1 | E_p^{-1})$
$\Rightarrow \cdots$
$\Rightarrow (I | E_1^{-1} \cdots E_p^{-1})$
$\Rightarrow (I | A^{-1})$.

Facts: A^{-1} can be obtained from A by performing e.r.o's.

ex. $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$. $T(f(x)) = f(x) + f'(x) + f''(x)$.
Determine if T is invertible. If so, find T^{-1}.

-sol-. <法I>. Let $f(x) = a_0 + a_1 x + a_2 x^2$
$\Rightarrow T(f) = (a_0 + a_1) + (a_1 + 2a_2)x + a_2 x^2$
$\triangleq b_0 + b_1 x + b_2 x^2$.



$$\therefore \begin{cases} a_2 = b_2 \\ a_1 + 2a_1 = b_1 \\ a_0 + a_1 + 2a_2 = b_0 \end{cases}$$

$$\therefore \begin{cases} a_2 = b_2 \\ a_1 = b_1 - 2b_2 \\ a_0 = b_0 - a_1 - 2a_2 = b_0 - b_1 \end{cases}$$

$$\Rightarrow T^{-1}(b_0 + b_1 x + b_2 x^2) = a_0 + a_1 x + a_2 x^2$$
$$= (b_0 - b_1) + (b_1 - 2b_2)x + b_2 x^2$$
$$\Rightarrow T^{-1}(a_0 + a_1 x + a_2 x^2) = (a_0 - a_1) + (a_1 - 2a_2)x + a_2 x^2$$

〈法二〉 Let $\beta = \{1, x, x^2\}$.

$$\therefore \begin{cases} T(1) = 1 + 0 + 0 = 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) = x + 1 \quad = 1 + 1 \cdot x + 0 \cdot x^2 \\ T(x^2) = x^2 + 2x + 2 = 2 + 2 \cdot x + 1 \cdot x^2 \end{cases}$$

$$\therefore [T]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$[T]_\beta^{-1}$

$$\Rightarrow [T_\beta]^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow [T(f(x))]_\beta = [T^{-1}]_\beta [f(x)]_\beta$$

I am OKAY.

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_0 - a_1 \\ a_1 - 2a_2 \\ a_2 \end{pmatrix}$$

$$\Rightarrow T^{-1}(a_0 + a_1 x + a_2 x^2) = (a_0 - a_1) + (a_1 - 2a_2)x + a_2 x^2 \quad \#$$

Note: A: nxn matrix

(1) A invertible:

$$(A|I) \xrightarrow[\text{eco}]{\text{ero or}} (I|A^{-1})$$

(2) A not invertible

$$(A|I) \xrightarrow[\text{or eco}]{\text{ero}} \left(\begin{matrix} X \\ 0 \cdots 0 \cdots 0 \\ X \end{matrix} \middle| B\right)$$ 發不回 I.

ex. $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

(1) Find A^{-1} If exists

(2) Write A as a product of elementary matrices if possible.

-sol-.

$$(A|I) = \left(\begin{matrix} 1 & 2 \\ 3 & 5 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \rfloor \times(-3)$$

$$\Rightarrow \left(\begin{matrix} 1 & 2 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ -3 & 1 \end{matrix}\right) \times(-1)$$

(In fact, $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$)

$\hookrightarrow E_1$ $\quad$ $\hookrightarrow A$

$$\Rightarrow \left(\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 3 & -1 \end{matrix}\right) \rfloor \times(-2).$$

$\bar{E}_1$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right) = (I \mid A^{-1}) \Rightarrow A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$\Rightarrow \overline{E_3}\,\overline{E_2}\,\overline{E_1}\,I = A$. $\overline{E_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\overline{E_3} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$ #

I am OKAY.

§3-3 Systems of Linear Equations - Theoreti

Def. Let A be an $m \times n$ matrix
" x " $n \times 1$ " (未知数)
then $Ax=b$ is called linear system
of m equations with n unknowns ($b: m \times 1$)

ex. $\begin{cases} x+y+z=2 \\ x-y-z=3 \end{cases}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Def. ① If $Ax=b$ has a solution, then the system is said to be consistent.
Otherwise, it is said to be inconsistent
② If $b=0$, then $Ax=b$ is said to be homogenous. Otherwise, $Ax=b$ is said to be nonhomogenous

Thm 3.8. Let $K_H = \{x : Ax=0\}$. Then
(1) $K_H = N(L_A)$
(2) $\dim K_H = n - \text{rank}(A)$

Cor. If $n>m$, then $K_H - \{0\} \neq \emptyset$.

pf. $\because \text{rank}(A) < \min\{m, n\} = m$
$\therefore \dim(K_H) = n - \text{rank}(A) > 0$
$\Rightarrow K_H - \{0\} \neq \emptyset$

Q.E.D.

·cal Aspects

Thm 3.9 Let $K=\{x\in F^n : Ax=b\}$ and let s be a solution of $Ax=b$. Then $K=s+K_H$.
$\triangleq \{s+\bar{x} \mid \bar{x}\in K_H\}$
$\Rightarrow Ax=b$ 的一般解 = 一个特殊解 + 齐次解.

pf: (1) "$\subseteq$". Let $x\in K$, then $Ax=b$.
$\because As=b$
$\therefore A(x-s)=0$
$\Rightarrow x-s\in K_H$.
$\Rightarrow \exists\, \bar{x}\in K_H$ s.t. $x-s=\bar{x}$
$\Rightarrow x=\bar{x}+s\in K_H$, $\forall x\in K$
$\Rightarrow K\subseteq s+K_H$

"$\supseteq$" If $x\in s+K_H$, then $\exists\, \bar{x}\in K_H$ with $x=s+\bar{x}$
$\Rightarrow Ax=A(s+\bar{x})=As+A\bar{x}=b+0=b$
$\Rightarrow x\in K$, $\forall x\in s+K_H$
$\Rightarrow s+K_H\subseteq K$
$\Rightarrow K=s+K_H$

Q.E.D.

ex. $\begin{cases} x_1+2x_2+x_3+x_4=1 \\ x_2-x_3+x_4=1 \end{cases}$ —(*)

(1) The dimension of the solution set of its homogeneous equation and its basis
(2) all solutions to (*)

I am OKAY.

-sol-.

(1) Let $\begin{cases} x_1+2x_2+x_3+x_4=0 \\ x_2-x_3+x_4=0 \end{cases}$

Let $x_3=t_1,\ x_4=t_2$

$$\Rightarrow \begin{cases} x_1+2x_2=-t_1-t_2 \\ x_2=t_1-t_2 \end{cases}$$

$$\Rightarrow \begin{cases} x_1=-3t_1+t_2 \\ x_2=t_1-t_2 \\ x_3=t_1 \\ x_4=t_2 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3t_1+t_2 \\ t_1-t_2 \\ t_1 \\ t_2 \end{pmatrix}$$

$$= t_1\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix},\ t_1, t_2 \in \mathbb{R}$$

$\Rightarrow \dim K_H = 2$, basis for K_H =

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(2) 一般解

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t_1\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix},\ t_1, t_2 \in \mathbb{R}$$

隨便找的

Thm 3.10. Let A be an $n \times n$ matrix. Then $Ax=b$ has a unique solution $\Leftrightarrow$ A is invertible.

pf. $\Rightarrow$) Let s be the unique solution of $Ax=b$

$\therefore K = S + K_H = \{s\}$ by Thm 3.9.

$\therefore K_H = \{0\}$

$\Rightarrow Null(L_A) = \{0\}$

$\Rightarrow L_A$ is 1-1

$\because A$ is an $n \times n$ matrix

$\therefore L_A$ is onto

$\Rightarrow A$ is invertible

$\Leftarrow$) Let s be any solution to $Ax=b$

$\Rightarrow As=b$

$\Rightarrow s = A^{-1}b$, which is unique.

Q.E.D.

Thm 3.11 $Ax=b$ is consistent $\Leftrightarrow$ $rank(A) = rank(A|b)$

pf. $Ax=b$ is consistent

$\Leftrightarrow b \in Range(A)$

$\Leftrightarrow span(A) = span(A|b)$

$\Leftrightarrow rank(A) = rank(A|b)$

Q.E.D.

I am OKAY.

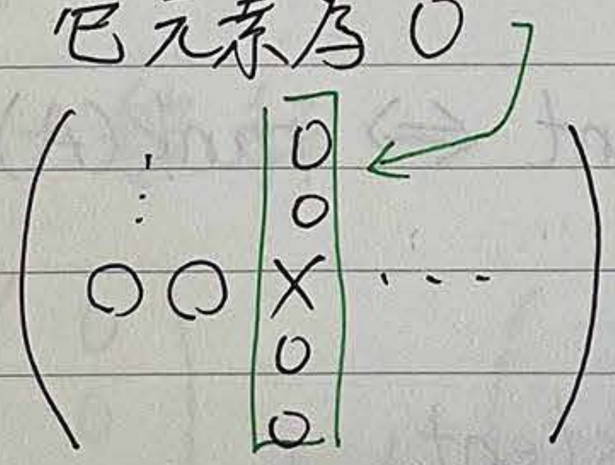
§3-4 System of Linear Equations -

Def. Two linear systems are said to be equivalent if their solu-

Thm. Let C be an invertible $m \times m$ matrix. Then the linear system $Ax=b$ is equivalent to $(CA)x = Cb$.

Recall: e.r.o's $\leftrightarrow$ elementary matrices $\leftrightarrow$ invertible
$\Rightarrow$ 可把C視為一連串的 elementary matrices.
$\Rightarrow$ CA愈簡單愈好

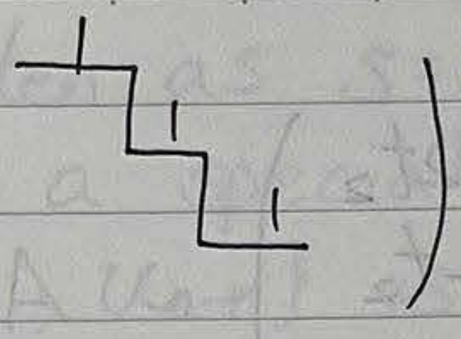
Def. Reduced Row Echelon Form (RREF)
A matrix is said to be in RREF if the following 3 conditions are satisfied:
(1) 若有全0的列，一定要在非零列下方
(2) 任意列的第一个非零元素，其相对的行之其它元素为0



(3). 任意列的第一个非零元素必为1，愈上列的1要在愈左边.

ex. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ 不滿足 (2), (3) $\rightarrow$ 非 RREF.

Computational Aspects

要表得像 

ex.

$$(A|b)=\left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right)$$

e.r.o. $\Rightarrow$ REF

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & \\ 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

forward pass

e.r.o. $\Rightarrow$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

backward pass

此称「高斯消去法」 → 若 $A \in M_{n\times m}(\mathbb{R})$ 且 $n \geq m$, 约需 $\frac{2}{3}n^3 \in O(n^3)$

大型 system → Iterative Methods.

$$\Rightarrow \begin{cases} x_1 + 2x_3 - 2x_5 = 3 \\ x_2 - x_3 + x_5 = 1 \\ x_4 - 2x_5 = 2 \end{cases}$$

$\Rightarrow$ Let $x_3 = t_1$, $x_5 = t_2$, then $\begin{cases} x_4 = 2 + 2t_2 \\ x_2 = 1 + t_1 - t_2 \\ x_1 = 3 - 2t_1 + 2t_2 \end{cases}$,

I am OKAY.

$t_1, t_2 \in \mathbb{R}$.

$A \geq$ kernel.

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3-2t_1+2t_2 \\ 1+t_1-t_2 \\ t_1 \\ 2+2t_2 \\ t_2 \end{pmatrix}$

$=$ $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

$=$ A basis for the solution set of the homo. system

特殊解 ($t_1 = t_2 = 0$ 時)

Remark. ① $(A|b) \xrightarrow{ero} (B|b')$ RREF

number of 非零列 of B = rank of A

$\Rightarrow \exists M$: invertible of A s.t. $B = MA$

② The set of homo. system $= Null(A) \Rightarrow$ Nullity(A) $= n - p$

Thm 3.14. Gaussian elimination transforms any matrix into its reduced row echelon form.

Thm 3.15 Let $A \in M_{m\times n}(F)$. 若:

(1) A 有 r 行非 0 的列

(2) $rank(A|b) = rank(A)$ (以保證 $Ax=b$ 有解)

(3) $(A|b)$ is in r.r.e.f.

則: (1) $rank(A) = r$.

(2) The general solutions of $Ax=b$

can be written as $S = S_0 + d_1u_1 + \cdots + d_{n-r}u_{n-r}$, where S_0 is a special solution of $Ax=b$, and $\{u_1, \ldots, u_{n-r}\}$ is a basis for homogeneous system, $d_i \in F$.

Thm 3.16. Let $A \in M_{m\times n}(F)$, $\operatorname{rank}(A)=r$.
If B is the r.r.e.f. of A, then:
(1) $\operatorname{rank}(B)=r$
(2) $\exists\, b_{j_i} = e_i \ (i=1,2,\ldots,r)$
(3) $\{a_{j_i}\}_{i=1}^{r}$ is linearly independent.
(4) If $b_k = \sum_{i=1}^{r} d_i e_i$, then $a_k = \sum_{i=1}^{r} d_i a_{j_i}$

Here, $A = \left(\begin{smallmatrix}|&&|\\ a_1 & \cdots & a_n \\ |&&|\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}|&&|\\ b_1 & \cdots & b_n \\ |&&|\end{smallmatrix}\right)$

ex.
$$(A|b) = \left(\begin{array}{ccccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right)$$

$$\to (B|b') = \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & \end{array}\right)$$

$e_1 = b_1$, $e_2 = b_2$, $e_3 = b_4$; $a_{j_1} = a_1$, $a_{j_2} = a_2$, $a_{j_3} = a_4$
(b_{j_1}, b_{j_2}, b_{j_3})

I am OKAY.

則 $b_5 = \begin{pmatrix} -2 \\ 1 \\ -2 \\ 0 \end{pmatrix} = -2e_1 + 1e_2 - 2e_3$

$a_5 = \begin{pmatrix} -9 \\ -3 \\ -5 \\ -8 \end{pmatrix} = -2a_1 + 1a_2 - 2a_4$

pf.(3) Let $\sum_{i=1}^{r} c_i a_{j_i} = 0$. Let M be a matrix

s.t. $MA = B$, then

$\because \sum_{i=1}^{r} c_i M(a_{j_i}) = 0 = \sum_{i=1}^{r} c_i b_{j_i} = \sum_{i=1}^{r} c_i e_i$

$\therefore c_i = 0, \forall i = 1, \cdots, r.$

(4) $\because b_k = \sum_{i=1}^{r} d_i b_{j_i}$

$\therefore Ma_k = \sum_{i=1}^{r} d_i M a_{j_i} = M(\sum_{i=1}^{r} d_i a_{j_i})$

$\because M$ is invertible

$\therefore a_k = \sum_{i=1}^{r} d_i a_{j_i}$

Q.E.D.

ex. Let $S = \{2x^3 + x^2 + x + 2,\ 3x^3 + x^2 + x + 2,\ x^3 + x^2 + x + 2,$
$4x^3 + x^2 + 2x + 3,\ -9x^3 - 3x^2 - 5x - 8\}$

從 S 中找出「最大」的線性獨立子集.

-sol- 以 $\{1, x, x^2, x^3\}$ 為 basis，則每個多項式都可表達為向量，且恰為前例中的矩陣之行

$\Rightarrow$ 可取 $\{2x^3+x^2+x+2,\ x^3+x^2+x+2,\ 4x^3+x^2+2x+3\}$

ex. $V=\{(x_1,\dots,x_5)\in\mathbb{R}^5 : x_1+7x_2+5x_3-4x_4+2x_5=0\}$

Show that (1) V is a subspace of $\mathbb{R}^5$

(2) $S=\{(-2,0,0,-1,-1),(1,1,-2,-1,1),(-5,1,0,1,1)\}$ is a linearly indep. subset of V.

(3) Find a basis of V containing S.

-sol-.

$x_1+7x_2+5x_3-4x_4+2x_5=0$ ↓齊次

思考：
$\Rightarrow$ 自由度為 4 $\Rightarrow \dim V=4$
$\Rightarrow$ 若 S 為 l.i.，找一向量加入 S 中即成 V 之基底

Let $A\in M_{5\times1}(\mathbb{R})$, $\because A=(1\ 7\ 5\ -4\ 2)$ is of r.r.e.f.

$$\therefore \begin{cases} x_1=-7t_1+5t_2-4t_3+2t_5 \\ x_2=t_1 \\ x_3=t_2 \\ x_4=t_3 \\ x_5=t_4 \end{cases}, \quad t_1,t_2,t_3,t_4\in\mathbb{R}$$

$$\Rightarrow \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = \begin{pmatrix} -7t_1-5t_2+4t_3-2t_4\\ t_1\\ t_2\\ t_3\\ t_4 \end{pmatrix}$$

I am OKAY.

$$= t_1 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$u_1 \quad u_2 \quad u_3 \quad u_4$

$$\Rightarrow \begin{pmatrix} -2 & 1 & -5 & -7 & -5 & 4 & -2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

r.r.e.f.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad -(*)$$

$\Rightarrow S \cup \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of V. $-(**)$

Ans: (1) is obvious

(2) by (*), S is a linear indep. subset of V by Thm 3.15

(3) $\because$ by (*), only u_3 is linear indep. with S

$\therefore S \cup \{u_3\}$ is a linearly indep. basis for V containing S by (**)

#

Ch.4. Determinants

*本章常用證明技巧：

(1) 數學歸納法

(2) elementary matrices. ∵證明对 elementary matrix 对 ⇒ invertible 也对.

§4.1. Determinants of Order 2.

Def. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is defined by $\det(A)=|A| \triangleq ad-bc$

Thm. Determinant is a linear function of each row when the other row is fixed. That is:

(1) $\begin{vmatrix} u+kv \\ w \end{vmatrix} = \begin{vmatrix} u \\ w \end{vmatrix} + k \begin{vmatrix} v \\ w \end{vmatrix}$

(2) $\begin{vmatrix} w \\ u+kv \end{vmatrix} = \begin{vmatrix} w \\ u \end{vmatrix} + k \begin{vmatrix} w \\ v \end{vmatrix}$

$\left(A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u \\ w \end{pmatrix}, u, w 为列向量 \right)$

pf. 直接代入 a, b, c, d 即可.

Thm 4.2 $A_{2\times 2}$ is invertible $\Longleftrightarrow \det(A) \neq 0$.

Moreover, $A^{-1}=\frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$

pf. $\Rightarrow$) $\because$ A is invertible

$\therefore \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}, \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ is linearly indep.

$\Rightarrow A_{11}^2 + A_{21}^2 \neq 0$.

(1) If $A_{11} \neq 0$. Apply e.r.o. on A

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \xrightarrow{\times(-\frac{A_{21}}{A_{11}})} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{12}\frac{A_{21}}{A_{11}} \end{pmatrix}$$

$\Rightarrow A_{22} - \frac{A_{12}A_{21}}{A_{11}} \neq 0 \Rightarrow A_{11}A_{22} - A_{12}A_{21} \neq 0$

$= \det(A)$

$\Rightarrow \det(A) \neq 0$

(2) If $A_{21} \neq 0$, 同理可證.

$\Leftarrow$) Let $M = -\frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$

It can be shown $AM = MA = I$ directly.

Q.E.D.

Def. Given $\{u, v\}$ an ordered basis for $\mathbb{R}^2$.

Define the orientation of $\{u, v\}$ by.

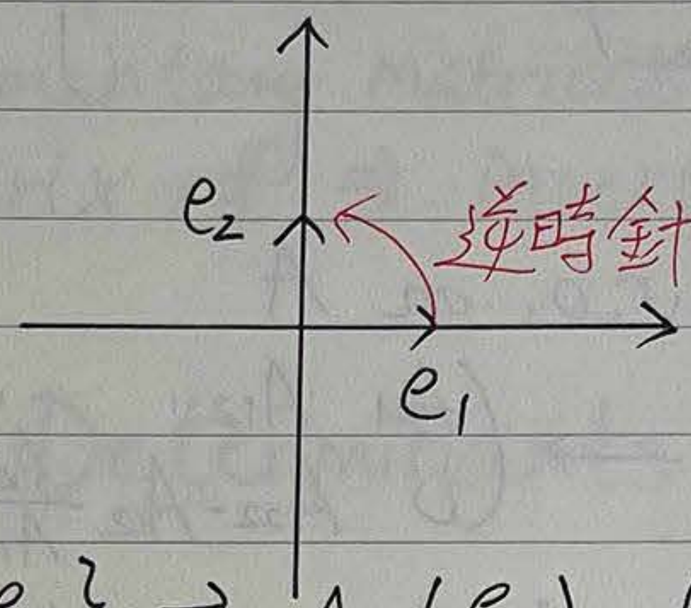
$$O\begin{pmatrix} u \\ v \end{pmatrix} = a \triangleq \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|} \quad \begin{matrix} \because \text{是 basis} \\ \Rightarrow \text{不會是 } 0 \end{matrix} = \begin{cases} 1 \\ -1 \end{cases}$$

(1) If $a = 1$, then $\{u, v\}$ is a right-handed coordinate.

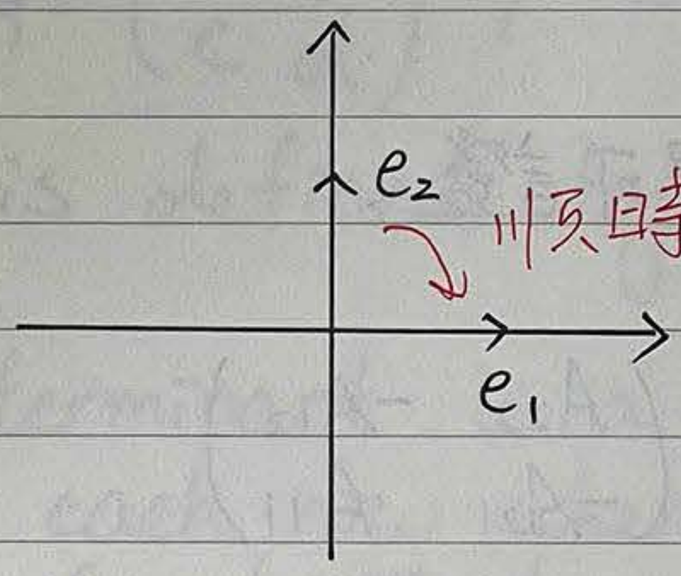
(2) If $a = -1$, " left-handed coordinate

I am OKAY.

ex. $\{e_1, e_2\} \Rightarrow A = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

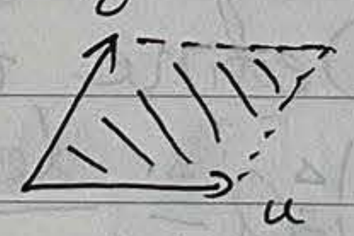


$\{e_2, e_1\} \to A = \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



△行列式值的几何意義

Thm. $\det\begin{pmatrix} u \\ v \end{pmatrix} = O\begin{pmatrix} u \\ v \end{pmatrix} A\begin{pmatrix} u \\ v \end{pmatrix}$, $u, v \in \mathbb{R}^2$

where $A(u, v)$ is the area of 

pf. We need ex11. first.

Claim: (ex11.) Let $\delta: M_{2\times 2}(F) \to F$ be a function with the following 3 properties:

(1) δ is linear

(2) $\delta(I_2) = 1$

(3) $\delta\begin{pmatrix} u \\ u \end{pmatrix} = 0$

then $\delta = \det$, that is, $\delta(A) = ad - bc$.
We assume the result is true.

Let $\delta\begin{pmatrix} u \\ v \end{pmatrix} = O\begin{pmatrix} u \\ v \end{pmatrix} A\begin{pmatrix} u \\ v \end{pmatrix}$,

$O\begin{pmatrix} u \\ v \end{pmatrix} = \begin{cases} 1, & \text{if } \{u, v\} \text{ is right-handed} \\ -1, & \text{" left-"} \end{cases}$

then (2), (3) is obviously satisfied.

Now we want to prove:

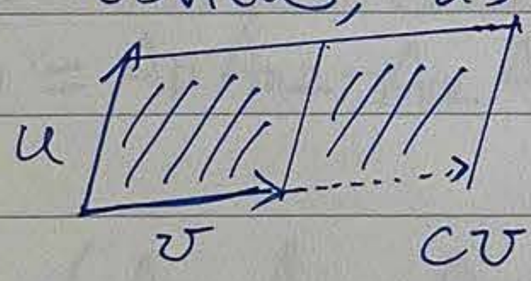
(a) $\delta\begin{pmatrix} u \\ cv \end{pmatrix} = c\delta\begin{pmatrix} u \\ v \end{pmatrix}$, $c \in R$.

(b) $\delta\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}$

(c) $\delta\begin{pmatrix} cu \\ v \end{pmatrix} = c\delta\begin{pmatrix} u \\ v \end{pmatrix}$

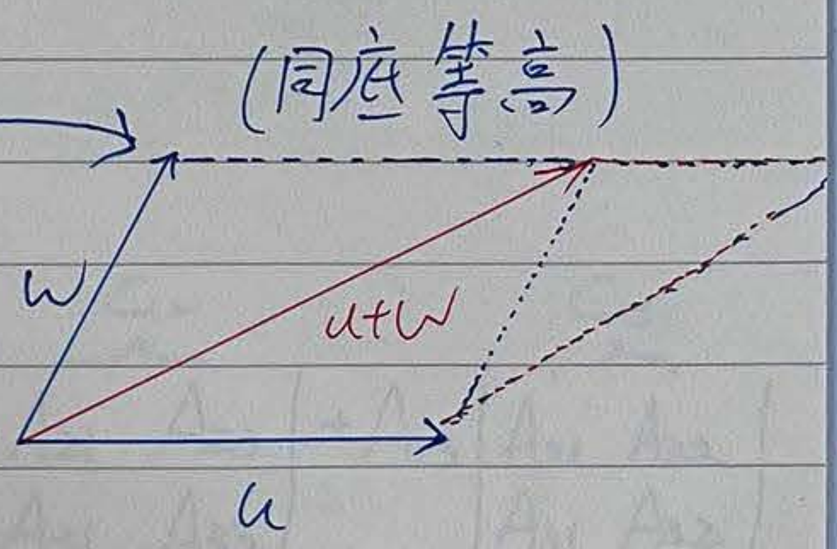
(d) $\delta\begin{pmatrix} u_1 + u_2 \\ v \end{pmatrix} = \delta\begin{pmatrix} u_1 \\ v \end{pmatrix} + \delta\begin{pmatrix} u_2 \\ v \end{pmatrix}$

(a). Is obvious, as the following diagram shows:



(b). Claim: $\delta\begin{pmatrix} u \\ w \end{pmatrix} = \delta\begin{pmatrix} u \\ u+w \end{pmatrix}$ ①

and $\delta\begin{pmatrix} u \\ au+bw \end{pmatrix} + b\delta\begin{pmatrix} u \\ w \end{pmatrix}$ ②



I am OKAY.

proof of claim: If $a=0$, then the claim is true. If not,

$$\delta\begin{pmatrix} u \\ au+bw \end{pmatrix} = \delta\begin{pmatrix} u \\ a(u+\frac{b}{a}w) \end{pmatrix} \overset{(a)}{=} a\,\delta\begin{pmatrix} u \\ u+\frac{b}{a}w \end{pmatrix}$$

$$\overset{①}{=} (a)\left(\frac{b}{a}\right)\delta\begin{pmatrix} u \\ w \end{pmatrix} = b\,\delta\begin{pmatrix} u \\ w \end{pmatrix}, \text{ Done.}$$

Now pick w so that $\{u, w\}$ forms a basis for $\mathbb{R}^2$. Let $\begin{cases} v_1 = a_1u + b_1w \\ v_2 = a_2u + b_2w \end{cases}$

$$\Rightarrow \delta\begin{pmatrix} u \\ v_1+v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ (a_1+a_2)u + (b_1+b_2)w \end{pmatrix}$$

$$\overset{②}{=} (b_1+b_2)\,\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

$$= b_1\,\delta\begin{pmatrix} u \\ w \end{pmatrix} + b_2\,\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

$$\overset{(a)}{=} \delta\begin{pmatrix} u \\ b_1w \end{pmatrix} + \delta\begin{pmatrix} u \\ b_2w \end{pmatrix}$$

$$\overset{②}{=} \delta\begin{pmatrix} u \\ a_1u+b_1w \end{pmatrix} + \delta\begin{pmatrix} u \\ a_2u+b_2w \end{pmatrix}$$

Q.E.D.

§4.2 Determinants for Order n.

Let $A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$

Def. (1) If $A=(A_{11})$, then $\det(A_{11}) = A_{11}$

(2) $\det(A) = |A| = \sum_{j=1}^{n} A_{ij}(-1)^{1+j} \det(\tilde{A}_{ij})$

$= \sum_{j=1}^{n} A_{ij} C_{ij}$ = cofactor expansion along the first row of A. Here, $\tilde{A}_{ij}$ = the matrix obtained from A by deleting the first row

ex. $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$$\det(A) = A_{11}(-1)^{1+1}\det\tilde{A}_{11} = A_{12}(-1)^{1+2}\det(\tilde{A}_{12})$$
$$= A_{11}\det(A_{22}) - A_{12}\det(A_{22})$$
$$= A_{11}A_{22} - A_{12}A_{21}.$$

ex. $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$$\det(A) = A_{11}\underbrace{\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}}_{C_{11}} - A_{12}\underbrace{\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix}}_{C_{12}} + A_{13}\underbrace{\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}}_{C_{13}}$$

I am OKAY.

ex. $I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$$\det(I) = 1 \cdot \det(I_{n-1}) = 1 \cdot 1 \cdot \det(I_{n-2})$$
$$= \cdots = 1^{n-1} \det((1)) = 1.$$

Thm 4.3 $\det(A)$ is n-linear. That is,

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}}_{B} + k \cdot \det\underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}}_{C}$$

pf. Induction on n.

(1) $n=1$ is obviously true

(2) For any n, if the Thm is true for $n-1$, then it is true for n

① If $r=1$, $\det(A) = \sum_{j=1}^{n} A_{1j}(-1)^{1+j} \det(\tilde{A}_{1j})$

$$= \sum_{j=1}^{n} (B_{1j} + kC_{1j})(-1)^{1+j} \det(\tilde{A}_{1j})$$

$$= \sum_{j=1}^{n} B_{1j}(-1)^{1+j} \det(\tilde{B}_{1j}) + k\sum_{j=1}^{n} C_{1j}(-1)^{1+j} \det(\tilde{C}_{1j})$$

$\uparrow$

$\tilde{A}_{1j} = \tilde{B}_{1j} = \tilde{C}_{1j}$

$$= \det(B) + k\det(C)$$

② If $r>1$

$$\det(A)=\sum_{j=1}^{n} A_{1j}(-1)^{1+j}\det(\tilde{A}_{1j})$$

$\because A_{1j}=B_{1j}=C_{1j}$

$\therefore \det(A_{ij})=\det(\tilde{B_{ij}})+k\det(\tilde{C_{ij}})$

$$\Rightarrow \det(A)=\sum_{j=1}^{n} A_{1j}(-1)^{1+j}\left(\det(\tilde{B_{ij}})+k\det(\tilde{C_{ij}})\right)$$

↑ induction hypothesis

$$=\sum_{j=1}^{n} A_{1j}(-1)^{1+j}\det(\tilde{B_{ij}})+k\sum_{j=1}^{n} A_{1j}(-1)^{1+j}\det(\tilde{C_{ij}})$$

$$=\sum_{j=1}^{n} B_{1j}(-1)^{1+j}\det(\tilde{B_{ij}})+k\sum_{j=1}^{n} C_{1j}(-1)^{1+j}\det(\tilde{C_{ij}})$$

↑ $\tilde{A_{1j}}=\tilde{B_{1j}}=\tilde{C_{1j}}$

$$=\det B+k\det(C).$$

⇒ by the induction process, the Thm is true for $n\times n$ matrices.

Q.E.D.

Lemma. Let $B=\begin{pmatrix} b_1 \\ \vdots \\ b_{r-1} \\ e_k \\ b_r \\ \vdots \\ b_n \end{pmatrix}$ → 第r列, $e_k=(0\cdots0\,|\,0\cdots0)$ ↑ k^{th}

then $\det(B)=1\cdot(-1)^{r+k}\det(B_{rk})$ = 可沿著第r列展開

pf: Similar to Thm 4.3

I am OKAY.

Cor. If $A \in M_{n\times n}(F)$ has a row $(0 \cdots 0)$, then $\det(A) = 0$.

Thm 4.4. $\det(A) = \sum_{j=1}^{n} A_{ij}(-1)^{i+j}\det(\tilde{A}_{ij})$ for any

$A \in M_{n\times n}(F)$. ⇒ 可沿任一列展開

pf. Let $A = \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ a_n \end{pmatrix}$, $a_i = A_{i1}e_1 + A_{i2}e_2 + \cdots + A_{in}e_n$

$$\Rightarrow \det(A) = \det\begin{pmatrix} a_1 \\ \vdots \\ \sum_{j=1}^{n} A_{ij}e_j \\ \vdots \\ a_n \end{pmatrix}$$

Thm 4.3 ↓

$$= \sum_{j=1}^{n} A_{ij}\det\begin{pmatrix} a_1 \\ \vdots \\ e_j \\ \vdots \\ a_n \end{pmatrix} \leftarrow \text{第 } i \text{ 列}$$

lemma. ↓

$$= \sum_{j=1}^{n} A_{ij}(-1)^{i+j}\det(\tilde{A}_{ij})$$

Q.E.D.

問題在於這种降階法 $\in O(n!)$，不太實際.

Q: How the e.r.o's on A affect the determinant of A?

Prop. (1).

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} = -\det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$$

(交換一次行或列一次負号)

(2). $A=\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \to \det\begin{pmatrix} a_1 \\ \vdots \\ ka_r \\ \vdots \\ a_n \end{pmatrix} = k\det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$

(3) $$\det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ ka_r+a_s \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix}$$

Lemma. Let $B\in M_{n\times n}(F)$, $n\geq 2$. If

$$B=\begin{pmatrix} & & \downarrow k^{th}\text{ column} & & \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix} \leftarrow i^{th}\text{ row}$$

, then $\det(B)$ 可沿第 i 列展開. That is, $\det(B)=(-1)^{i+k}\det(B_{ik})$.

pf: 太長, 置於章尾

Cor. $$\det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} = 0$$

pf. (1) $n=2 \to$ obvious

(2) For $n>3$, Suppose the Cor. is true for $(n-1)$, then

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} = \text{沿不一樣的列展開} \overset{\text{induction hypothesis}}{=} 0$$

Q.E.D.

I am OKAY.

§4.3 Properties of Determinants

Prop. Properties of determinant ($A, B \in M_{n\times n}(F)$)

(1). n-linear

(2) Can expanded along any row (column)

(3) If A has two identical rows, then $\det(A)=0$

(4) Let E_1 be a type I e.r.o. elementary matrix $\Rightarrow -\det(A) = \det(E_1 A)$

(5) " E_2 "
$\Rightarrow k\det(A) = \det(EA)$

(6) " E_3 "
$\Rightarrow k\det(A) = \det(EA)$

(7) $\det(AB) = \det(A)\det(B)$ (Thm 4.7)

(8) $\det(A) = \det(A^t)$

Remark. 满足(1),(3),以及 $\delta: M_{n\times n}(F) \to F$, $\delta(I_n) \triangleq 1$ 之函数必为行列式 (prove in 4.5)

pf. (1)(2)(3) 已证

(4) Let
$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} \longrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} a_1 \\ \vdots \\ a_r+a_s \\ \vdots \\ a_r+a_s \\ \vdots \\ a_n \end{pmatrix} \overset{(3)}{=} 0$$

$$\overset{(1)}{=} \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} + \det(A) + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$$

$$\overset{(3)}{=} \det(A) + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$$

$$\Rightarrow -\det(A) = \det(E_i A).$$

(待續)

ex $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{\text{e.r.o.}} \begin{pmatrix} \square & & \\ & \square & \\ 0 & & \square \end{pmatrix}$ → 对角线乘积 = det

$$\overset{\times(-3)}{\underset{\hookrightarrow}{\left[\right.}} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \overset{\times(-2)}{\left.\right]} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & -7 & -5 \end{pmatrix} \overset{\times(-\frac{1}{5})}{\left.\right]}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & 0 & -18/5 \end{pmatrix} \Rightarrow \det\begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & 0 & -18/5 \end{pmatrix}$$ 沿第三列展開

$$= \left(-\frac{18}{5}\right)\begin{vmatrix} 1 & 3 \\ 0 & -5 \end{vmatrix}(-1)^{3+3} = \left(-\frac{18}{5}\right) \times 1 \times (-5) \overset{(1)}{=} 18 = \det(A) \quad \#$$

⇒下三角矩陣的行列式值 = 对角线相乘!

I am OKAY.

ex. Find $\det(A)$.

$$A = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}_{(n+1)(n+1)}$$

-sol-. Consider the function $F: \mathbb{R} \to M_{(n+1)\times(n+1)}(\mathbb{R})$

$$F(x; c_0, \ldots, c_n) = \begin{pmatrix} 1 & c_0 & \cdots & c_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_{n-1} & \cdots & c_{n-1}^n \\ 1 & x & \cdots & x^n \end{pmatrix}$$

$\Rightarrow \det F(x; c_0, \ldots, c_n)$ 為 $-n$ 次多項式，其根為 $c_0, c_1, \ldots, c_{n-1}$.

$\Rightarrow \det F(x; c_0, \ldots, c_n) = a(x-c_0)(x-c_1)\cdots(x-c_n)$, for some $a \in \mathbb{R}$.

假設延著最後一列展開，

$\det(F) = 1 \cdot (-1)^{1+(n+1)}(F_{1j}) + x(-1)^{2+(n+1)}(\tilde{F}_{2j}) + \cdots$

$$+ x^n(-1)^{(n+1)+(n+1)} \begin{vmatrix} 1 & c_0 & \cdots & c_0^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_{n-1} & \cdots & c_{n-1}^{n-1} \end{vmatrix} = a.$$

$\Rightarrow \det(A) = \det F(c_n; c_0, c_1, \ldots, c_n)$

$= a(c_n - c_0)\cdots(c_n - c_{n-1})$

$= a'(c_{n-1} - c_0)\cdots(c_{n-1} - c_{n-2})(c_n - c_0)\cdots(c_n - c_{n-1})$

$= a''(c_{n-2} - c_0)\cdots(c_{n-2} - c_{n-3})(c_{n-1} - c_0)\cdots(c_{n-1} - c_{n-2})$

$(c_n - c_0)\cdots(c_n - c_{n-1})$

$$= \cdots = \prod_{0 \le i \le j \le n} (c_j - c_i)$$

pf. (5~7) (case i) If A is an elementary matrix.

(a). Let $A = E_1$, $\because E_1$ is obtained by interchanging two rows of I_n

$\therefore \det(E_1) = -1$ by Prop(1).

$\Rightarrow \det(AB) = \det(E_1 B) = -\det(B) = \det(E_1)\det(B)$ by (4).

(b). $A = E_2$, then by (1), $\det(E_2) = k$ for some k.

$\Rightarrow \det(AB) = \det(E_2 B) = k\det(B) = \det(A)\det(B)$

(c) If $A = E_3$, then $\det(E_3) = 1$ by Prop(3).

$\Rightarrow \det(AB) = \det(E_3 B) = \det(B) = \det(A)\det(B)$

(case ii) If A is an invertible matrix, then by Thm, $\exists F_1, \ldots, F_p$ are elementary matrices s.t. $A = F_1 \cdots F_p$

$$\begin{aligned} \Rightarrow \det(AB) &= \det(F_1 \cdots F_p B) \\ &= \det(F_1)\det(F_2 \cdots F_p B) \\ &= \det(F_1 \cdots F_p)\det(B) \\ &= \det(AB). \end{aligned}$$

(case iii) If A is not invertible, then

$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is not full rank

I am OKAY.

$\Rightarrow \exists\ i$ s.t. $a_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} c_j a_j$

$\Rightarrow A \xrightarrow{\text{e.r.o's}} \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 0 \\ a_i \\ \vdots \\ a_n \end{pmatrix}$ Cor $\Rightarrow \det(A) = 0$.

$\Rightarrow \det(A) = 0$

$\because$ AB is also not full rank

$\therefore \det(AB) = 0 = \det(A)\det(B)$.

(8) (Case i). If A is an elementary matrix

(a) If $A = E_1$ is a type I elementary matrix, then $E_1^t = E_1$

$\Rightarrow \det(E_1) = \det(E_1^t)$

(b) $\because E_2 = E_2^t$, where E_2 " II "

$\therefore \det(E_2) = \det(E_2^t)$

(c) If $A = E_3$ is a type III elementary matrix, then E_3 is a lower triangular matrix

$\Rightarrow E_3^t$ is an upper "

$\Rightarrow \det(E_3) = \det(E_3^t) =$ product of the diagonal $= 1$.

(Case ii) Let A be an invertible matrix, then $\exists$ elementary matrices $F_1, \ldots, F_p$ s.t.

$A = F_1 \cdots F_p$

$\Rightarrow A^t = F_p^t \cdots F_1^t$

$\Rightarrow \det(A) = \det(F_1) \cdots \det(F_p)$

(7)

(1)

$$= \det(F_1^t) \cdots \det(F_p)^t$$
$$= \det(F_p^t) \cdots \det(F_1^t)$$
$$= \det(F_p^t \cdots F_1^t)$$
(7) $$= \det(A^t)$$

(case iii) If A is not invertible, then $\det(A)=0$

$\because$ A^t is not invertible, too

$\therefore \det(A^t)=0$

$\Rightarrow \det(A) = \det(A^t)=0$

Q.E.D.

Cor. A is invertible $\Leftrightarrow \det(A)\neq 0$. If this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

pf. $\Rightarrow$) $\because$ A is invertible

$\therefore AA^{-1}=I$

$\Rightarrow \det(AA^{-1}) = \det(I) = 1 = \underbrace{\det(A)}_{\neq 0}\det(A^{-1})$ (7)

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

$\Leftarrow$) Suppose A is not invertible, then A is not full rank. Let $A=\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$

$\Rightarrow \exists\, i$ s.t. $a_i = \sum_{j=1, j\neq i}^{n} c_j a_j$

$\Rightarrow \begin{pmatrix} | & | & & | & & | \\ a_1 & a_2 & \cdots & a_i & \cdots & a_n \\ | & | & & | & & | \end{pmatrix}$, we can transform

$\times(-c_1)$ from a_i to a_1; $\times(-c_2)$ from a_i to a_2; $\times(-c_n)$ from a_i to a_n

I am OKAY.

A into $\begin{pmatrix} | & | & & | & & | \\ d_1 & d_2 & \cdots & 0 & \cdots & d_n \\ | & | & & | & & | \end{pmatrix}$

$\Rightarrow$ by Cor, $\det(A) = 0$ $\rightarrow\leftarrow$

$\Rightarrow$ A is ~~A~~ invertible

Q.E.D.

Thm. (Cramer's Rule)

Let $Ax=b$ be the matrix form of a system of n linear equations in n unknowns.

$\Rightarrow$ If $\det(A) \neq 0$, then $Ax=b$ has a unique solution. Moreover, $x_k = \dfrac{\det(M_k)}{\det(A)}$,

$$M_k = \begin{pmatrix} | & | & & | & | & | & & | \\ a_1 & a_2 & \cdots & a_{k-1} & b & a_{k+1} & \cdots & a_n \\ | & | & & | & | & | & & | \end{pmatrix}$$

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

pf. $\because \det(A) \neq 0$

$\therefore A$ is invertible

$\Rightarrow Ax=b$ implies $x = A^{-1}b$, a unique solution.

Write $M_k = \begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{k-1} & b & a_{k+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}$

$$= \begin{pmatrix} | & & | & | & | & & | \\ Ae_1 & \cdots & Ae_{k-1} & Ax & Ae_{k+1} & \cdots & Ae_n \\ | & & | & | & | & & | \end{pmatrix}$$

$$= A\begin{pmatrix} | & | & & | & | & | & | \\ e_1 & e_2 & \cdots & e_{k-1} & x & e_{k+1} & \cdots & e_n \\ | & | & & | & | & | & | \end{pmatrix} \triangleq M'$$

$$\Rightarrow \det(M_k) = \det(A)\det(M')$$

$\because$

$$M' = \begin{pmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ 0 & 0 & \cdots & x_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_k & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n & \cdots & 1 \end{pmatrix}$$

$$\Rightarrow \det(M') = x_k(-1)^{k+k}\det(I_{n-1}) = x_k$$

$$\Rightarrow \det(M_k) = \det(A)x_k$$

$$\Rightarrow x_k = \frac{\det(M_k)}{\det(A)}$$

Q.E.D.

Remark. 实际运算中不会用这方法解方程式

Prop. If $A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}$, $A \in M_{3\times 3}(\mathbb{R})$, then

$|\det(A)| =$ a_1, a_2, a_3 所张的平行六面体体积 $\triangleq V(A)$

$\det(A) = \mathcal{O}(A)V(A)$, $\mathcal{O}(A) =$ orientation of a_1, a_2, a_3

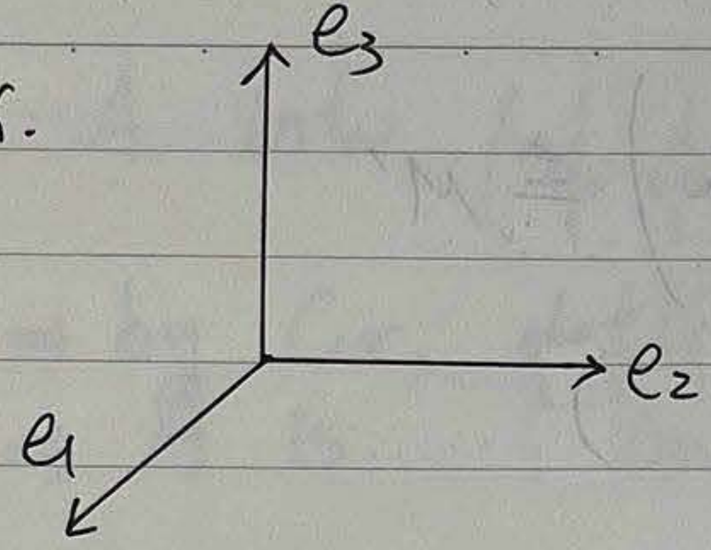
Def. Let $(a_1, a_2, a_3) \in \mathbb{R}^3$ be an ordered basis of $\mathbb{R}^3$

$\det(A) > 0 \Rightarrow$ right-handed coordinate system

$\det(A) < 0 \Rightarrow$ left "

I am OKAY.

ex.



right-handed: $\{e_1, e_2, e_3\}$ ⇒用右手定則
e_1 轉到 e_2, e_3 與右手大姆指同向.

e_2" e_3, e_1"

e_3" e_1, e_2"

left-handed: $\{e_2, e_1, e_3\}$: 用左手定則,
e_2 轉到 e_1, e_3 與左手大姆指同向

e_1" e_3, e_2"

e_3" e_2, e_1"

§4.4 Summary - Important Facts of Deleminants.

Prop. Properties of Determinant (revisit).

Let $A, B \in M_{n\times n}(F)$.

(1) det is n-linear (Thm4.3)

(某行(列)加减可分開，可合併)

(2) alternating properties (Thm4.7)

(兩行(列)互換，其值变号)

(3) $\det(AB) = \det(A)\det(B)$ (Thm4.7)

(列行内積)

(4) $\det(A) = \det(A^t)$ (Thm4.8)

(行列互换，其值变号)

(5) $\det(A)$ 可沿任一行 or 列展開 (Thm4.4)

(遞迴降階)

(6). $\det(E_1A) = \det(AE_1) = -\det(A)$, E_1:

elementary matrix of type I. (Thm4.7)

(兩行(列)互换，其值变号)

(7) $\det(E_2A) = \det(AE_2) = k\det(A)$, E_2:

elementary matrix of type II

(某行(列)乘k倍可提出)

(8) $\det(E_3A) = \det(AE_3) = \det(A)$, E_3:

elementary matrix of type III. (Thm4.7)

(某一行(列)可乘k倍加到另一行(列)，其值不变)

Cor3 If δ satisfies (1), (2), (3), then

$\delta(E_1) = -\delta(I) = -1 = \det(E_1)$

$\delta(E_2) = k\delta(I) = k = \det(E_2)$

$\delta(E_3) = \delta(I) = 1 = \det(E_3)$

I am OKAY.

§4.5 A Characterization of the Determinant.

In this section, we want to prove that, if a function $\delta: M_{n\times n}(F) \to F$ is

(1) n-linear (3) $\delta(I_n) = 1$

(2) alternating

then $\delta(A) = \det(A)$, $\forall\, A \in M_{n\times n}(F)$

Thm 4.10 If δ satisfies (1), (2), then:

(a)
$$\det\begin{pmatrix} \vdots \\ -a_r- \\ \vdots \\ -a_s- \\ \vdots \end{pmatrix} = -\det\begin{pmatrix} \vdots \\ -a_s- \\ \vdots \\ -a_r- \\ \vdots \end{pmatrix}$$

(b).
$$\det\begin{pmatrix} \vdots \\ -u- \\ \vdots \\ -u- \\ \vdots \end{pmatrix} = 0.$$

pf. (a) 可先將 a_r 依序与 $a_{r+1}, a_{r+2} \cdots a_{r+s-1}$ 交換，再和 a_s 交換後，(一步)^2 把 a_s 換到原本 a_r 在的地方

$$\Rightarrow \det\begin{pmatrix} \vdots \\ -a_r- \\ \vdots \\ -a_s- \\ \vdots \end{pmatrix} = (-1)^{2(r-s)-1} \det\begin{pmatrix} \vdots \\ -a_s- \\ \vdots \\ -a_r- \\ \vdots \end{pmatrix}$$

$$= -\det\begin{pmatrix} \vdots \\ -a_s- \\ \vdots \\ -a_r- \\ \vdots \end{pmatrix}$$

(b), by (A),

$$\det\begin{pmatrix}\vdots\\ -u-\\ -u-\\ \vdots\end{pmatrix} = -\det\begin{pmatrix}\vdots\\ -u-\\ -u-\\ \vdots\end{pmatrix} \Rightarrow \det\begin{pmatrix}\vdots\\ -u-\\ -u-\\ \vdots\end{pmatrix} = 0$$

Q.E.D.

Corl If δ satisfies (1), (2), then $\delta(E_3A) = \delta(A)$.

pf. Let $A = \begin{pmatrix}-a_1-\\ \vdots\\ -a_r-\\ \vdots\\ -a_s-\\ \vdots\\ -a_n-\end{pmatrix}$, $E_3A = \begin{pmatrix}-a_1-\\ \vdots\\ -a_r-\\ \vdots\\ -ka_r+a_s-\\ \vdots\\ -a_n-\end{pmatrix}$

$$\delta(E_3A) \overset{(1)}{\underset{(a)}{=}} k\delta\begin{pmatrix}-a_1-\\ \vdots\\ -a_r-\\ \vdots\\ -a_r-\\ \vdots\\ -a_n-\end{pmatrix} + \delta\begin{pmatrix}-a_1-\\ \vdots\\ -a_r-\\ \vdots\\ -a_s-\\ \vdots\\ -a_n-\end{pmatrix}$$

$\overset{(b)}{=} \delta(A)$

Q.E.D.

Cor2. Let $A \in M_{n\times n}(F)$ be not full rank
$\rightarrow \delta(A) = 0$.

pf. This follows directly from Cor1.

Cor3. If δ satisfies (1), (2), (3), then
$\delta(E_1) = -\delta(I) = -1 = \det(E_1)$
$\delta(E_2) = -k\delta(I) = k = \det(E_2)$
$\delta(E_3) = \delta(I) = 1 = \det(E_3)$

I am OKAY.

Thm 4.12. If δ satisfies (1), (2), (3), then
$\delta(A) = \det(A)$, $\forall A \in M_{n\times n}(F)$.

pf. (1) If A is not invertible, $\delta(A) = 0 = \det(A)$

(2) If A is invertible, then $\exists$ elementary matrices $E_1, \dots, E_p$ s.t. $A = E_1 \cdots E_p$

$\Rightarrow$ by Thm 4.11 (補述於後),

$$\begin{aligned}\delta(A) &= \delta(E_1)\delta(E_2)\cdots\delta(E_p)\\ &= \det(E_1)\det(E_2)\cdots\det(E_p) \quad (\text{by (1)})\\ &= \det(E_1E_2\cdots E_p)\\ &= \det(A).\end{aligned}$$

Q.E.D.

Thm 4.11 Let δ satisfies (1), (2), (3), then
$\delta(AB) = \delta(A)\delta(B)$, $A, B \in M_{n\times n}(F)$.

pf. exercise.

* 附錄: proof of lemma. in §4.2.

lemma. Let $B \in M_{n\times n}(F)$.

$$B = \begin{pmatrix} 0 \cdots 0 \,|\, 0 \cdots 0 \end{pmatrix} \rightarrow i^{th}$$

$$\uparrow k^{th}$$

$\Rightarrow \det(B) = (-1)^{i+k} \det(\widetilde{B_{ik}})$ (*) = 沿著 $(0 \cdots 0 \,|\, 0 \cdots 0)$ 展開

pf. We prove by induction on n.

(1). If $n=2$.

(i) $i=1$, by def, it is true.

(ii) $i=2$, then B has 2 possibility:

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B_{11} & B_{12} \\ 1 & 0 \end{pmatrix}$$

$\det B = B_{11}$ $\qquad$ $\det B = B_{12}$.

$\Rightarrow$ The result is true.

(2). Assume that $\det(B) = (-1)^{i+k} \widetilde{B_{ik}}$ is true for a matrix of size $(n-1)\times(n-1)$, $n \geq 3$.

Wanted: (*) is true for a matrix of size $n \times n$.

I am OKAY.

Give a matrix B of size $n \times n$ with the following structure.

$$B = \begin{pmatrix} 0 \cdots 0 & 1 & 0 \cdots 0 \end{pmatrix} \to i^{th}$$

$$\downarrow k^{th}$$

(i) $i = 1$. $\Rightarrow$ (*) is true by def.

(ii) $i > 1$. Then

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ 0 & 0 & \cdots 0 \; 1 \; 0 \cdots & 0 \end{pmatrix} \to i^{th}$$

$$\uparrow k^{th}$$

$$\Rightarrow \det B = \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det \tilde{B}_{1j}$$

$$= \sum_{j<k \; (j=1)}^{(n)} (-1)^{1+j} B_{1j} \det \tilde{B}_{1j} \quad \text{—(I)}$$

$$+ (-1)^{1+k} B_{1k} \det \tilde{B}_{1k} \quad \leftarrow j = k. \quad \text{—(II)}$$

$$+ \sum_{j>k \; (j=k+1)}^{(n)} (-1)^{1+j} B_{1j} \det \tilde{B}_{1j} \quad \text{—(III)}$$

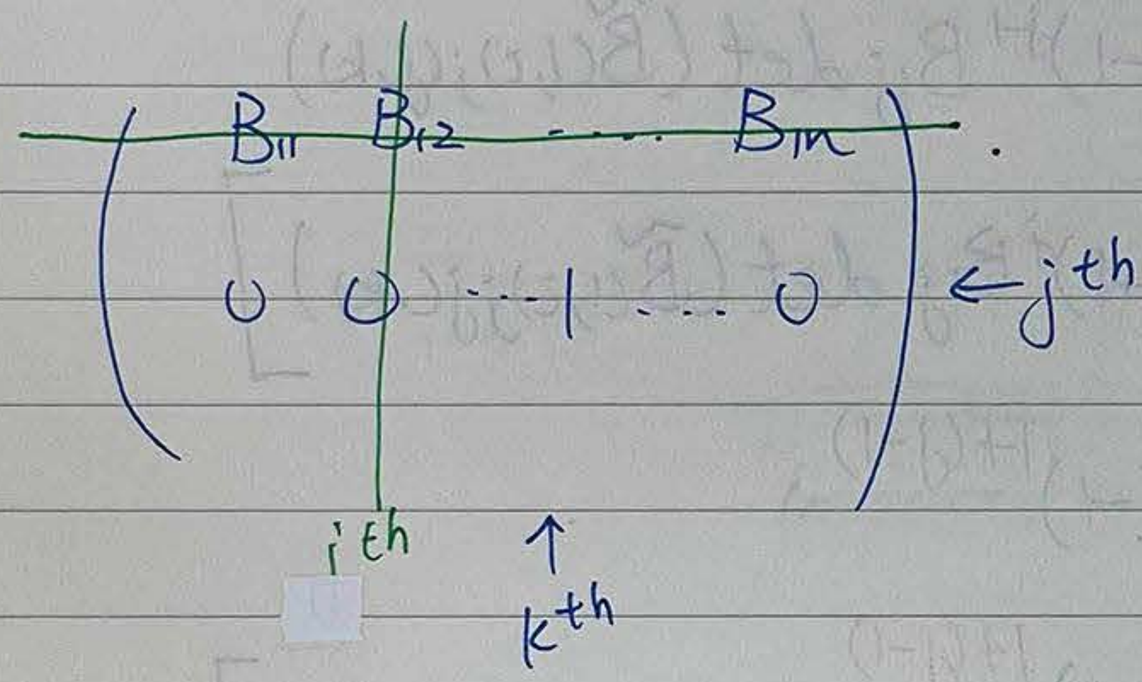
$$= (\text{I}) + (\text{II}) + (\text{III})$$

(II) : $\because \widetilde{B}_{ik}$ has a row which is all zero

$\therefore$ (II) = 0.

(I) : $\because j<k$

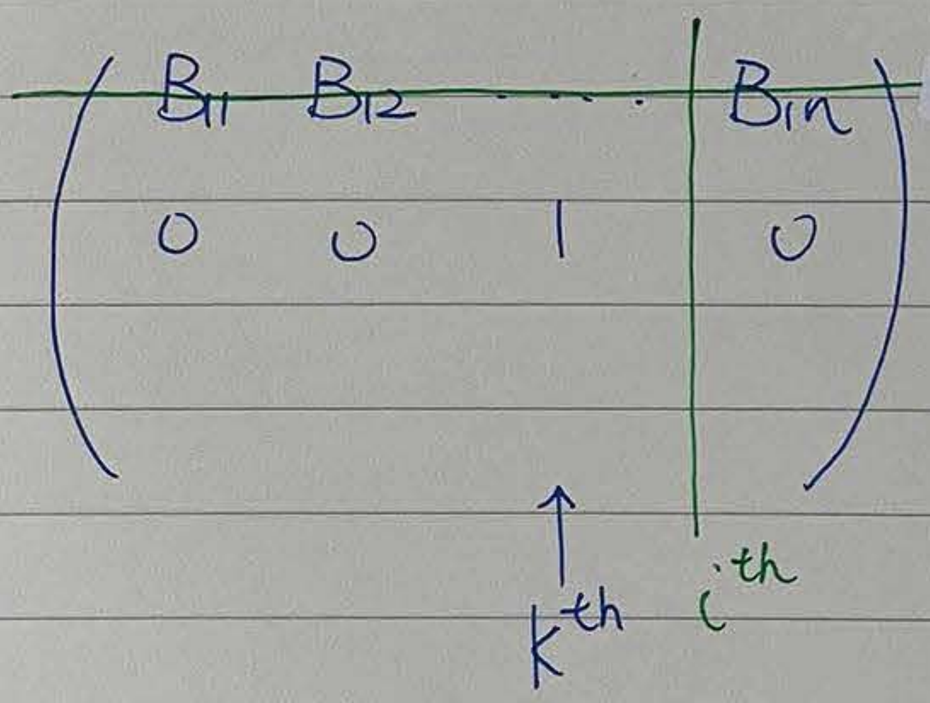
$\therefore \det(B_{ij}) \overset{\text{induction hypothesis}}{=} (-1)^{(i-1)+(k-1)} \det \widetilde{B}_{(i,i);(j,k)}$



where $\widetilde{B}_{(i,i);(j,k)}$ is the matrix obtained by deleting the first and j^{th} row and i^{th} and k^{th} column.

(III) : $\because j>k$

$\therefore \det(B_{ij}) \overset{\text{induction hypothesis}}{=} (-1)^{(i-1)+k} \det \widetilde{B}_{(i,i);(j,k)}$



I am OKAY.

$$\Rightarrow \det(B) = (I) + (II) + (III)$$

$$= \sum_{j<k} (-1)^{i+k-1} B_{ij} \det(\widetilde{B}_{(1,i)j(j,k)})$$

$$+ \sum_{j>k} (-1)^{i+k+j} B_{ij} \det(\widetilde{B}_{(1,i)j(j,k)})$$

$$= (-1)^{i+k} \left[\sum_{j<k} (-1)^{j-1} B_{ij} \det(\widetilde{B}_{(1,i)j(j,k)}) \right.$$

$$\left. + \sum_{j>k} (-1)^{j} B_{ij} \det(\widetilde{B}_{(1,i)j(j,k)}) \right]$$

$$= (-1)^{i+k} \left[\sum_{j<k} (-1)^{(j+1)} \text{''} \right.$$

$$\left. + \sum_{j>k} (-1)^{1+(j-1)} \text{''} \right]$$

$$= (-1)^{i+k} \det(\widetilde{B}_{ik})$$

$\Rightarrow$ By the induction process, we conclude that $(*)$ holds, $\forall n \in \mathbb{N}$.

Q.E.D.

Ch5. Diagonalization

§5-1 Eigenvalue and Eigenvectors

Def. (1) Let $T: V \to V$ be a L.T., $\dim(V) < \infty$.
Then T is called diagonalizable if $\exists$ an ordered basis β s.t. $[T]_\beta$ is a diagonal matrix

(2) Let $A \in M_{n\times n}(F)$, then A is said to be diagonalizable if L_A is diagonalizable.
Here $L_A: F^n \to F^n$ and $L_A(x) = Ax$, $x \in F^n$.

Def. (1) Let $T: V \to V$ be a L.T., then $\lambda \in F$ is called an eigenvalue of T, if
$\exists\, v \in V$, $v \neq 0$ with $T(v) = \lambda v$.
In this case, then v is called an eigenvector correspondence to λ.

(2) $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is called an eigenvalue of A if $\exists\, v \in F^n$, $v \neq 0$ with $Av = \lambda v$.
In this case, v is called an eigenvector corresponding to λ.

Remark. $\because$ T (or L_A) is linear,
$\therefore$ If v is an eigenvector, then so is av for any $a \in F$. ($T(av) = \lambda(av)$)

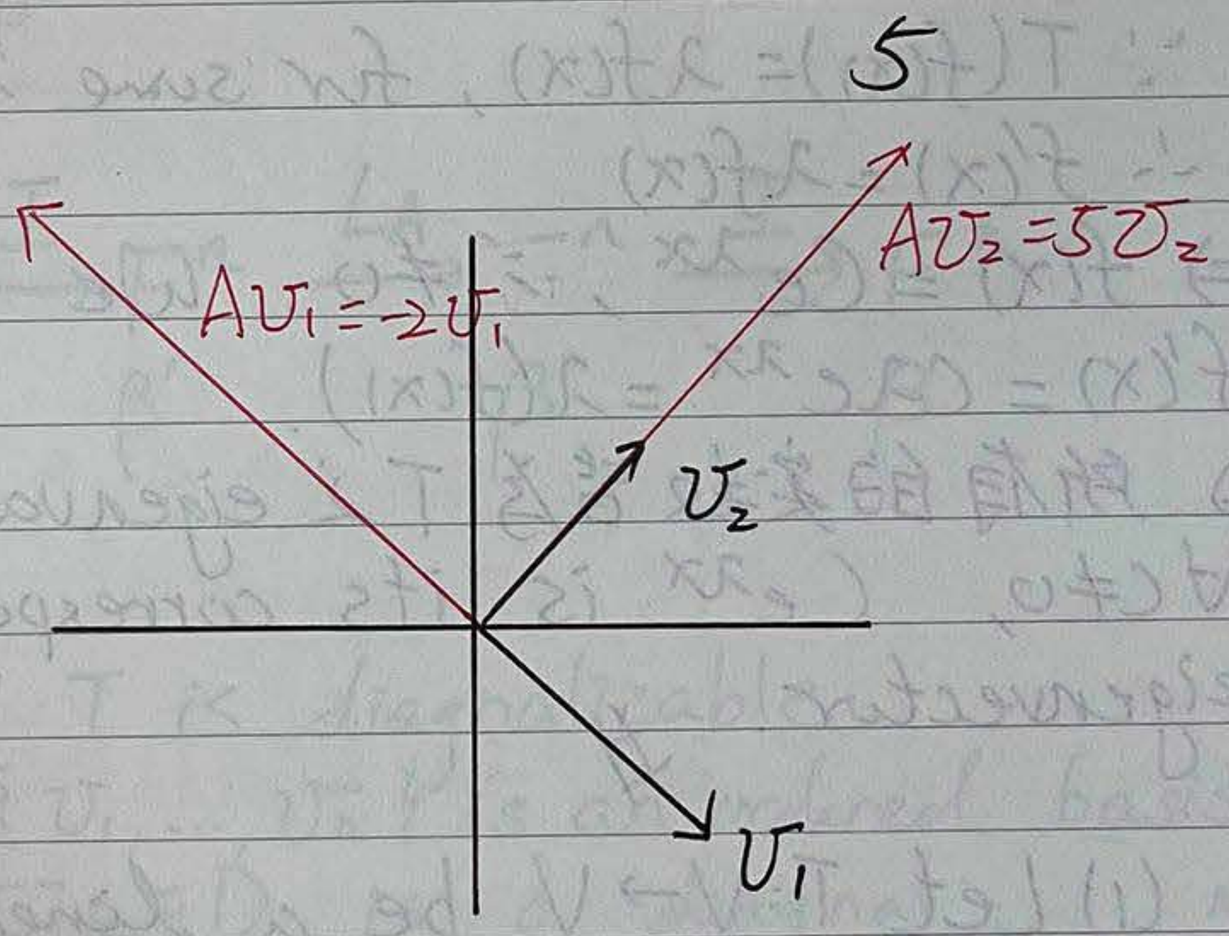
ex1. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\because A\vec{v}_1 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

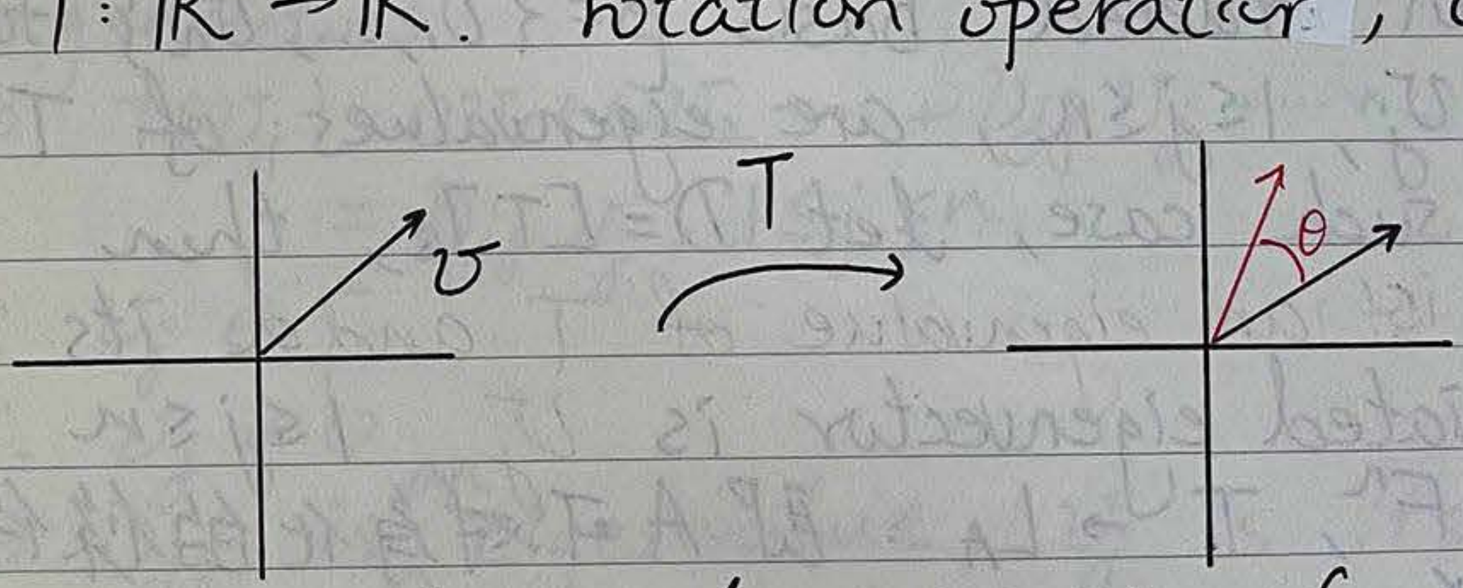
$\therefore \lambda = -2$ is an eigenvalue, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigenvector corresponding to -2

$\because A\vec{v}_2 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\therefore \lambda = 5$ is an eigenvalue, $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$"



ex2. $T: \mathbb{R}^2 \to \mathbb{R}^2$. rotation operation, $0 < \theta < \pi$



$\Rightarrow$ No eigenvalue and eigenvector for T!

I am OKAY.

Let $\beta = \{e_1, e_2\} = \{(1,0), (0,1)\}$

$$[T]_\beta = \begin{pmatrix} \cos\theta & \cos(\theta+\frac{\pi}{2}) \\ \sin\theta & \sin(\theta+\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

However! If we view $[T]_\beta \in M_{2\times 2}(\mathbb{C})$

$\Rightarrow$ $\exists$ eigenvalues!

ex. $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is defineed by $T(f(x)) = f'(x)$. Find eigenvecter of T.

(光滑 (可微 ∞ 多次))

-sol-. $\because T(f(x)) = \lambda f(x)$, for some $\lambda \in \mathbb{R}$,

$\therefore f'(x) = \lambda f(x)$

$\Rightarrow f(x) = Ce^{\lambda x}$, $C \neq 0$. We have

$f'(x) = C\lambda e^{\lambda x} = \lambda(f(x))$.

$\Rightarrow$ 所有的实数皆为 T 之 eigenvalue, and $\forall C \neq 0$, $Ce^{\lambda x}$ is its corresponding eigenvector. //

Thm 5.1 (1) Let $T: V \to V$ be a linear transform, $\dim < \infty$. Then T is diagnalizable

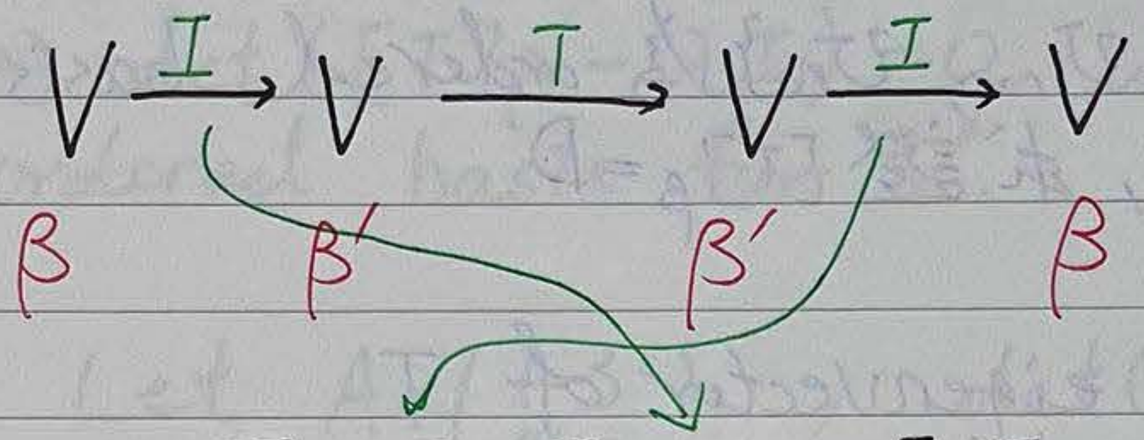
$\Leftrightarrow$ $\exists$ an ordered basis $\beta = \{v_1, \dots, v_n\}$ for V s.t. v_j, $1 \le j \le n$, are eigenvalues of T.

In such case, let $D = [T]_\beta$, then

D_{jj} is an eigenvalue of T and its associated eigenvector is v_j, $1 \le j \le n$

(2). $V = F^n$, $T \hookrightarrow L_A$: 即 A 可对角化的條件是一樣

(3). Let β' be another ordered basis for V. Then $[T]_\beta = [I]_{\beta'}^{\beta} [T]_{\beta'} [I]_{\beta}^{\beta'}$ (Thm 2.23)

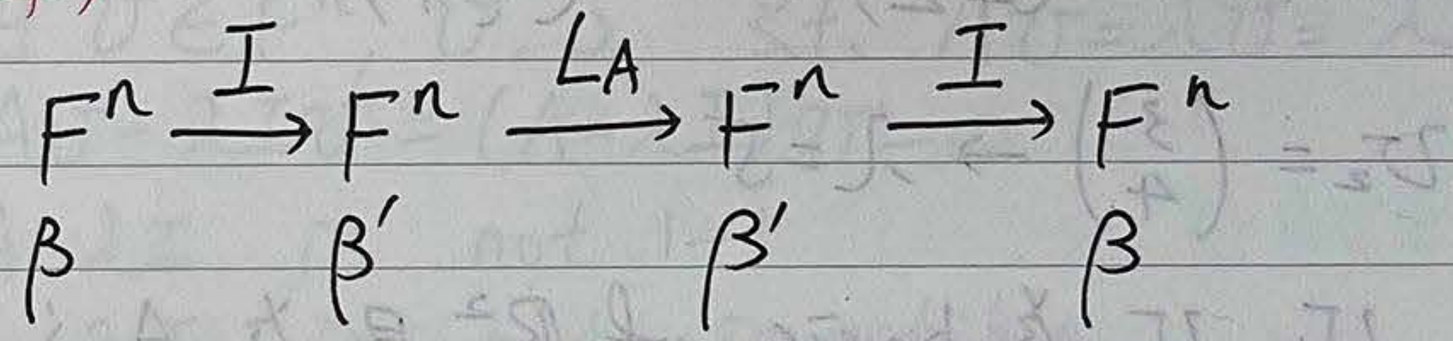


$T = I \circ T \circ I \Rightarrow [T]_\beta = [I]_{\beta'}^{\beta} [T]_{\beta'} \cdot [I]_{\beta}^{\beta'}$

In particular, if β' = the standard basis for F^n; $V = F^n$ and $T = L_A$. then:

$D = Q^{-1}AQ$, where $Q = \begin{pmatrix} | & & | \\ v_1' & \cdots & v_n' \\ | & & | \end{pmatrix}$, eigenvectors

$[L_A]_\beta$ (对角线矩阵) $[L_A]_{\beta'}$



pf. (1) $\Rightarrow$) $\because$ T is diagnalizable

$\therefore \exists \beta = \{v_1, \ldots, v_n\}$ is an ordered basis for V s.t. $[T]_\beta = D$ is a diagnal matrix.

(We want to prove that $v_1, \ldots, v_n$ are eigenvectors.

$\Rightarrow T(v_j) = 0v_1 + 0v_2 + \cdots + D_{jj}v_j + \cdots + 0v_n$

$= D_{jj}v_j$, $1 \le j \le n$

$\because v_j \in \beta$

$\therefore v_j \neq 0$

$\Rightarrow D_{jj}$ is an eigenvalue, v_j is its associated eigenvectors.

I am OKAY.

⇐) 即：已知 $\beta=\{v_1,\dots,v_n\}$ 為 ordered basis + eigenvectors，求證 $[T]_\beta=D$.

∵ v_j is an eigenvector of T

∴ $T(v_j)=\lambda_j v_j$ for some $\lambda_j\in F$

$\Rightarrow T(v_j)=0v_1+\cdots+\lambda_j v_j+\cdots+0v_n,\ 1\le j\le n$

$\Rightarrow [T]_\beta=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

Q.E.D.

ex4. Let $A=\begin{pmatrix}1&3\\4&2\end{pmatrix}$, $v_1=\begin{pmatrix}1\\-1\end{pmatrix}\to\lambda=-2$

$v_2=\begin{pmatrix}3\\4\end{pmatrix}\to\lambda=5$

∵ v_1, v_2 為 basis of $\mathbb{R}^2$ 且為 A 之 eigenvector

∴ Let $\beta=\{v_1,v_2\}$,

$[L_A]_\beta=\begin{pmatrix}-2&0\\0&5\end{pmatrix}=Q^{-1}[L_A]_\beta Q=Q^{-1}AQ$

$\begin{cases} Q=\begin{pmatrix}| & |\\ v_1 & v_2\\ | & |\end{pmatrix}=\begin{pmatrix}1&3\\-1&4\end{pmatrix} \\ Q^{-1}=\frac{1}{7}\begin{pmatrix}4&-3\\1&1\end{pmatrix}\end{cases}$

$\Rightarrow \frac{1}{7}\begin{pmatrix}4&-3\\1&1\end{pmatrix}\begin{pmatrix}1&3\\4&2\end{pmatrix}\begin{pmatrix}1&3\\-1&4\end{pmatrix}=\begin{pmatrix}-2&0\\0&5\end{pmatrix}$

Thm 5.2 (1) Let $A\in M_{n\times n}(F)$, then $\lambda\in F$ is an eigenvalue of $A\iff \det(A-\lambda I)=0$

(2) $T: V\to V$: linear, $\dim(V)<\infty$. Then λ is an eigenvector of T

$\Leftrightarrow \det([T]_\beta - \lambda I) = 0$, where β is an ordered basis for V.

ex. Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $\det(A - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$

$\Rightarrow \lambda = 5, -2$ //

pf. (1) λ is an eigenvalue of A

$\Leftrightarrow \exists\ v \in F^n$, $v \neq 0$, s.t. $Av = \lambda v = \lambda I v$

$\Leftrightarrow Av - \lambda I v = (A - \lambda I)v = 0$.

$\Leftrightarrow A - \lambda I$ is not 1-1

$\Leftrightarrow A - \lambda I$ is not invertible

$\Leftrightarrow \det(A - \lambda I) = 0$.

Q.E.D.

Def. (1) $f(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial (CP) of A.

The zeros of the CP are eigenvalues of A.

(2) Let $T: V \to V$ be a linear transform, $\dim(V) < \infty$, β is an ordered basis for V. Then $f(\lambda) = \det([T]_\beta - \lambda I)$ is called the CP of T.

(相似矩陣 CP 相同)

Q: Is such def. well-defined?

A: Yes!

I am OKAY.

If $\underbrace{[T]_\beta}_{A} = Q^{-1}\underbrace{[T]_{\beta'}}_{B}Q$, then $[T]_\beta$ & $[T]_{\beta'}$ is similar.

Fact: All similar matrices have the same eigenvalues.

pf. $\det(A-\lambda I) = \det(Q^{-1}BQ-\lambda I)$
$= \det[Q^{-1}(B-\lambda I)Q]$
$= \det(Q^{-1})\det(B-\lambda I)\det(Q)$
$= \det(B-\lambda I)$

Q.E.D.

Thm 5.3. Let $f(t)$ be the CP of T (or A)
(1) 首項係數 $(-1)^n$
(2) has at most n distinct eigenvalue

pf: exercise.

Thm 5.4 (如何找 eigenvectors)
Let T be a linear operator on a vector space T. Then v is an eigenvector of T corresponding to $\lambda \Leftrightarrow v \neq 0$ and $v \in N(T-\lambda I)$

pf. $\Rightarrow$) $\because T(v) = \lambda v$, $v \neq 0$
$\therefore T(v) = \lambda I(v)$
$\Rightarrow T(v) - \lambda I(v) = 0$
$\Rightarrow (T-\lambda I)(v) = 0$
$\Rightarrow v \in N(T-\lambda I)$, $v \neq 0$

⇐) 即反之向推回

Q.E.D.

ex1. $\begin{pmatrix}1&3\\4&2\end{pmatrix}$ 的特徵值為？特徵向量為？

-sol-.

$$\det\left(\begin{pmatrix}1&3\\4&2\end{pmatrix}-t\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=\det\begin{pmatrix}1-t&3\\4&2-t\end{pmatrix}$$

$=(1-t)(2-t)-12=t^2-3t-10\triangleq 0$

$\Rightarrow (t-5)(t+2)=0\Rightarrow t=5$ or -2.

→一定要是相依方程式

When $t=5$, $\begin{pmatrix}-4&3\\4&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow\begin{cases}-4x_1+3x_2=0\\4x_1-3x_2=0\end{cases}\Rightarrow$ 可取 $\begin{cases}x_1=1\\x_2=4/3\end{cases}$

$\Rightarrow \forall a\neq 0$, $a\begin{pmatrix}1\\4/3\end{pmatrix}$ is an eigenvector

corresponding the eigenvalue $=5$

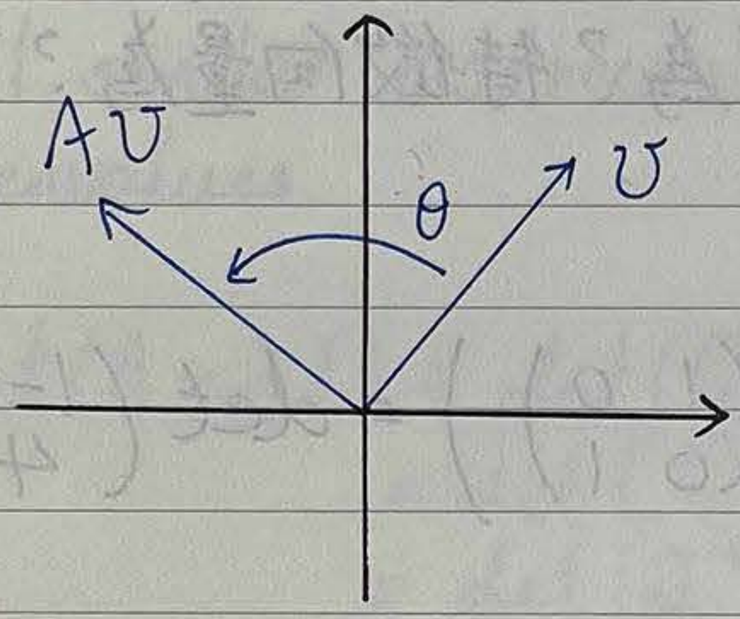
When $t=-2$, $\begin{pmatrix}3&3\\4&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow x_1+x_2=0\Rightarrow a\begin{pmatrix}1\\-1\end{pmatrix}, a\neq 0$ is an eigenvector"

→ ※

I am OKAY.

ex2. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$



$C.P. = \begin{vmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{vmatrix} = t^2 - (2\cos\theta)t + 1$

$\because$ 判別式 $4\cos^2\theta - 4 < 0$

$\therefore$ 此矩陣無 eigenvalue.

However, if we let $\underset{A}{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \in M_{2\times 2}(\mathbb{C})$

$\Rightarrow \begin{vmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{vmatrix} = t^2 - 2(\cos\theta)t + 1 = 0$

$\Rightarrow t = \cos\theta \pm i\sin\theta.$

① When $t = \cos\theta + i\sin\theta$.

$A - (\cos\theta + i\sin\theta)I = \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -i\sin\theta x_1 - \sin\theta x_2 = 0$

$\Rightarrow$ 令 $x_1 = 1,\ x_2 = -i$

$\Rightarrow \forall a \neq 0,\ a\begin{pmatrix}1\\-i\end{pmatrix},\ a\in\mathbb{C}$ is an eigenvector corresponds to eigenvalue $t = \cos\theta + i\sin\theta$. #

ex3. $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f(x) + (x+1)f'(x)$

求 T 之 eigenvalue, eigenvector?

〈法I〉: 找出 $C.P. = \det([T]_\beta - tI)$, $\beta = \{1, x, x^2\}$

$\Rightarrow$ 找 C.P. 之根

$\Rightarrow$ 找 $[T]_\beta$ 之 eigenvector.

$\Rightarrow$ 將向量還原為多項式

$1 \xleftrightarrow{\phi_\beta} \begin{pmatrix}1\\0\\0\end{pmatrix},\ x \longleftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix},\ x^2 \longleftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$

〈法II〉 $T(f(x)) = \lambda f(x),\ f(x) \neq 0$.

$\Rightarrow f(x) + (x+1)f'(x) = \lambda f(x)$

$\Rightarrow (\lambda - 1) f(x) = (x+1) f'(x)$

$\Rightarrow \dfrac{\lambda - 1}{x+1} = \dfrac{f'(x)}{f(x)}$

$\Rightarrow \int \dfrac{\lambda-1}{x+1}\,dx = \int \dfrac{f'(x)}{f(x)}\,dx$

$\Rightarrow \ln|(x+1)^{\lambda-1}| = \ln|f(x)|$

其中一解 $\Rightarrow f(x) = (x+1)^{\lambda-1}$

可直接拿掉"ln"跟ln, 因為T是線性算子, f(x)加了係數仍是eigenvector

I am OKAY.

∵ $f(x) \in P_2(\mathbb{R})$

∴ $\lambda = 3, 2, 1.$

$\Rightarrow \forall a \neq 0,\quad \lambda=1 \longleftrightarrow a$

$\lambda=2 \longleftrightarrow a(x+1)$

$\lambda=3 \longleftrightarrow a(x+1)^2$ #

§5-2 Diagonalizability

Thm 5.5 Let T be a linear operator on a vector space V, $\{\lambda_i\}_{i=1}^{i=k}$ be distinct eigenvalues of T. If $v_1, \dots, v_k$ are eigenvectors of T s.t. λ_i corresponds to v_i, then $\{v_1, \dots, v_k\}$ is linearly independent.

ex. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 2$

$\Rightarrow \{v_1, v_2\}$ is l.i.

pf. by Induction.

(1) $\because k=1 \therefore v_1 \neq 0 \therefore \{v_i\}$ is l.i.

(2) Assume the Thm is true for $k-1$.

Wanted: The Thm is true for $k \Leftrightarrow \{v_1, \dots, v_k\}$ is l.i.

Assume $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$.

$\Rightarrow (T - \lambda_k I)(a_1v_1 + \cdots + a_kv_k) = 0$

$\Rightarrow a_1(T-\lambda_k I)v_1 + a_2(T-\lambda_k I)v_2 + \cdots + a_k(T-\lambda_k I)v_k = 0$.

$\Rightarrow a_1(\lambda_1 v_1 - \lambda_k v_1) + a_2(\lambda_2 v_2 - \lambda_k v_2) + \cdots + a_k(\lambda_k v_k - \lambda_k v_k) = 0$

$\Rightarrow a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_2 - \lambda_k)v_2 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0$.

$\therefore$ by induction hypothesis

I am OKAY.

$$\therefore \begin{cases} a_1(\lambda_1 - \lambda_k) = 0 \\ a_2(\lambda_2 - \lambda_k) = 0 \\ \vdots \\ a_{k-1}(\lambda_{k-1} - \lambda_k) = 0 \end{cases}$$

$\Rightarrow a_k v_k = 0 \Rightarrow a_k = 0$, this completes the proof.

Q.E.D.

Remark. The inverse is not true.

ex. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$

but $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Cor. Let $\dim(V) = n$ or $A \in M_{n\times n}(F)$. If $T(A)$ has n distinct eigenvalues, then $T(A)$ is diagonalizable

Note that the inverse is not true,

ex: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda_1 = \lambda_2 = 1$.

Q: If C.P. has repeated roots, is the matrix diagonalizable?

Def. A polynomial $f(t) \in P(F)$ is said to split over F if $f(t) = c(t-a_1)\cdots(t-a_n)$ where $a_i \in F$, $i = 1, 2, \ldots, n$.

ex. $f(t)=(t+1)(t^2+1)$ does not split over $\mathbb{R}$, but splits over $\mathbb{C}$.

Thm 5.6 $T(A)$: diagonalizable $\Rightarrow$ C.P. splits in F.

pf. C.P. $= \det([T]_\beta - \lambda I)$

$$= \det\begin{pmatrix} \lambda_1 - t & & \\ & \ddots & \\ & & \lambda_n - t \end{pmatrix}$$

$$= (\lambda_1 - t)\cdots(\lambda_n - t)$$

Q.E.D.

Def. Let T be a linear transform over V, $A=[T]_\beta$, $f(t)$ be C.P. of T, λ be eigenvalues of T.

(1) If $f(t)=(t-\lambda)^k g(t)$, where k is the largest integer so that $(t-\lambda)^k \mid f(t)$, then k is called the algebraic multiplicity ($am(\lambda)$) of λ.

(2) The eigenspace of λ is

$E_\lambda = \{v \in V : T(v)=\lambda v\}$ (所有 eigenvector $\neq 0$ 之集合)

$= \{v \in V : (T-\lambda I)(v)=0\}$

Note that E_λ is a subspace of V.

(3) The geometric multiplicity of $\lambda = gm(\lambda) = \dim E_\lambda$

I am OKAY.

ex. $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $f(t)=C.P.=\begin{vmatrix}1-t&0\\0&1-t\end{vmatrix}$

$=(1-t)^2$

$\Rightarrow am(1)=2$, $gm(1)=\dim E_1=\dim \mathbb{R}^2=2$.

ex. $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $CP=f(t)=(1-t)^2$

$\Rightarrow am(1)=2$

$E_1=\{v:(A-I)v=0\}$

$\because \operatorname{rank}(A-I)=1$

$\therefore$ Nullity $(A-I)=1$

$\Rightarrow gm(1)=1$

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$\Rightarrow \begin{cases}x_1+x_2=x_1\\x_2=x_2\end{cases} \Rightarrow 取 \begin{cases}x_1=1\\x_2=0\end{cases}$$

$$\Rightarrow E_1=\left\{t\begin{pmatrix}1\\0\end{pmatrix}:t\in\mathbb{R}\right\}$$

Thm 5.7 $1\le gm(\lambda)\le am(\lambda)$

pf. Let $gm(\lambda)=p$, let $\{v_1,\dots,v_p\}$ be an ordered basis of E_λ. Extend $\{v_1,\dots,v_p\}$ so that $\beta=\{v_1,\dots,v_p,v_{p+1},\dots,v_n\}$ is an ordered basis for V. Note that $v_1,\dots,v_p$ are eigenvalues corresponding to λ.

$$\Rightarrow \begin{cases} T(v_1) = \lambda v_1 \\ T(v_2) = \lambda v_2 \\ \vdots \\ T(v_p) = \lambda v_p \end{cases} \Rightarrow [T]_\beta = \begin{pmatrix} \overbrace{\begin{matrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \lambda \end{matrix}}^{p} & \begin{matrix} \\ \\ \\ \\ ? \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{matrix} & \begin{matrix} \\ ? \end{matrix} \end{pmatrix}$$

$$\Rightarrow [T]_\beta = \left(\begin{array}{c|c} \lambda I_p & C \\ \hline 0 & D \end{array}\right)$$

$$\Rightarrow C.P. = \det([T]_\beta - tI_n)$$
$$= (\lambda - t)^p g(t), \text{ for some } g(t)$$

$$\Rightarrow gm(\lambda) = p \leq am(\lambda)$$

Q.E.D.

Thm 5.8 Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T. Let S_i be a finite l.i. subset of the eigenspace E_{λ_i}, $i = 1, \dots, k$, then $S = S_1 \cup \cdots \cup S_k$ is a l.i. subset of V.

pf. Let $\begin{cases} S_1 = \{v_{11}, \dots, v_{1i_1}\}, & w_1 = a_{11}v_{11} + \cdots + a_{1i_1}v_{1i_1} \\ S_2 = \{v_{21}, \dots, v_{2i_2}\}, & w_2 = a_{21}v_{21} + \cdots + a_{2i_2}v_{2i_2} \\ \vdots & \vdots \\ S_k = \{v_{k1}, \dots, v_{ki_k}\}, & w_k = a_{k1}v_{k1} + \cdots + a_{ki_k}v_{ki_k} \end{cases}$

If $w_1 + \cdots + w_k = 0$, then $w_i \in S_i$, $i = 1, \dots, k$, we have 2 cases.

I am OKAY.

① If $w_i \neq 0$, then w_i are eigenvector of λ_i

$\Rightarrow$ by Thm 5.5, $w_i = 0$ for all i.

$\Rightarrow a_{ij} = 0, \ \forall i, j.$

要注意在哪了field.

Thm 5.9. Let T be a linear operator on a finite-dimensional vector space V s.t. the characteristic polynomial of T splits. Let $\lambda_1, \dots, \lambda_k$ be eigenvalues (distinct) of T. then: (1) T is diagonalizable $\iff$ $am(\lambda_i) = gm(\lambda_i)$, $i = 1, 2, \dots, k$. (k可小於n)

(2) If T is diagonalizable and β_i is an ordered basis for E_λ, for each i, then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.

pf. (1) $\Rightarrow$) Let β (基底 + eigenvectors) s.t. 都是T之eigenvector且l.i. $[T]_\beta$ is diagonalizable. Let $\beta_i = \beta \cap E_{\lambda_i} \neq \emptyset$, $gm(\lambda_i) = g_i$, $am(\lambda_i) = a_i$, and $|\beta_i| = d_i$.

$\because a_i \geq g_i \geq d_i \geq 1$

$\therefore n = \sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} g_i \geq \sum_{i=1}^{k} d_i = n \ (\geq k)$

$\Rightarrow a_i = g_i = d_i$ for all i.

$\Leftarrow$) Let β_i be an ordered basis for E_{λ_i}

Claim: $\beta = \bigcup_{i=1}^{k} \beta_i$ is an ordered basis consisting of eigenvectors. (This is true from 已知2 & Thm 5.8)

pf of claim: $\because d_i = gm(\lambda_i) = am(\lambda_i)$

$\therefore |\beta| = \sum_{i=1}^{k} am(\lambda_i) = n$ — ①

I am OKAY.

Also, β is l.i. by Thm 5.8 — ②

$\Rightarrow$ by ①, ②, and Thm 5.5, β is as claimed.

$\Rightarrow$ (2) is also proved.

Q.E.D.

ex. Let $A=\begin{pmatrix}3&1&0\\0&3&0\\0&0&4\end{pmatrix}$

A可被对角线化吗?

-sol- $\because f(t)=\det\begin{pmatrix}3-t&1&0\\0&3-t&0\\0&0&4-t\end{pmatrix}=(3-t)^2(4-t)$

$\therefore am(4)=gm(4)=1$

$am(3)=2$

$\because A-3I=\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}$

$\therefore rank(A-3I)=2$

$\Rightarrow Nullity(A-3I)=3-2=1=gm(3)$.

$\Rightarrow$ A無法被对角线化. #

ex. $T: P_2(\mathbb{R})\to P_2(\mathbb{R})$

$T(f(x))=f(1)+f'(0)x+(f'(0)+f''(0))x^2$

Is T diagonalizable?

-sol- Take $\beta=\{1,x,x^2\}$ as a basis,

$\because T(1)=1,\ T(x)=1+x+x^2,\ T(x^2)=1+2x^2$

$$\therefore [T]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\Rightarrow CP = \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1 & 2-t \end{vmatrix} = (1-t)^2(2-t)$$

$$\Rightarrow am(1)=2,\ am(2)=1$$

$$\because A-I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},\ rank(A-I)=1$$

$$\therefore Nullity(A-I)=2=gm(1)$$

$$\because 1 \le gm(2) \le am(2) \quad (Thm\ 5.7)$$

$$\therefore gm(2)=1$$

$$\Rightarrow am(1)=gm(1)=2,\ am(2)=gm(2)=1$$

$$\Rightarrow T \text{ is diagonalizable.} \quad \#$$

ex. 承上例，試找一基底 γ s.t. $[T]_\gamma$ = diagonal matrix.

-sol- 由 $[T]_\beta$ 找 Q 再找 γ : $Q^{-1}[T]_\beta Q = D$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_2 + x_3 = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ are eigenvectors correspond to } 1.$$

I am OKAY.

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x_1+x_2+x_3=0 \\ x_2=0 \end{cases}$$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponed to 2

$$\Rightarrow Q=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \gamma=\{1, -x+x^2, 1+x^2\}$$

△ Applicatins.

(1) Compute A^n
(2) linear ODEs
(3) 高階 linear ODE

ex. 設 $A=\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$, $\exists Q$ s.t. $Q^{-1}AQ=D$

$\Rightarrow A^2=(QDQ^{-1})(QDQ^{-1})=QD^2Q^{-1}$

$\Rightarrow A^n=QD^nQ^{-1}$

可解得 eigenvalue 為 $\begin{cases} 1 \to \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ 2 \to \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{cases}$

$\Rightarrow Q=\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$, $D^{2012}=\begin{pmatrix} 1^{2012} & 0 \\ 0 & 2^{2012} \end{pmatrix}$ #.

ex. 解 $\begin{cases} x_1'=3x_1+x_2+x_3 \\ x_2'=2x_1+4x_2+2x_3 \\ x_3'=-x_1-x_2+x_3 \end{cases}$

-sol-. $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

令 $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$，对角化後，$Q = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$，

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. 再令 $x' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x' = QDQ^{-1}x \Rightarrow Q^{-1}x' = DQ^{-1}x$

$\Rightarrow$ 令 $Q^{-1}x = y$，则 $y' = Dy$

$\Rightarrow \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Rightarrow y = \begin{pmatrix} C_1 e^{2t} \\ C_2 e^{2t} \\ C_3 e^{4t} \end{pmatrix}$

$\Rightarrow x = Qy = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 e^{2t} \\ C_2 e^{2t} \\ C_3 e^{4t} \end{pmatrix} = \cdots$

$= e^{2t}\left[C_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right] + e^{4t}\left[C_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right]$ #

ex. 解 $x'' + 2x' + x = 0$.

-sol-. $\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$，可仿照上一例去解 #

(Obvious → $x'' = -x - 2x'$)

I am OKAY.

△ Direct Sums.

Def. Let W_i, $i=1,\dots,k$ be subspaces of a vector space V.

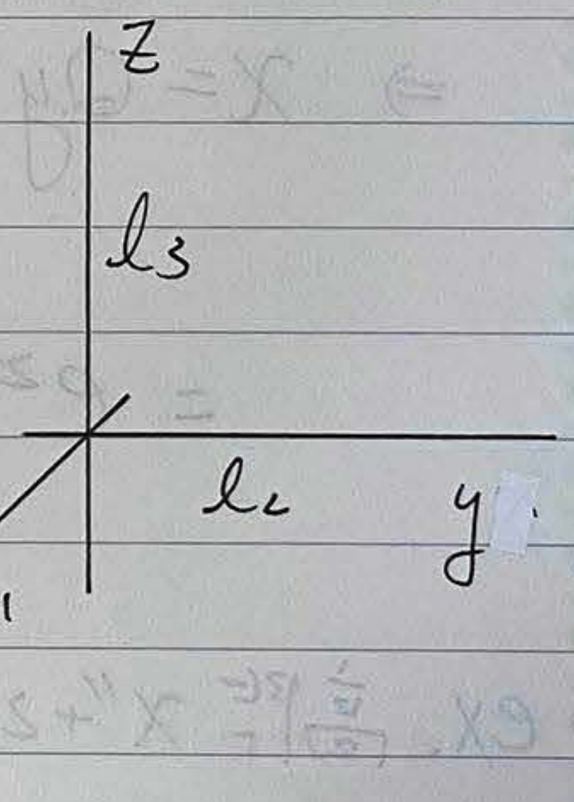
(1) The sum of W_i, denoted by $\sum_{i=1}^{k} W_i$, is defined to be $\sum_{i=1}^{k} W_i = \{v_1+\cdots+v_k : v_i \in W_i,\ i=1,2,\dots,k\}$

(2) V is called the direct sum of W_i, $i=1,\dots,k$. denoted by $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if

(a) $V = \sum_{i=1}^{k} W_i$

(b) $W_j \cap \sum_{\substack{i=1 \\ i\neq j}}^{k} W_i = \{0\}$, $\forall\ 1 \le j \le k$.

ex. Let $V = \mathbb{R}^3$, $l_1 = \{(x,0,0) : x \in \mathbb{R}\}$
$l_2 = \{(0,y,0) : y \in \mathbb{R}\}$
$l_3 = \{(0,0,z) : z \in \mathbb{R}\}$



$\Rightarrow V = \mathbb{R}^3 = l_1 \oplus l_2 \oplus l_3$, since

$\begin{cases} (1)\ l_1 + l_2 + l_3 = \mathbb{R}^3 \\ (2)\ l_1 + l_2 = \text{the } xy \text{ plane} \end{cases}$

$l_3 \cap l_1 + l_2 = \{0\}$, so is the other combinations.

Thm 5.11 Let $T: V \to V$ be a L.T., $\dim(V) < \infty$. Then T is diagonalizable $\iff$ V is the direct sum of its eigenspaces.

$\lambda_1, \cdots\cdots, \lambda_k$: distinct

$\downarrow$ $\downarrow$

$E_{\lambda_1} \oplus \cdots\cdots \oplus E_{\lambda_k} = V$.

I am OKAY.

§5.3 Matrix Limit and Markov Chain

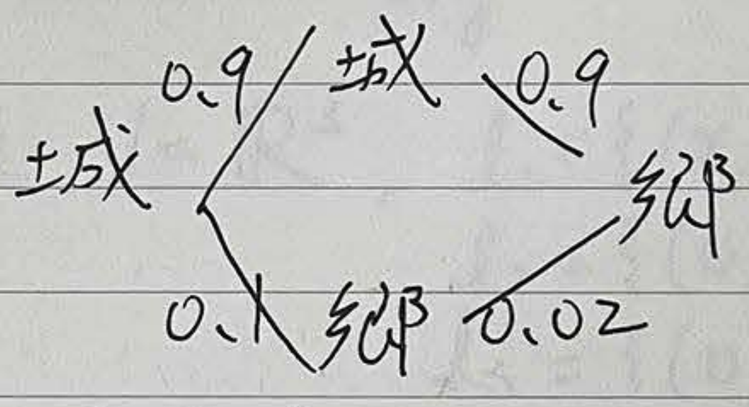
Q:

明年 \ 今年	城市	郷鎮
城市	0.9	0.02
郷鎮	0.1	0.98

問：①兩年後，原本住城市，仍在城市 的机率？
②多"
城市，跑到郷村
郷村，仍在郷村
跑到城市

③若原本有70%的人住城鎮，30%住郷村，則長期之後，有多少人住城鎮，多少人住郷村？

A：①



$$\Rightarrow 0.9 \cdot 0.9 + 0.1 \cdot 0.02 = P(\text{城} \to \text{城}) = 0.83$$

$$A \triangleq \begin{pmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} (A^2)_{11} & (A^2)_{12} \\ (A^2)_{21} & (A^2)_{22} \end{pmatrix}$$

②即求 A^m

③令 $P = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \Rightarrow$ 即求 $\lim_{m \to \infty} A^m P$

Problems: Objects ex. 人
States: ex. 城市, 鄉村
States evolves into another states over time by probabilities

Markov Chain:
(1) Finite states
(2) current state → next state by a probability
* Markov Process states 可以是 infinite states

Def. A matrix A is called a transition (or stockastic) matrix if
(1) $A_{ij} \geq 0$, $\forall i, j$
(2) $\sum_{i=1}^{n} A_{ij} = 1$ for all j (行和為1)

Remarks: (1) $A_{ij} = P(\text{state } j \to \text{state } i \text{ in one unit time})$
(2) Let $P = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}$ be a vector. Then P is called a probability vector if $P_i \geq 0$, for all i and $\sum_{i=1}^{n} P_i = 1$

(3) A is called regular if $\exists\, n \in \mathbb{N}$ s.t. $(A^n)_{ij} > 0$, $\forall i, j$.

I am OKAY.

Thm 5.20. Let A be an $n \times n$ regular transition matrix, then

(1) 1 is an eigenvalue of A and $gm(1)=1$. All other eigenvalues have absolute value less than 1.

(2) A is diagonalizable and $\lim_{m\to\infty} A^m$ exists, say, equals to L.

(3) $AL=LA=L$

(4) $L=\begin{pmatrix} | & & | \\ v & \cdots & v \\ | & & | \end{pmatrix}$, where v is a probability vector. Moreover, $Av=v$ (i.e., v is an eigenvector corresponding to eigenvalue 1).

(5) For any probability vector w, $\lim_{m\to\infty} A^m w = v$

ex. $A=\begin{pmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{pmatrix}$, $w=\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$. $\lim_{m\to\infty} A^m w = ?$

-sol-

$$\det(A-tI)=\begin{vmatrix} 0.9-t & 0.02 \\ 0.1 & 0.98-t \end{vmatrix} = \begin{vmatrix} 0.9-t & 0.02 \\ 1-t & 1-t \end{vmatrix}$$

$$=(1-t)\begin{vmatrix} 0.9-t & 0.02 \\ 1 & 1 \end{vmatrix} = (1-t)(0.88-t)$$

when $t=1$, $\begin{pmatrix} 0.9 & 0.02 \\ 0.1 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

When $t=0.88$, $\begin{pmatrix} 0.02 & 0.02 \\ 0.1 & 0.1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow Q=\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$, $Q^{-1}=-\frac{1}{6}\begin{pmatrix} -1 & -1 \\ -5 & 1 \end{pmatrix}=\frac{1}{6}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$

$\Rightarrow D=Q^{-1}AQ=\begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix}$

$\Rightarrow A^m=QD^mQ^{-1}=\frac{1}{6}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix}^m\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$

$\Rightarrow \lim_{m\to\infty} A^m = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}\lim_{m\to\infty}\begin{pmatrix} 1 & 0 \\ 0 & 0.88^m \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$

$\uparrow ?$

$=\frac{1}{6}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}=\begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{pmatrix}$

△ Eigenvalue 之估計

Thm 5.16 (Gerschgorin 圓盤定理)

Let $r_i=\sum_{\substack{k=1 \\ k\neq i}}^{n}|A_{i,k}|$, $C_i=\{z\in\mathbb{C}:|z-A_{kk}|\le r_i\}$.

(a) If λ is an eigenvalue of A, then

$\lambda\in\bigcup_{i=1}^{n} C_i$ → 不知在哪個圓

(b). $|\lambda|\le \min\{\rho(A), \nu(A)\}$, where $\rho(A)$ = the row sum of $A \triangleq \max_{1\le i\le n}\rho_i(A)$

and $\rho_i(A)=\sum_{k=1}^{n}|A_{ik}|$, $i=1,2,\dots,n$

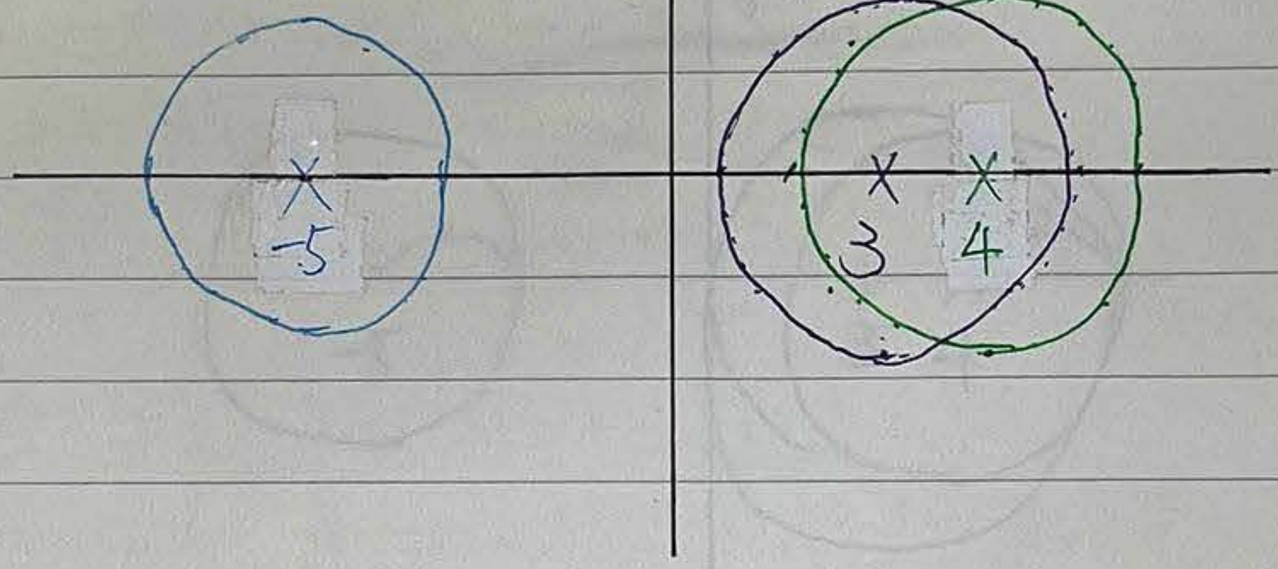
I am OKAY.

$\nu(A) = \max_{1\le j\le n} \nu_j(A)$ = the column sum of A.

and $\nu_j(A) = \sum_{k=1}^{n} |A_{kj}|$, $j = 1, 2, \dots, n$

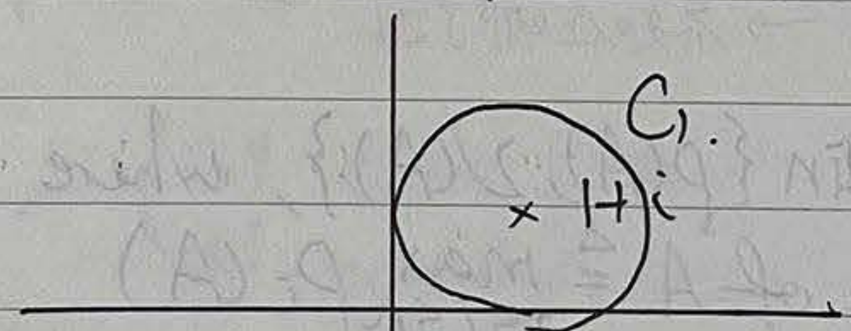
ex. $\begin{pmatrix} 5 & 1 & -1 \\ 1 & 4 & 1 \\ -2 & 0 & 3 \end{pmatrix}$ $\nu(A) = \max_{1\le j\le 3}\{\nu_1(A)=8, \nu_2(A)=5, \nu_3(A)=5\} = 8$

$\rho(A) = \max_{1\le j\le 3}\{\rho_1(A)=7, \rho_2(A)=6, \rho_3(A)=5\} = 7$



$\Rightarrow 0$ is not an eigenvalue of A
$\Rightarrow A$ is invertible

ex. $A = \begin{pmatrix} 1+i & & \\ & 1-i & \\ & & 1+\sqrt{3}\,i \end{pmatrix}$



pf of Thm:

(a). Let $Av=\lambda v$, $v=\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Let $v_k=\max_{1\le i\le n}|v_i|$

we want $|\lambda-A_{kk}|\Rightarrow \lambda\in C_k$.

$\because |v_k||\lambda-A_{kk}|=|\lambda v_k-A_{kk}v_k|$

$=\left|\sum_{j=1}^{n}A_{kj}v_j-A_{kk}v_k\right|$

$=\left|\sum_{j=1,j\neq k}^{n}A_{kj}v_j\right|$

(思考 $Av=\lambda v$

$k\begin{pmatrix} \\ \\ \\ \end{pmatrix}\begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix}=\begin{pmatrix} \lambda v_k \\ \\ \\ \end{pmatrix}$)

$\le\sum_{j=1,j\neq k}^{n}|A_{kj}||v_j|$ (△不等式)

$\le|v_k|\sum_{j=1,j\neq k}^{n}|A_{kj}|$

$=|v_k||r_k|$

$\therefore |\lambda-A_{kk}|\le r_k$.

$\Rightarrow \lambda\in C_k$

(续)

Remark. 入在哪个园，由pf中可得知.

pf of part(b):

$\because |\lambda-A_{kk}|\le r_k$ by (a)

$\therefore |\lambda|-|A_{kk}|\le r_k$

I am OKAY.

$\Rightarrow |\lambda| \le \rho_K(A) \le \rho(A)$

We want $|\lambda| \le \nu(A)$.

$\because$ A and A^t have the same eigenvalues

$\therefore$ λ is also an eigenvalue of A^t

$\Rightarrow |\lambda| \le \rho(A^t) = \nu(A)$

$\Rightarrow |\lambda| \le \min\{\rho(A), \nu(A)\}$

Q.E.D.

Thm 5.18 If (1) $A \in M_{n\times n}(\mathbb{C})$, $A_{ij} > 0$, $\forall i, j$, and

(2) λ is an eigenvalue of A where $\lambda = \rho(A)$

$\Rightarrow$ (a) $\lambda > 0$

(b) $u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvalue of A corresponding to λ and $gm(\lambda) = 1$

Such A is called a positive matrix

*反例：如果 A is not positive，(a)会对吗？

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho(A) = 1$, 但 $\lambda = \pm 1$.

pf. $\because Av = \lambda v$, $v = (v_1 \dots v_n)^t$, define $|v_k|$ as before.

$\therefore |\lambda||v_k| = |\lambda v_k| = \left|\sum_{j=1}^{n} A_{kj} v_j\right|$

$\le \sum_{j=1}^{n} |A_{kj}||v_j|$ (by Δ ineq) ①

$$\overset{②}{\leq} |v_k| \sum_{j=1}^{n} |A_{kj}|$$

$$= |v_k| \rho_k(A)$$

$$\overset{③}{\leq} |v_k| \rho(A).$$

⇒ 由已知不等式，可發現①，②，③ 應該都要成等式.

① $|a_1 + a_2 + a_3| = |a_1| + |a_2| + |a_3|$.

"=" 成立 ⇒ a_i 皆同方向.

⇒ $A_{kj} v_j = c_j z$, where $c_j \geq 0$, $|z| = 1$, $\forall j = 1, \dots, n$

② $|v_j| = |v_k|$, $\forall j = 1, \dots, n$

③ $\rho(A) = \rho_k(A)$

$$\because v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \frac{c_1 z}{A_{k1}} \\ \vdots \\ \frac{c_n z}{A_{kn}} \end{pmatrix} = \begin{pmatrix} |v_k| z \\ \vdots \\ |v_k| z \end{pmatrix}$$

$$\left(|v_k| = \left| \frac{c_j z}{A_{kj}} \right| = \left| \frac{c_j}{A_{kj}} \right| \right)$$

$$= |v_k| z \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = |v_k| z u, \text{ where } u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\therefore u$ is also an eigenvector of A w.r.t. λ

$\Rightarrow Au = \lambda u$.

$\Rightarrow \lambda > 0$

I am OKAY.

$\because v = |v_k| z u$

$\therefore g_m(\lambda) = 1$.

Q.E.D.

△ Summary of the properties of A

(I): transition (or stochastic)

(II): transition + regular ($\exists m \in \mathbb{N}$ s.t. $(A^m)_{ij} > 0$)

Thm I.(a) A^m is transition for any m

Ap is a probability vector, for any probability vector p. (Thm 3.15)

(b). $|\lambda| \leq 1$, λ is an eigenvalue of A (Cor)

(c) 1 is an eigenvalue of A (Thm 5.17)

Thm II.(a) $|\lambda| \leq 1$ and if $|\lambda| = 1$, then $\lambda = 1$ and $g_m(1) = 1$ (Thm 5.19)

(b) $a_m(1) = 1$ (ex21 of Ch7)

pf (a). Recall Thm 5.18: If $A_{ij} > 0$, $\forall i, j$, and $|\lambda| = \rho(A)$ or $\nu(A)$, then

(i) $\lambda > 0$

(ii) $g_m(\lambda) = 1$

Claim: by Thm 5.15 and its Cor, A^m is a transition matrix

pf of claim: let $u^t = (1, \dots, 1)$, then $u^t A = u^t$.

$\Rightarrow u^t A^m = (u^t A) A^{m-1} = u A^{m-1} = \cdots = u A = u^t$.

$\Rightarrow A$ is a transition.

∵ A is regular

∴ ∃ s s.t. $(A^s)_{ij} > 0$, ∀ i, j

∵ If λ is an eigenvalue of A, then λ^s is an eigenvalue of A^s (check it)

∴ $|\lambda^s| = 1$

⇒ by Thm 5.18, $\lambda^s = \lambda^{s+1} = 1$

⇒ $\lambda^s(\lambda - 1) = 0$

⇒ $\lambda = 1$.

Let E_λ be the eigenspace of A corresponding to $\lambda = 1$

" $E_{\lambda'}$ " " A^s "

$\lambda = 1$

⇒ by Thm 5.18, $\dim(E_{\lambda'}) = 1$

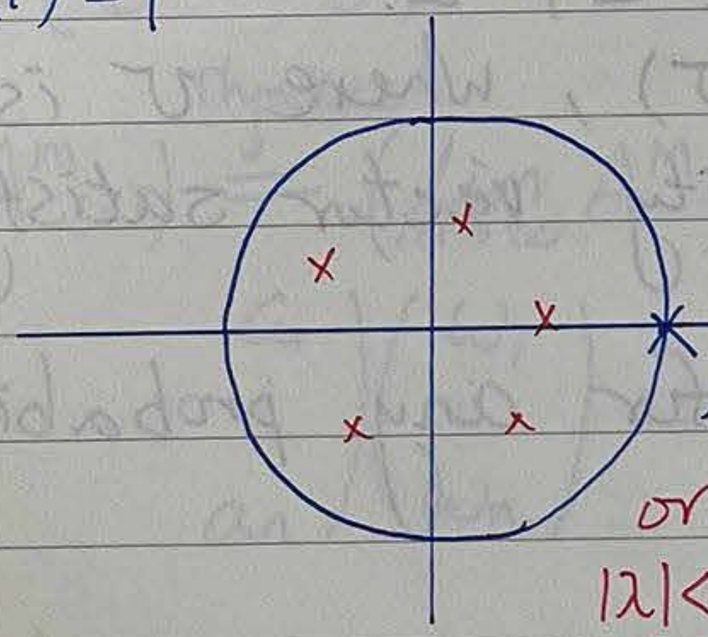
∵ If $v \in E_\lambda$, then $(A - \lambda I)v = 0$

∴ $(A^s - \lambda^s I)v = 0$

⇒ $v \in E_{\lambda'}$

⇒ $E_\lambda \subset E_{\lambda'}$

⇒ $gm(1) = 1$



(a) 的意思就是說：
若 A 是 transition matrix 又是 regular matrix，則所有 eigenvalue 要馬只有 1，要馬都在單位圓內(不会在边界)

Q.E.D.

I am OKAY.

Thm. $\lim_{m\to\infty} QB^mP = Q\lim_{m\to\infty} B^mP$ provided that $\lim_{m\to\infty} B^m$ exist.

Thm. (a) $\lim_{m\to\infty} A^m$ exists $\overset{①}{\iff}$ $|\lambda|<1$ or $\lambda=1$ and ② $g_m(1) = a_m(1)$

(b) If Thm 5.14(1) holds and A is diagonalizable, then $\lim_{m\to\infty} A^m$ exists.

這2个 Thm 要到 Ch7 才能 pf.

Thm. 5.20 If A is a transition matrix and is regular, then:

(1) $|\lambda|<1$ or $\lambda=1$. In fact, $\lambda=1$ is an eigenvalue of A, and $g_m(1)=1$ (Thm 5.15 ~ 5.19 + Cor)

(2) $a_m(1)=1$ (ex21 of Ch7.2)

(3) $\lim_{m\to\infty} A^m$ exists (Thm 5.12, 5.13充要, 5.14充1份)

(4) Let $L = \lim_{m\to\infty} A^m$, then ① L is a transition matrix and ② $AL=LA=L$.

(5) $L = (v, \dots, v)$, where v is the unique probability vector satisfying $Av = \lambda v$ $(\lambda=1)$

(6) $\lim_{m\to\infty} A^m w = v$, for any probability vector w.

pf (4) ①: Let $u=(1 \dots 1)^t$, we want $u^tL = u^t$

$\because u^tL = u^t(\lim_{m\to\infty} A^m)$

$= \lim_{m\to\infty} u^tA^m$ (by Thm 5.12)

$$= \lim_{m\to\infty} u^t A A^{m-1}$$

$$= \lim_{m\to\infty} u^t A^{m-1}$$

$$= \cdots = u^t$$

$\therefore$ L is a transition matrix

(2) $L = \lim_{m\to\infty} A^{m+1} = \lim_{m\to\infty} A A^m$

Thm 5.12 $= A \lim_{m\to\infty} A^m$

$= AL$

by an analogous process, we can prove $L = LA$.

(5). Let $L = (v_1 \cdots v_n)$

$\because A(v_1 \cdots v_n) = (v_1 \cdots v_n)$ by (4), and $gm(1) = 1$, and L is a transit.

$\therefore v_i = \alpha v$, for $i = 1, \ldots, n$ and v.

$\therefore$ L is a transition matrix

$\therefore \alpha = 1$

$\Rightarrow L = (v \cdots v)$

(6) $\lim_{m\to\infty}(A^m w) \overset{\text{Thm 5.12}}{=} (\lim_{m\to\infty} A^m) w = Lw$

$$= \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & & \vdots \\ a_n & \cdots & a_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_1(w_1 + \cdots + w_n) \\ \vdots \\ a_n(\underbrace{w_1 + \cdots + w_n}_{=1}) \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = v.$$

Q.E.D.

I am OKAY.

how

ex. $A=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ C S W ↓下一年 C S W $\Rightarrow (A^2)_{ij}>0$

$w=\begin{pmatrix} 3/6 \\ 2/6 \\ 1/6 \end{pmatrix}$, $\lim_{m\to\infty} A^m w = ?$

-sol- $\because$ $(A-I)v=\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore \begin{cases} -x_1+\frac{1}{2}x_2+\frac{1}{2}x_3=0 \\ \frac{1}{2}x_1-x_2+\frac{1}{2}x_3=0 \\ \frac{1}{2}x_1+\frac{1}{2}x_2-x_3=0 \end{cases}$ (对称方程式)

要是probability vector ↓

$\Rightarrow x_1=x_2=x_3$ is a solution. Take $v=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$

$\Rightarrow \lim_{m\to\infty} A^m w=\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$, $\lim_{m\to\infty} A^m=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ //

ex. Hardy-Weinberg law

下一代 $\begin{matrix} \\ GG \\ Gg \\ gg \end{matrix}\begin{matrix} GG & Gg & gg \end{matrix}$

$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix}=A$ 父:GG 母:Gg

$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}=B$ 父:Gg 母:Gg

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix} = C$$

父：gg
母：Gg

$$W_0 = \begin{pmatrix} GG \\ Gg \\ gg \end{pmatrix} \overset{\text{比例各佔}}{\equiv} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad p+q+r=1$$

M = transition matrix $= pA + qB + rC$

$\Rightarrow$ ① $M_0 = M_1 = M_2$ (每一代的 transition matrix 都一樣)

② W_0：這一代的分佈

MW_0：下一代的分佈

M^2W_0：下下一代的分佈，$M(MW_0) = MW_0$

$\Rightarrow$ 一旦 generation 就穩定了.

$$M_0 = \begin{pmatrix} p+\frac{1}{2}q & \frac{1}{2}p+\frac{1}{4}q & 0 \\ \frac{1}{2}q+r & \frac{1}{2} & p+\frac{1}{2}q \\ 0 & \frac{1}{4}q+\frac{1}{2}r & \frac{1}{2}q+r \end{pmatrix} = \begin{pmatrix} a & \frac{a}{2} & 0 \\ b & 1 & a \\ 0 & \frac{b}{2} & b \end{pmatrix}$$

$$\Rightarrow \begin{cases} a+b=1 \\ a=p+\frac{1}{2}q \\ b=\frac{1}{2}q+r \end{cases}$$

$$\Rightarrow W_1 = M_0 W_0 = \begin{pmatrix} p' \\ q' \\ r' \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix}$$

I am OKAY.

令 M_0 中的 p 改为 p'，q 改为 q'，a 改为 a'，
b 改为 b'

⇒ 可发现 $a=a'$，$b=b'$，故一代之后就会稳定

5-4. Invariant Subspaces and the Cayley-Hamilton Theorem.

ex. $T = L_A$, β: standard basis, $[L_A]_\beta = A$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

$$\Rightarrow f(t) = C.P. = \begin{vmatrix} 1-t & 2 \\ 3 & 4-t \end{vmatrix} = t^2 - 5t - 2t^0$$

$$\Rightarrow f(A) = A^2 - 5A - 2I = O$$

Def. $T: V \to V$ is a linear transform. Let $W \subseteq V$ be a subspace, W is called T-invariant if $T(W) \subset W$

ex. (1) $W = \{0\}$
(2) $W = V$
(3) $W = N(T)$
(4) $W = R(T)$
(5) $W = E_\lambda$

ex. Let $v \in V$, $v \neq 0$. Let $W = \text{span}(\{v, T(v), T^2(v), \dots, \})$. Then W is a T-invariant subspace of V.

ex. $f(a,b,c) = (a+b, b+c, c+a)$, $f: \mathbb{R}^3 \to \mathbb{R}^3$

$v = (1,0,0) = e_1$

$\Rightarrow T(v) = (1,0,1)$, $T^2(v) = (1,1,2)$, $\dots$

$\Rightarrow W = \text{span}(\{e_1, T_1(e_1), T_2(e_1), \dots\}$

$= \text{span}(\{e_1, T_1(e_1), T_2(e_1)\}$

$= \mathbb{R}^3$

I am OKAY.

ex. $f: P(\mathbb{R}) \to P(\mathbb{R})$, $T(f(x)) = f'(x)$

Let $v = f(x) = x^2$

$W = \text{span}(\{\underset{\parallel}{x^2}, \underset{\parallel}{2x}, \underset{\parallel}{2}\}) = P_2(\mathbb{R})$

$\quad\quad v \quad T(v) \quad T^2(v)$

lemma. W is the smallest T-invariant subspace containing v.

pf. ① subspace: of course

② T-invariant: if $w \in W$, we want to show that $T(w) \in W$

$\Rightarrow \exists\, a_i$ (可以是0) s.t. $w = \sum_{i=0}^{n} a_i T^i(v)$

$\Rightarrow T(w) = \sum_{i=1}^{n} a_i T^{i+1}(v)$

$\Rightarrow T(w) \in W$

③ The smallest: If Z is T-invariant subspace and $v \in Z$, then $W \subset Z$

$\because v \in Z$

$\therefore T(v) \in Z \Rightarrow T^j(v) \in Z,\ \forall j \in \mathbb{N}$

Q.E.D.

Def. $W = \text{span}(\{v, T(v), T^2(v), \dots\})$ is called a T-cyclic subspace generated by v.

Thm. Let $T: V \to V$ be a L.T., $\dim(V) = n < \infty$. Let W be a T-invariant subspace of V. Let $f(t)$ and $g(t)$ be the C.P. of T and T_W, respectively, then $g(t)$ divides $f(t)$.

($T_W = T|_W$)

pf. Let $\beta=\{v_1,\dots,v_k\}$ be an ordered basis for W.
$\beta'=\{v_1,\dots,v_k,v_{k+1},\dots,v_n\}$ be an ordered basis for V. Let $A=[T]_{\beta'}$ and $B=[T]_\beta$, then $A=\left[\begin{array}{c|c} B_{k\times k} & C \\ \hline O & D \end{array}\right]_{n\times n}$, since

$$\begin{cases} T(v_1)=a_{11}v_1+a_{21}v_2+\cdots+a_{n1}v_n \\ T(v_2)=\cdots \\ \vdots \\ T(v_n)=\cdots \end{cases}$$

(k項)

and that W is T-invariant

$$\Rightarrow f(t)=\left|\begin{array}{c|c} B-tI & C \\ \hline O & D-tI \end{array}\right|=|B-tI|\cdot|D-tI|$$

↑ 上課的HW

$$=g(t)\cdot g(t),\ \text{for}\ g(t)=|D-tI|.$$

$$\Rightarrow g(t)\mid f(t)$$

Q.E.D.

Thm 5.22 Let $T:V\to V$ be a L.T. $\dim(V)=n<\infty$.
Let $\dim W=k$, then: (assume $v\neq 0$) (β is a basis for W)
(1) $\{v, T(v),\dots,T^{k-1}(v)\}$ is linear independent
(2) Let $T^k(v)=-a_0v-a_1T(v)-\cdots-a_{k-1}T^{k-1}(v)$,
then $g(t)=(-1)^k(t^k+a_{k-1}t^{k-1}+\cdots+a_1t+a_0)$
Consequently, $g(T)v=0$.

pf. (1): ∵ $\beta'\triangleq\{v, T(v),\dots,T^{k-1}(v), T^k(v)\}$ is linearly dependent
∴ $\exists a_i$ not all zero s.t. $\sum_{i=0}^{k}a_iT^i(v)=0$
$\Rightarrow a_k=0$, Otherwise since $v, T(v),\dots,T^{k-1}(v)$ is l.i., then $a_0=a_1=\cdots=a_{k-1}=0$

I am OKAY.

$$\Rightarrow a_0 v + a_1 T(v) + \cdots + a_l T^l(v) = 0$$

$$\Rightarrow T^l(v) = \sum_{i=0}^{l-1} \left(-\frac{a_i}{a_l}\right) T^i(v)$$

$$\Rightarrow T^l(v) \in \text{span}(\beta)$$

Let $w \in Z = \text{span}(\beta) \Rightarrow T(w) \in Z$

$$\Rightarrow w = \sum_{i=0}^{l-1} b_i T^i(v)$$

$$\Rightarrow T(w) = \sum_{i=0}^{l-1} b_i T^{i+1}(v)$$

$\Rightarrow T(w) \in Z$ by step 1.

$\Rightarrow Z$ is T-invariant

$\Rightarrow$ by lemma, $\because Z \supseteq W$

$\because W \supseteq Z$

$\therefore W = Z$

$\Rightarrow l = k$.

(2). $\because$ $\begin{cases} T(v) = 0v + 1T(v) + 0T^2(v) + \cdots \\ T(T(v)) = 0v + 0T(v) + 1T^2(v) + \cdots, \cdots, \\ T(T^{k-1}(v)) = T^k(v) \end{cases}$

$\therefore$ $$[T_W]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix} = B$$

$$\Rightarrow g(t) = |B - tI| = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & \vdots & \\ \vdots & \vdots & & -t & \\ 0 & 0 & & 1 & -a_{k-1} - t \end{vmatrix} = ?$$

Claim: $g(t)=(-1)^k(t^k+a_{k-1}t^{k-1}+\cdots+a_1t+a_0)$ —(*)

pf of claim:

$k=1 \Rightarrow B=(-a_0)$

assume (*) is true for dimension $\le k-1$

$\Rightarrow$ When $k=n$, $g(t)=(-t)(-1)^{1+1}(t^{k-1}+a_{k-1}t^{k-2}+\cdots$ ← 沿第一列展開 (induction hypothesis)

$+a_2t+a_1)+(-a_0)(-1)^{1+k}$.

$=(-1)^k(t^k+a_{k-1}t^{k-1}+\cdots+a_2t^2+a_1t+a_0)$

$\because T^k(v)=-a_0v-a_1T(v)-\cdots-a_{k-1}T^{k-1}(v)$

$\therefore a_0v+a_1T(v)+\cdots+a_{k-1}T^{k-1}(v)+T^k(v)=0$

$\Rightarrow (-1)^k g(T)v=0$.

$\Rightarrow g(T)v=0$.

Q.E.D.

Thm 5.23 (Cayley-Hamilton Thm).

Let T be a linear operator on a finite dimensional vector space V, and $f(t)$ be its C.P., then $f(T)=0$ (zero function)

pf. ① $v=0 \Rightarrow$ of course!

② $v\neq 0$: Let $W=\text{span}(\{v, T(v), \cdots\})$, and $f(t)$ be the C.P. of T, $g(t)$ be the C.P. of T_W

$\Rightarrow$ by Thm5.21, $f(t)=g(t)q(t)$ for some $q(t)$

$\Rightarrow f(T)=g(T)q(T)$

$\Rightarrow f(T)v=(g(T)q(T))v=g(T)(q(T)v)$

$=g(T)\cdot 0=0$ (zero vector)

↑ Thm5.22

Q.E.D.

I am OKAY.

Cor. $f(A)=0$ (零矩陣

pf. Let $T=L_A$, $[L_A]_\beta=A$ in Thm 5.23, where β is the standard basis, done.

Q.E.D.

Ch6 Inner Product Spaces

§6-1 Inner Product and Norms

Def. Let V be a vector space with a field F.
An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to F$, satisfying $\forall x, y, z \in V$ and $c \in F$.

(1) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(2) $\langle cx, y \rangle = c\langle x, y \rangle$

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (共軛)

(4) $\langle x, x \rangle > 0$ if $x \neq 0$

ex. $V = F^n$ ($\mathbb{C}^n$ or $\mathbb{R}^n$)

$\langle x, y \rangle = \sum_{i=1}^{n} \overline{y_i} x_i$,

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. This is called the standard inner product on F^n.

ex2. $V = M_{n\times n}(F)$, $\langle A, B \rangle = tr(B^* A)$. Here $(B^*)_{ij} = \overline{B_{ji}}$. and B^* is called the adjoint of B. (complex conjugate)

$B = \begin{pmatrix} 1+i & 2-i \\ 3 & 2+i \end{pmatrix} \Rightarrow B^* = \begin{pmatrix} 1-i & 3 \\ 2+i & 2-i \end{pmatrix}$

This is called Frobenious inner product.

I am OKAY.

ex3. $V = C[0,1]$, F = The set of real-valued cont. functions on $[0,1]$.
Define $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt = \int_0^1 f(t)g(t)\,dt$

Def. Let V be an inner product space,
(1) $x, y \in V$ is said to be perpendicular or orthogonal if $\langle x, y \rangle = 0$
(2) $S \subset V$ is orthogonal if $\langle x, y \rangle = 0$, $\forall\, x, y \in S$
(3) $S \subset V$ is orthonormal if S is orthogonal and $\|x\| = 1$, $\forall\, x \in S$
(4) Let $x \in V$, the length or norm of x is defined to be $\|x\| = \sqrt{\langle x, x \rangle}$

ex. $V = \mathbb{R}^3$, $\beta = \{e_1, e_2, e_3\}$ is an orthonormal basis.

ex. H = the set of complex-valued cont. functions on $[0, 2\pi]$. Let $f, g \in H$,

$$\langle f, g \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt, \quad t \in [0, 2\pi],$$

$f(t) \in \mathbb{C}$.

Let $S = \{f_n(t),\ f_n(t) = e^{int} = \cos(nt) + i\sin(nt),\ n \in \mathbb{Z}\}$. Claim: S is orthonormal.

pf of claim: $\|f_n(t)\| = \langle f_n(t), f_n(t) \rangle$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{int}e^{-int}\,dt = 1.$$

$$\langle f_n(t), f_m(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{int}e^{-imt}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n-m)t + i\sin(n-m)t\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n-m)t\,dt + \frac{i}{2\pi}\int_0^{2\pi} \sin(n-m)t\,dt$$

$$= 0.$$

Thm 6.1 Let V be an inner product space. then for all $x, y, z \in V$ and $c \in F$,

(1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(2) $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$ ⇒對第二個變量而言內積未必是線性

(3) $\langle x, 0\rangle = \langle 0, x\rangle = 0$

(4) $\langle x, x\rangle = 0 \iff x = 0$

(5) Given y and z, if $\langle x, y\rangle = \langle x, z\rangle$, $\forall x \in V$, then $y = z$. Consequently, if $\langle x, 0\rangle = 0$, $\forall x \in V$, then $x = 0$.

pf. (5): $\langle y-z, y-z\rangle = \langle y-z, y\rangle + \langle y-z, -z\rangle$ (by ①)

$= \langle y-z, y\rangle - \langle y-z, z\rangle$ (by ②)

$= \langle y-z, y-z\rangle$ (by ①)

$= 0$

$\therefore y-z=0$

$\Rightarrow y=z.$

(1) ~ (4) is obvious.

Q.E.D.

I am OKAY.

Thm 6.2 Let V be an inner product space over F, then $\forall x, y \in V$ and $c \in F$:

(1) $\|cx\| = |c| \|x\|$

(2) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Cauchy-Schwarz ineq.)

(3) $\|x+y\| \leq \|x\| + \|y\|$ (Triangular ineq.)

ex. $V = \mathbb{R}^2$, $x = (a_1, b_1)$, $y = (a_2, b_2)$

$\Rightarrow (a_1 a_2 + b_1 b_2)^2 \leq (a_1^2 + b_1^2)(a_2^2 + b_2^2)$

ex. $V = C([0,1])$, $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$.

$\Rightarrow \left(\int_0^1 f(t) g(t)\, dt \right)^2 \leq \left(\int_0^1 f(t)\, dt \right) \left(\int_0^1 g^2(t)\, dt \right)$

ex. $V = M_{n \times n}(\mathbb{R})$, $A, B \in V$

$\Rightarrow |tr(B^* A)| \leq$

pf of Thm 6.2: Let $x, y \in V$ and $c \in F$, then:

(2) $\langle x - cy, x - cy \rangle = \langle x, x \rangle + \langle -cy, x \rangle + \langle x, -cy \rangle + \langle -cy, -cy \rangle$

$= \langle x, x \rangle - c\langle y, x \rangle - c\langle x, y \rangle + |c|^2 \langle y, y \rangle$

① Let $y \neq 0$. Choose $c = \dfrac{\langle x, y \rangle}{\langle y, y \rangle}$

$\therefore \|x - cy\|^2 = \langle x, x \rangle - \dfrac{|\langle x, y \rangle|^2}{\langle y, y \rangle} - \dfrac{|\langle x, y \rangle|^2}{\langle y, y \rangle} + \dfrac{|\langle x, y \rangle|^2}{(\langle y, y \rangle)^2} \langle y, y \rangle$

$= \langle x, x \rangle - \dfrac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$

≥ 0.

$$\therefore \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \geq 0$$

$$\Rightarrow |\langle x,y\rangle| \leq \|x\| \cdot \|y\|$$

② let $y=0$, the ineq. still holds.

(3) $\because \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$

$\|x+y\|^2$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$\therefore \|x+y\| \leq \|x\| + \|y\|$.

Q.E.D.

I am OKAY.

§6-2 The Gram-Schmidt Orthogonalization

Def. Let V be an inner product space. A subset of V is an orthonormal basis for V if it is an ordered basis that is orthonormal.

ex. $V = \mathbb{R}^3$, $\beta = \{e_1, e_2, e_3\}$ is an orthonormal basis.

Q: Given an inner product space V. Does V have an orthonormal basis?
A: Yes! (Thm 6.5).

Thm 6.3 Let V be an inner product space. $S \doteq \{v_1, \dots, v_k\}$ is an orthogonal set of nonzero vectors. Let $W = \text{span}(S)$, then if $w \in W$, then $w = \sum_{i=1}^{k} \left(\frac{\langle w, v_i \rangle}{\|v_i\|^2} \right) v_i$

pf. Let $w = \sum_{i=1}^{k} a_i v_i = a_1 v_1 + \cdots + a_k v_k$

$\Rightarrow \langle w, v_j \rangle = \langle a_j v_j, v_j \rangle$ since S is orthogonal.

$\Rightarrow \langle w, v_j \rangle = a_j \langle v_j, v_j \rangle, \forall j$

$\Rightarrow a_j = \frac{\langle w, v_j \rangle}{\|v_j\|^2}$

Q.E.D.

Process and Orthogonal Complements

Cor 1. If S is orthonormal, then $a_j = \langle w, v_j \rangle$

Cor 2. S is linearly independent.

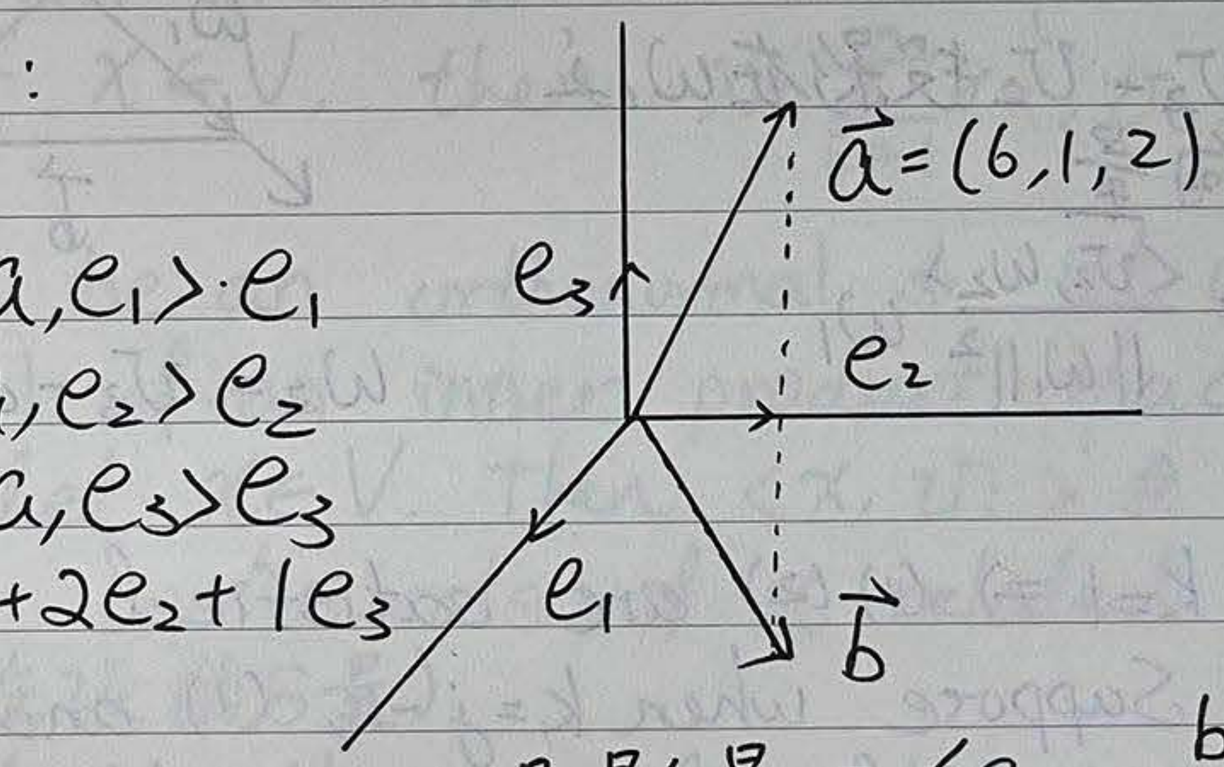
pf. $\because \sum_{i=1}^{n} a_i v_i = 0$

$\therefore a_i = 0, \forall i$, by Thm 6.3.

Q.E.D.

* 几何意義：

$(6,2,1) = \langle a, e_1 \rangle \cdot e_1$
$+ \langle a, e_2 \rangle e_2$
$+ \langle a, e_3 \rangle e_3$
$= 6e_1 + 2e_2 + 1e_3$



投影量 $= \langle a, \frac{b}{\|b\|} \rangle \cdot \frac{b}{\|b\|}$

向 $= \frac{\langle a, b \rangle}{\|b\|^2} b$

Thm 6.4 (Gram Schmidt)

Let V be an inner product space and $S = \{v_1, \dots, v_n\}$ be a linearly indep. subset of V. Define $S' = \{w_1, w_2, \dots, w_k\}$, where

(1) $v_1 = w_1$

(2) $w_j = v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, w_i \rangle}{\|w_i\|^2} w_i$, $2 \le i \le k$

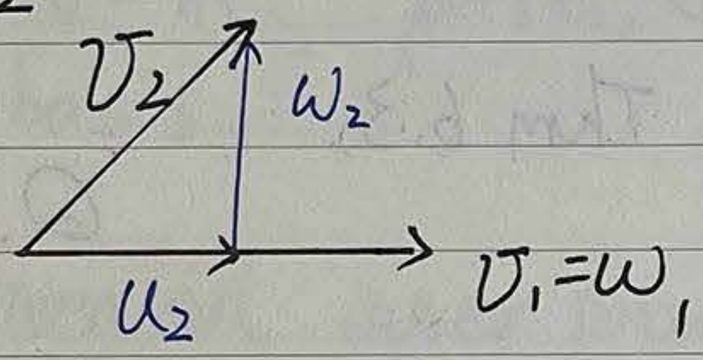
I am OKAY.

(即 $v_j - v_j$ 投影在 w_i $(1 \le i \le j-1)$ 之投影向量)

then (1) $S' = \{w_1, \dots, w_k\}$ is an orthogonal set of nonzero vectors

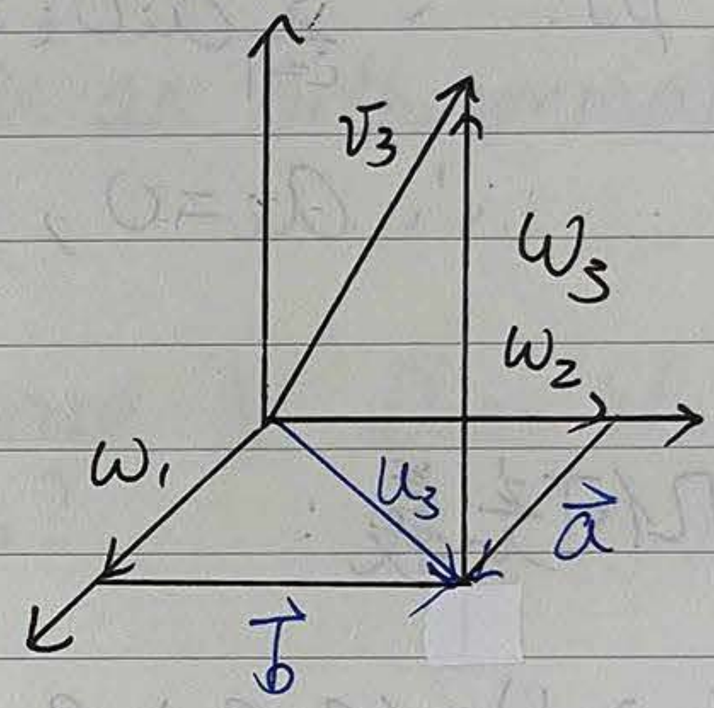
(2) $\text{span}(S) = \text{span}(S')$

$k=2$



$w_2 = v_2 - v_2$ 投影在 w_1 之向量

$$\Rightarrow u_2 = \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$



$$w_3 = v_3 - a - b$$

pf. When $k=1 \Rightarrow$ (1) (2) are satisfied.

Suppose when $k = j-1$, (1) and (2) are satisfied.

$\Rightarrow$ Let $w_j = v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, w_i \rangle}{\|w_i\|^2} w_i$, then

$\forall\, l = 1, \dots, j-1$,

$$\langle w_j, w_l \rangle = \langle v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, w_i \rangle}{\|w_i\|^2} w_i, w_l \rangle$$

$$= \langle v_j, w_l \rangle - \langle \frac{\langle v_l, w_l \rangle}{\|w_l\|^2} w_l, w_l \rangle$$

$$= \langle v_j, w_l \rangle - \langle v_j, w_l \rangle$$

$$= 0$$

Now we claim that $w_j \neq 0$.

∵ If $w_j = 0$, then v_j can be written in a form of linear combination of $\{v_i\}_{i=1}^{j-1}$ by induction hypothesis ×

∴ when $k=j$, (1) is true by the contradiction.

⇒ (2) is true for $k=j$.

Q.E.D.

Thm 6.5 Let V be a nonzero finite-dimensional inner product space. Then:

(1) $\exists$ an orthonormal basis $\beta = \{v_1, \dots, v_n\}$

(2) If $x \in V$, then $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$

Def. Let β be an orthonormal subset (possibly infinite) of an inner product space V and let $x \in V$. Then $\langle x, v_i \rangle \triangleq$ the Fourier coefficients of x related to β.
(x 投影在 v_i 的分量)

ex. Let $V = H =$ the set of continuous complex-valued functions on $[0, 2\pi]$. Define

$$\begin{cases} \langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\,dt \\ \beta = \{f_n(t) = e^{int} : n \in \mathbb{Z}\} : \text{orthonormal set} \end{cases}$$

$$\Rightarrow \langle f(t), f_n(t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\,dt$$

= the n^{th} Fourier coefficient of f related to β

I am OKAY.

Q: Does $f(x) = \sum_{n=-\infty}^{\infty} \langle f, f_n \rangle f_n(x)$?

A: 高徵会討論.

Def. Let S be a nonempty subset of an inner product space V. The orthogonal complement of S in V, say $S^\perp$ (S-perp), is defined to be $\{x \in V : \langle x, y \rangle = 0, \forall y \in S\}$.

ex. $S = \{0\} \Rightarrow S^\perp = V$

$S = V \Rightarrow S^\perp = 0$

Facts: (1) $S^\perp$ is a subspace of V

(2) $S \cap S^\perp = \{0\}$, where S is a subspace. (反例即, 故S要先是subspace)

pf: (1) is obvious

(2) Suppose $x \neq 0$ and $x \in S \cap S^\perp$

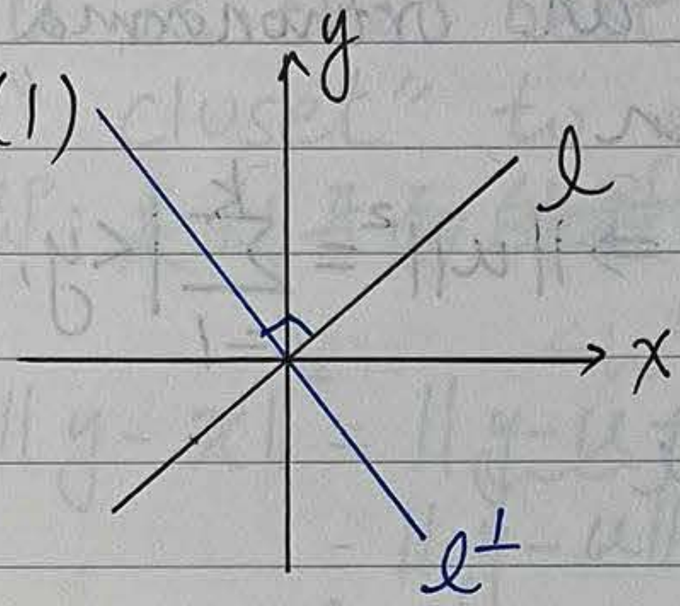
$\Rightarrow \langle x, x \rangle = 0$

$\Rightarrow x = 0 \quad \rightarrow\leftarrow$

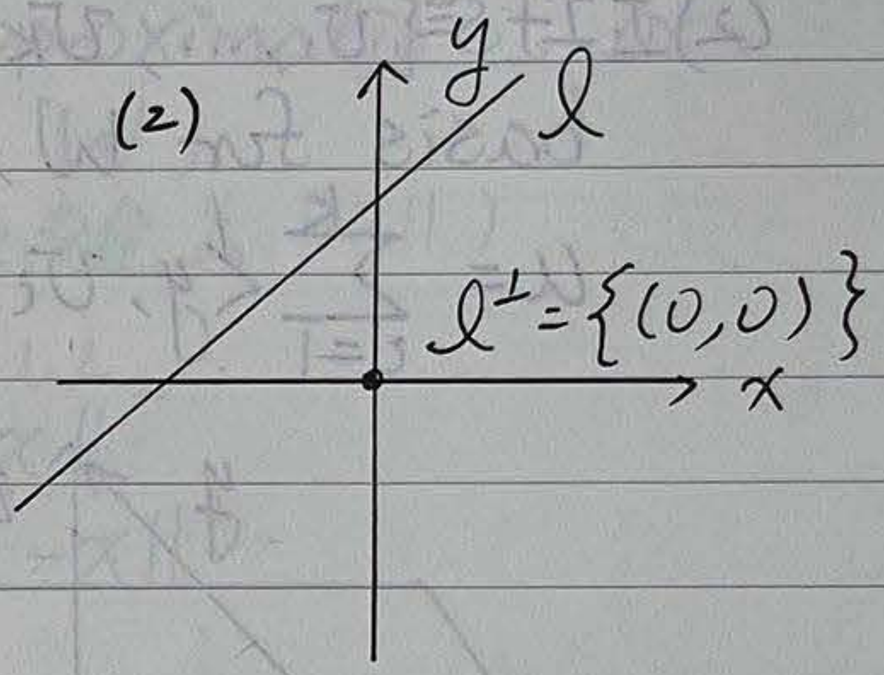
$\Rightarrow S \cap S^{\perp} = \{0\}$

Q.E.D.

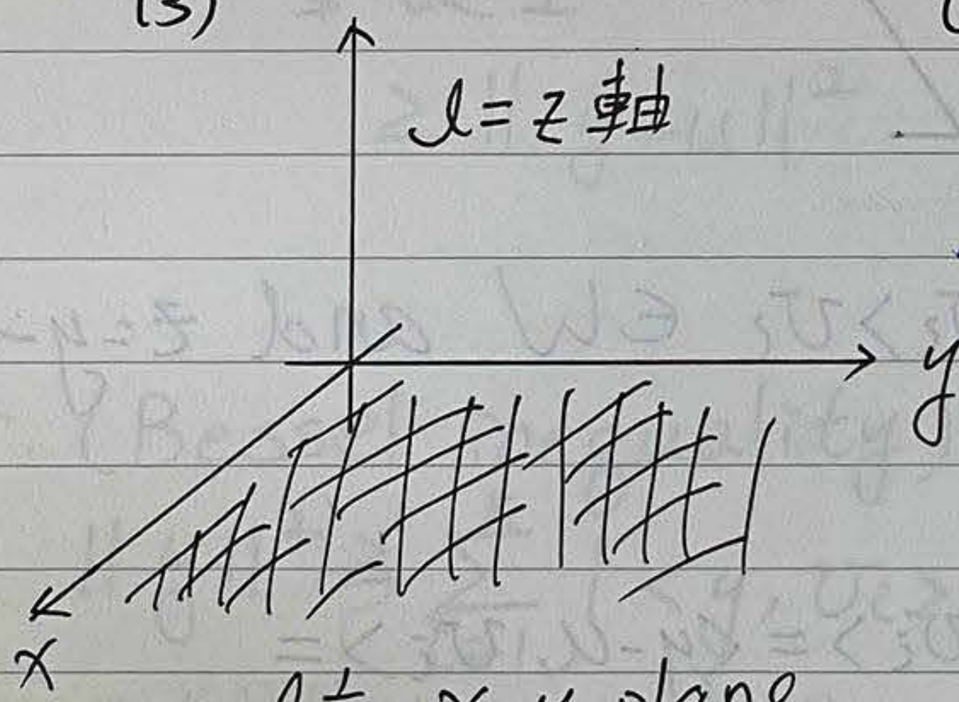
ex. (1)



(2)

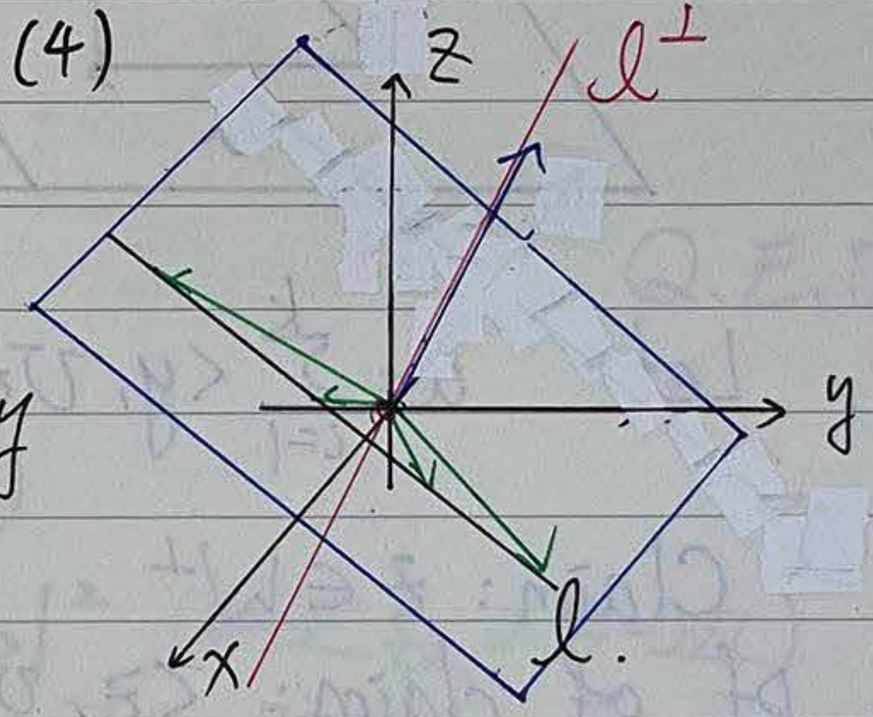


(3)



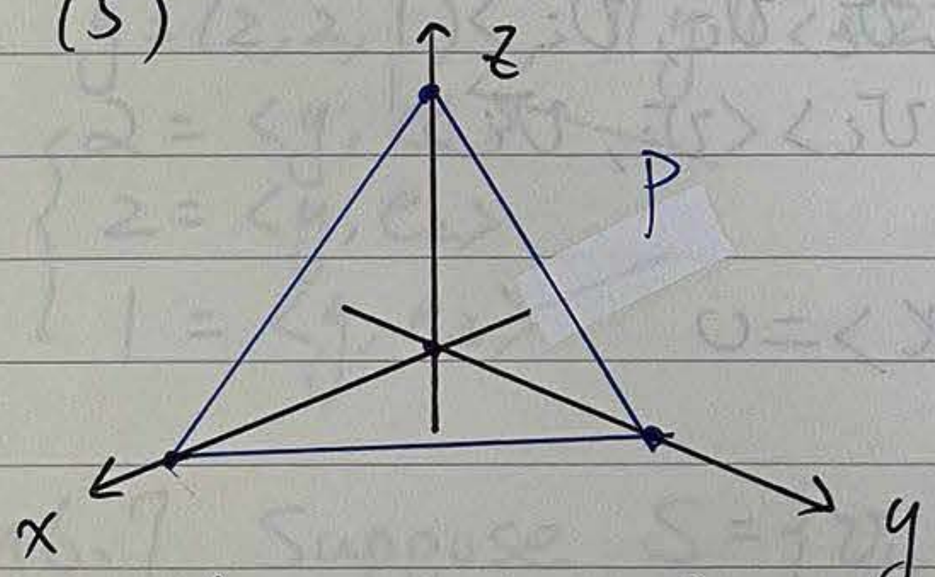
$\ell^{\perp}$ = x-y plane

(4)



$\ell^{\perp}$ = 过原点之直線，此直線之方向為「过原点及 ℓ 的平面」之法向量

(5)



$P^{\perp} = \{(0,0,0)\}$

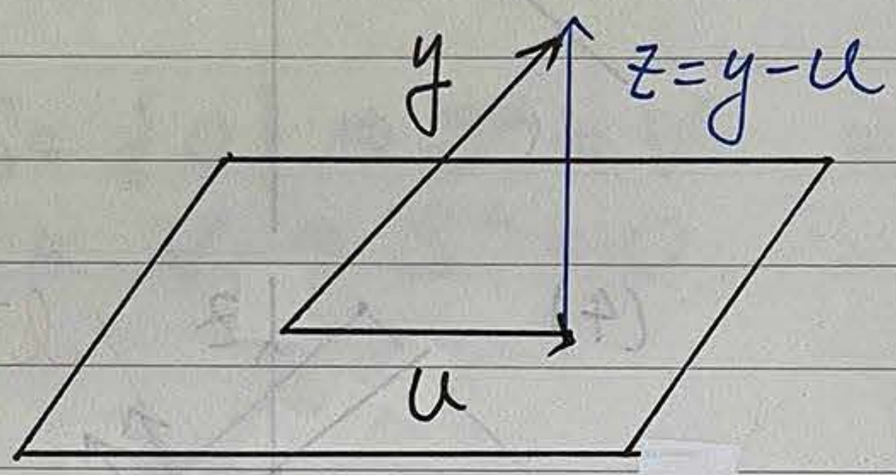
I am OKAY.

Thm 6.6 Let V be an inner product space, $\dim(V)<\infty$. Let W be a subspace of V, and $y\in V$, then:

(1) $\exists!\ u\in W$ and $z\in W^\perp$ s.t. $y=u+z$

(2) If $S\triangleq\{v_1,\dots,v_k\}$ be an orthonormal basis for W, then

$$u=\sum_{i=1}^{k}\langle y,v_i\rangle v_i \Rightarrow \|u\|^2=\sum_{i=1}^{k}|\langle y,v_i\rangle|^2$$



pf. Let $u=\sum_{i=1}^{k}\langle y,v_i\rangle v_i \in W$ and $z=y-u$

Claim: $z\in W^\perp$

pf of claim: $\langle z,v_i\rangle=\langle y-u,v_i\rangle$

$=\langle y,v_i\rangle-\langle u,v_i\rangle$

$=\langle y,v_i\rangle-\langle\langle y,v_i\rangle v_i,v_i\rangle$

$=\langle y,v_i\rangle-\langle y,v_i\rangle\langle v_i,v_i\rangle$ (1).

$=0$.

$\Rightarrow \forall x\in W,\ \langle z,x\rangle=0$

$\Rightarrow z\in W^\perp$

Let $y=u+z=u'+z'$, $u,u'\in W$ and $z,z'\in W$.

$\because u-u'=z'-z\in W\cap W^\perp=\{0\}$

$\therefore u=u',\ z=z'$

Q.E.D.

Cor. $\|y-u\| \le \|y-x\|$, $\forall x \in W$. The equality holds only if $u=x$.
(u is the best "approximation" or "closet" to y in W)
(即：点到平面的最短距离为 $\|z\|$)

pf. $\|y-x\|^2 = \|\underbrace{y-u}_{\in W^\perp} + \underbrace{u-x}_{\in W}\|^2$
$= \|y-u\|^2 + \|u-x\|^2$
↑ 毕氏定理 (Thm 6.2)
$\ge \|y-u\|^2$

Q.E.D.

Cor. (Bessel inequality)
$\|y\|^2 \ge \sum_{i=1}^{k} |\langle y, v_i\rangle|^2$ (斜边长 ≥ 另一边)
"=" 成立時, $x \in \text{span}(S)$ $= \|u\|^2$.

ex. $y=(2,2,1)$, $\|y\|^2 = 2^2+2^2+1^2 = 3^2$.
$\begin{cases} 2 = \langle y, e_1\rangle \\ 2 = \langle y, e_2\rangle \\ 1 = \langle y, e_3\rangle \end{cases}$

Thm 6.7. Suppose $S=\{v_1, \dots, v_k\}$ is an orthonormal set is an n-dimensional inner product space V, and $W=\text{span}(S)$.
then: (1) $V = W \oplus W^\perp$
(2) $\dim(V) = \dim(W) + \dim(W^\perp)$

I am OKAY.

Fact: $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

pf. Let $V = H$, define

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt, \quad \forall f, g \in H.$$

Take $y = f(t) = t$,

$$\Rightarrow \|y\|^2 = \langle f, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} t^2\, dt = \frac{4}{3}\pi^2$$

Let $S_n = \{ f_k(t) = e^{ikt}, -n \le k \le n \}$

$$\because \langle t, f_k(t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} t\, e^{-ikt}\, dt$$

$$= \frac{1}{2\pi} \left(\frac{t e^{-ikt}}{-ik} + \frac{e^{-ikt}}{k^2} \right) \Big|_0^{2\pi}$$

$$= \begin{cases} -\frac{1}{ik}, & k \neq 0 \\ \pi, & k = 0 \end{cases}$$

$\therefore$ By Bessel ineq,

$$\left(\frac{4\pi}{3}\right)^2 \ge 2 \sum_{k=1}^{n} \frac{1}{k^2} + \pi^2$$

$$\Rightarrow \frac{\pi}{6} \ge \sum_{k=1}^{n} \frac{1}{k^2}$$

$$\Rightarrow \frac{\pi}{6} \ge \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\because f(t) = t = \sum_{n=-\infty}^{\infty} \langle f, f_n \rangle f_n(t)$$

∴ by 高微, we can prove that "=" holds.

Q.E.D.

I am OKAY.

§6-3 The Adjoint of a Linear Operator

Def. Let V be a inner product space, $\dim(V) < \infty$.
Let $T: V \to V$ be a L.T.
Define T^*, the adjoint operator of T, s.t.
($T^*: V \to V$ is linear) and $\forall x, y \in V$,
we have $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$

Q: Is such definition well-defined?
(1) Existence of T^*
(2) Uniqueness.
(3) Is T^* linear?

Thm 6.8 Let $g: V \to F$ be linear, then $\exists! y \in V$
s.t. $g(x) = \langle x, y \rangle$

ex. Let $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x + y = \langle (x,y), (1,1) \rangle$

pf. Let $\beta = \{v_1, \dots, v_n\}$ be an orthonormal basis for V.

$\Rightarrow \exists a_i \in F$ s.t. $x = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$

$\Rightarrow g(x) = g\left(\sum_{i=1}^{n} a_i v_i\right)$

$= \sum_{i=1}^{n} a_i g(v_i)$ (since g is linear)

$= \sum_{i=1}^{n} \langle x, v_i \rangle g(v_i)$ (by Cor of Thm 6.3)

$= \left\langle x, \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \right\rangle$

We can take $y = \sum_{i=1}^{n} \overline{g(v_i)} v_i$

Uniqueness: Suppose $\exists\, y, y'$ s.t. $g(x) = \langle x, y \rangle$
$= \langle x, y' \rangle$
$\Rightarrow \langle x, y - y' \rangle = 0$
$\Rightarrow y - y' = 0$
$\Rightarrow y = y'$

Q.E.D

Thm 6.9. The def of T^* is well-defined.
That is, T^* exists, is unique and is linear.

pf. (1) Existence: Let y be arbitrary fixed.
Define $g(x) = \langle T(x), y \rangle$. Then $g: V \to F$ is linear.
$\because$ By Thm 6.8, $\exists!\, z \in V$ s.t. $g(x) = \langle T(x), y \rangle$
$= \langle x, z \rangle$
$\therefore$ z is a function of y
$\Rightarrow$ Define a function $T^*(y) = z$
$\Rightarrow \langle T(x), y \rangle = \langle x, T^*(y) \rangle$

(2) Uniqueness: Suppose $T^*, T_1 : V \to V$ s.t.
$\langle T(x), y \rangle = \langle x, T^*(y) \rangle = \langle x, T_1(y) \rangle$
$\Rightarrow \langle x, T^*(y) - T_1(y) \rangle = 0$, $\forall x, y \in V$.
$\Rightarrow T^*(y) - T_1(y) = 0$, $\forall y \in V$
$\Rightarrow T^*(y) = T_1(y)$, $\forall y \in V$
$\Rightarrow T^* = T_1$
$\Rightarrow T^*$ is unique

I am OKAY.

(3) Wanted: $T^*(cy_1+y_2) = cT^*(y_1) + T^*(y_2)$

$\iff \langle x, T^*(cy_1+y_2)\rangle = \langle x, cT^*(y_1) + T^*(y_2)\rangle$, $\forall x$

$\because \langle x, T^*(cy_1+y_2)\rangle = \langle T(x), cy_1+y_2\rangle$

$= \bar{c}\langle T(x), y_1\rangle + \langle T(x), y_2\rangle$

$= \bar{c}\langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle$

$= \langle x, cT^*(y_1) + T^*(y_2)\rangle$

$\therefore T^*(cy_1+y_2) = cT^*(y_1) + T^*(y_2)$

$\Rightarrow T^*$ is linear

Q.E.D.

Q: How to find T^*?

Thm 6.10. Let β be an orthonormal basis for V. Then $[T^*]_\beta = [T]_\beta^*$. Note that if $A=(a_{ij})$, then $A^* = (\overline{a_{ji}})$.

(A^* = A 轉置後，將每一元都以其共軛取代)

pf. Let $A=[T]_\beta$ and $B=[T^*]_\beta$.

$\Rightarrow B_{ij} = \langle T^*(v_j), v_i\rangle$

$= \overline{\langle v_i, T^*(v_j)\rangle}$

$= \overline{\langle T(v_i), v_j\rangle}$

$= \overline{A_{ji}}$

$= (A^*)_{ij}$

$\Rightarrow B = A^*$

Q.E.D.

Remarks. If β is not an orthonormal basis, then the Thm is not true!

Cor. $(L_A)^* = L_{A^*}$

pf: $\because [L_A]_\beta = A$

$\therefore$ by Thm 6.10, $[(L_A)^*]_\beta = [L_{A^*}]_\beta$

Q.E.D.

ex. Let $T: F^2 \to F^2$ be defined by $T(a_1, a_2)$
$= (2ia_1 - 3a_2,\ a_1 + a_2)$

Given $\beta = \{e_1, e_2\}$, then $[T]_\beta = ?$

$\because \begin{cases} T(e_1) = (2i, 1) = 2ie_1 + e_2 \\ \quad\quad = \langle T(e_1), e_1\rangle e_1 + \langle T(e_1), e_2\rangle e_2 \\ T(e_2) = (-3, 1) = -3e_1 + e_2 \\ \quad\quad = \langle T(e_2), e_1\rangle e_1 + \langle T(e_2), e_2\rangle \end{cases}$

$\therefore [T]_\beta = \begin{pmatrix} \langle T(e_1), e_1\rangle & \langle T(e_2), e_1\rangle \\ \langle T(e_1), e_2\rangle & \langle T(e_2), e_2\rangle \end{pmatrix} = \begin{pmatrix} 2i & -3 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow [T^*]_\beta = \begin{pmatrix} -2i & 1 \\ -3 & 1 \end{pmatrix}$

(Thm 3.10)

$\Rightarrow T^*(a_1, a_2) = (-2ia_1 + a_2,\ -3a_1 + a_2)$

Remark. If $\dim(V) = \infty$, then T^* may not exist.
(exercise 24 of 6-3)

I am OKAY.

Thm 6.11 Let V be an inner product space, and let T and U be linear operators on V. Then:

(1) $(T+U)^* = T^* + U^*$

(2) $(cT)^* = \bar{c}T^*$

(3) $(TU)^* = U^*T^*$

(4) $T^{**} = T$

(5) $I^* = I$.

Cor. 若 T, U 為 $n \times n$ 矩陣，(1)~(5) 亦成立.

pf. of (4): We want to show that $\langle x, T^{**}(y)\rangle = \langle x, T(y)\rangle$, $\forall x, y \in V$.

$\because \langle x, T^{**}(y)\rangle = \overline{\langle T^*(x), y\rangle}$

$= \overline{\langle y, T^*(x)\rangle}$

$= \overline{\langle T(y), x\rangle}$

$= \langle x, T(y)\rangle$, $\forall x, y \in V$

$\therefore T^{**}(y) = T(y)$, $\forall y \in V$

其他留做 exercise.

Q.E.D.

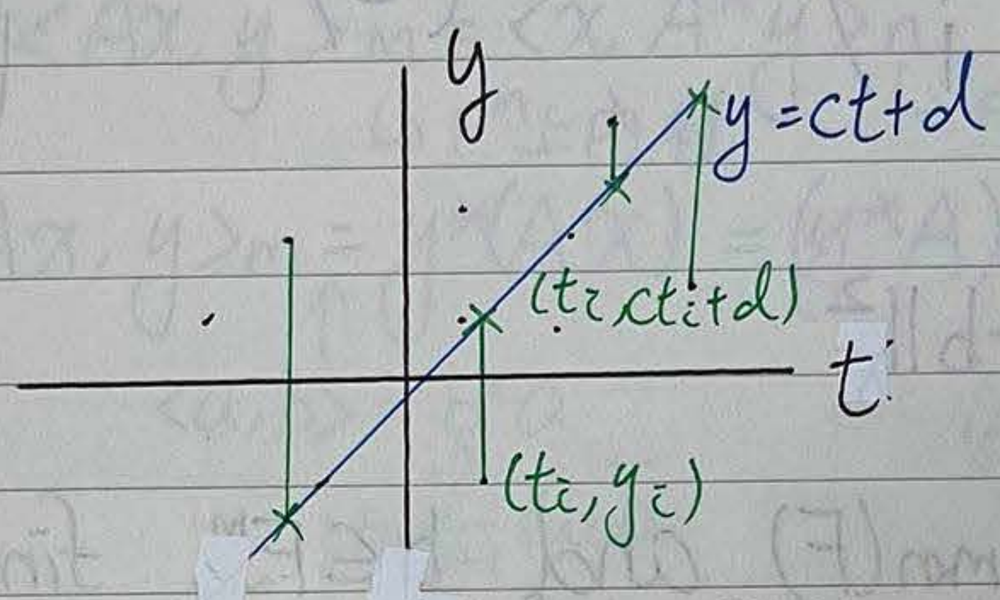
△ Applications:

(1) Data Fitting

(2) Data Analogs (Hibert Huang Transform)

△ Least Squares Approximation.

Q: Given (t_i, y_i), $i = 1, 2, \dots, m$. How to find a "best" line to represent those data?



Q: What is "best" mean?

A: 全部誤差 $E = \sum_{i=1}^{n} (y_i - ct_i - d)^2$ 為最小.

① 微積分的方法 (exercise 23)

⇒ 找 c 和 d s.t. E 最小. 即令

$f(c,d) \triangleq \sum_{i=1}^{n} (y_i - t_i c - d)^2$, 求解

$$\arg\min_{c,d} f(c,d) = ?$$

⇒ 用微分去解.

② 線代的方法.

Let

$$A = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_m & 1 \end{pmatrix}, \quad x = \begin{pmatrix} c \\ d \end{pmatrix}, \quad b = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

then $E = \|Ax - b\|^2$.

I am OKAY.

推廣：若所求 model 為 $y=at^2+bt+c$，則：

$$A=\begin{pmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ \vdots & \vdots & \vdots \\ t_m^2 & t_m & 1 \end{pmatrix},\quad x=\begin{pmatrix} a \\ b \\ c \end{pmatrix},\quad b=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

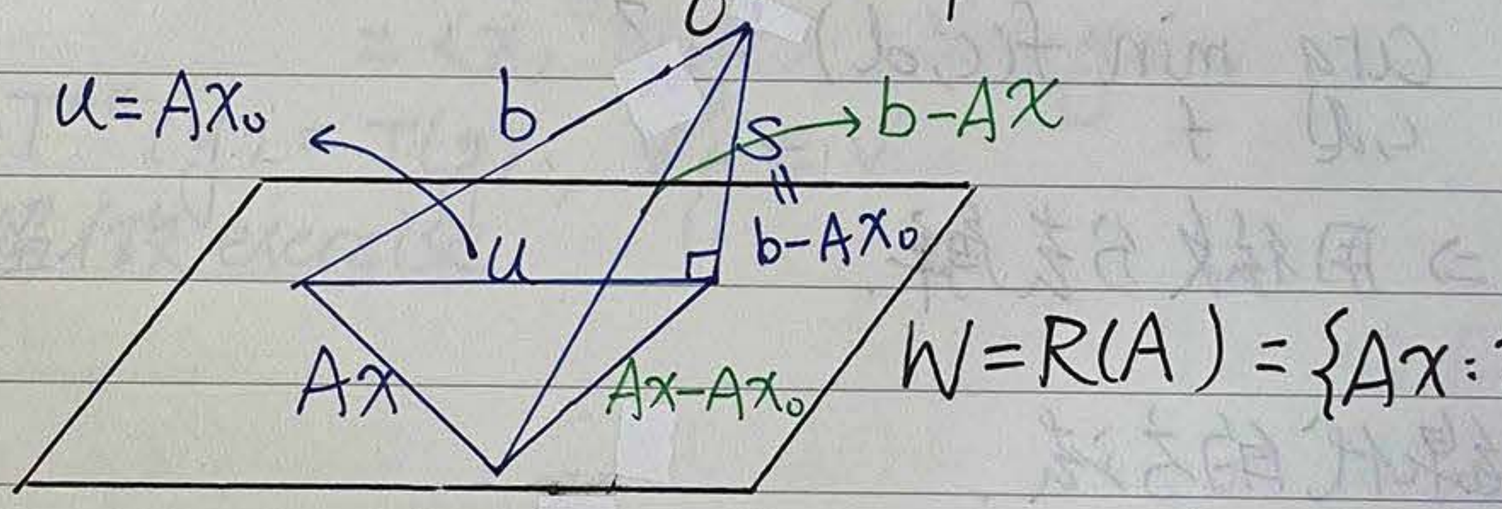
$\Rightarrow E=\|Ax-b\|^2$

Q: Given $A\in M_{m\times n}(F)$ and $b\in\mathbb{R}^m$, find an $x_0\in\mathbb{R}^n$ s.t. $\|Ax_0-b\|$ is a minimum

ex. Given a set $\{(t_i,y_i)\}_{i=1}^m$ of measurements, find a "best" polynomial of degree $n-1$ to represent such set of measurements.

這類問題稱為 least square problems.

解：



step 1. 將 b 投影到 $W=R(A)$ 之子空間得 u

2. $u=Ax_0$，x_0 即為解（不見得唯一）

3. 若 $\text{rank}(A)=n$，則有唯一解 $x_0=(A^*A)^{-1}A^*b$

Remark. If t_i are all distinct, assume $m\geq n$, then $\text{rank}(A)=n$.

若要證此解法之正確性，需先證幾個定理

lemma1. Let $A \in M_{n\times n}(F)$, $x \in F^n$, and $y \in F^m$.
Then $\langle Ax, y\rangle_m = \langle x, A^*y\rangle_n$
↳ F^m上內積　↳ F^n上內積

pf. $\langle Ax, y\rangle_m = y^*(Ax) = (y^*A)x = (A^*y)^*x$

$\langle a, b\rangle = b^*a$.
$= \sum_{i=1}^{n} a_i \overline{b_i}$

$(AB)^* = B^*A^*$
$A^{**} = A$

$= \langle x, A^*y\rangle_n$

Q.E.D.

lemma2. Let $A \in M_{m\times n}(F)$. Then $\text{rank}(A^*A) = \text{rank}(A)$

pf. by dim Thm,
$n =$ Nullity(A) + rank(A) = Nullity(A^*A) + rank(A^*A),
We claim that $N(A) = N(A^*A)$
pf of claim: $\because$ if $x \in N(A)$, then $Ax = 0$
$\therefore A^*Ax = 0$
$\Rightarrow x \in N(A^*A)$.
$\Rightarrow N(A) \subseteq N(A^*A)$.
On the other hand, if $x \in N(A^*A)$, then
$A^*Ax = 0$.
$\Rightarrow 0 = \langle x, A^*Ax\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$
$\Rightarrow Ax = 0$
$\Rightarrow x \in N(A)$
$\Rightarrow N(A) \supseteq N(A^*A)$
$\Rightarrow N(A) = N(A^*A)$

I am OKAY.

$\Rightarrow$ by claim, $\text{Nullity}(A) = \text{Nullity}(A^*A)$

$\Rightarrow$ $\text{rank}(A) = \text{rank}(A^*A)$

Q.E.D.

Cor. If $A \in M_{m\times n}(F)$ and $\text{rank}(A)=n$, then A^*A is invertible.

Thm 6.12. Let $A \in M_{m\times n}(F)$ and $b \in F^m$, then

① $\exists\, x_0 \in F^n$ s.t.

$$\begin{cases} (A^*A)x_0 = A^*b \\ \|Ax_0 - b\| \le \|Ax - b\|, \ \forall x \in F^n \end{cases}$$

② Further more, if $\text{rank}(A)=n$, then $x_0 = (A^*A)^{-1}A^*b$

pf. $\because$ by Thm 6.6 and its Cor, $\exists Ax_0 \in W$ closet to y, where $x_0 \in F$

$\therefore \|Ax_0 - b\| = \|Ax_0 - b + Ax - Ax\|$

$= \|Ax_0 - b\|^2 + \|Ax - Ax_0\|^2$ (by Thm 6.2)

$\ge \|Ax_0 - b\|^2$

Also, $\because$ by Thm 6.6 and its Cor,

$b - Ax_0 \in W^\perp$

$\therefore \langle Ax, b - Ax_0 \rangle = 0, \ \forall x \in F^n$

$\Rightarrow \langle x, A^*(b - Ax_0) \rangle = 0, \ \forall x \in F^n$

$\Rightarrow A^*(b - Ax_0) = 0$

$\Rightarrow A^*Ax_0 = A^*b$

② ∵ If $\text{rank}(A)=n$, then by Cor of lemma 2, A^*A is invertible

∴ $x_0=(A^*A)^{-1}A^*b$

Q.E.D.

△ Finding the min. solution of $Ax=b$

Q: Given a consistent system $Ax=b$, find a solution s s.t. $A\in M_{m\times n}$, $\|s\|\le\|x\|$ for any x satisfying $Ax=b$

lemma 3. Let $A\in M_{m\times n}(F)$, then

① $N(A)=(\text{Range}(A^*))^\perp$

② $N(A)^\perp=\text{Range}(A^*)$

③ $F^n=\text{Range}(A^*)\oplus N(A)$ (Thm 6.7)

pf. ① Let $x\in N(A)$, then $Ax=0$

$\Leftrightarrow \langle Ax,y\rangle=0,\ \forall y\in F^m$

$\Leftrightarrow \langle x,A^*y\rangle=0,\ \forall y\in F^m$

$\Leftrightarrow x\in R(A^*)^\perp$

$\Leftrightarrow N(A)\subseteq R(A^*)^\perp$

By the inverse of above, $N(A)\supseteq R(A^*)^\perp$

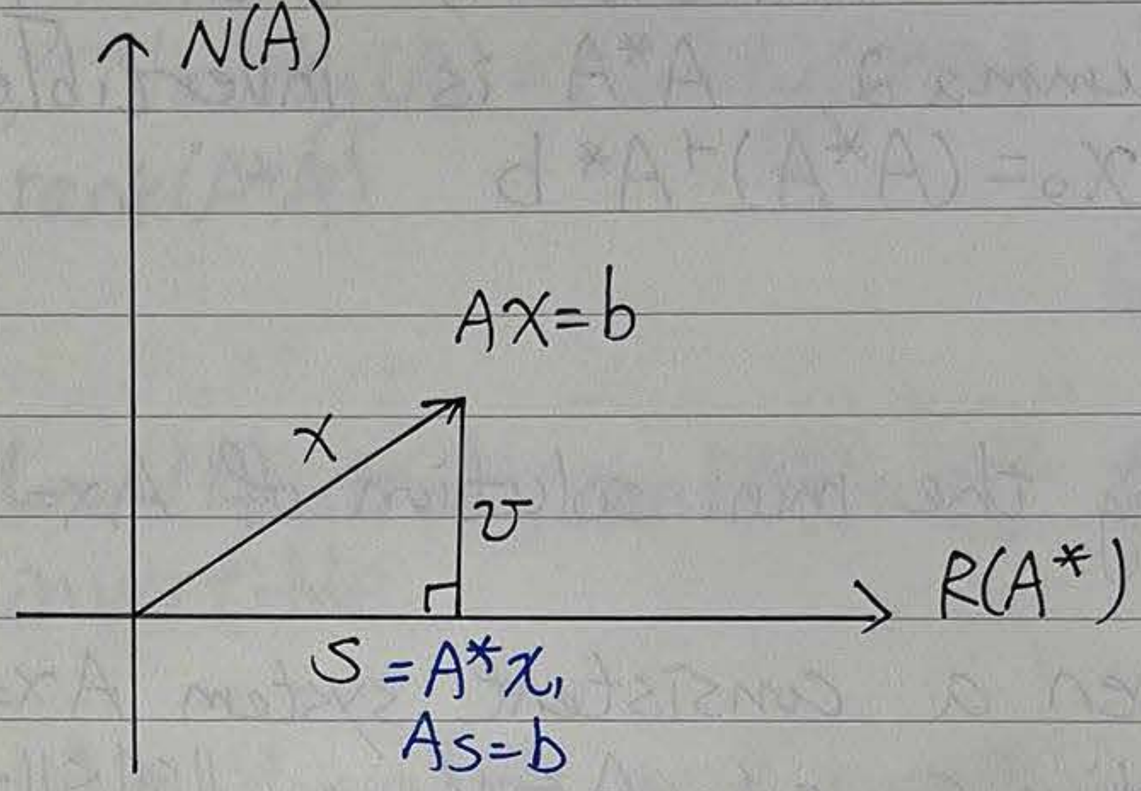
$\Rightarrow N(A)=R(A^*)^\perp$

② By $W^{\perp\perp}=W$, then $N(A)^\perp=R(A^*)$

③ Follows from that $V=W\oplus W^\perp$

I am OKAY.

* 图解：



解：① 找任一解 i.e. $Ax=b$

② 将 x 投影到 $R(A^*)$，得 S

③ S 即为唯一最小解

④ 实际上如何找 S

Q: ③ Why?

I.

A: 令 S 为最小解，x（任意解）$= s+v$, where $s\in R(A^*)$, $v\in N(A)$

$\Rightarrow Ax = A(s+v) = b$

$\because b = As+Av = A(s)$ since $v\in N(A)$

$\therefore$ s 是解

$\Rightarrow x-s\in N(A)$

$\Rightarrow \exists x_0\in N(A)$ s.t. $x-s=x_0$

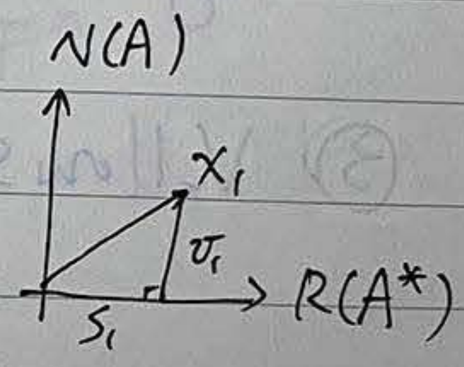
$\Rightarrow x = s+x_0$ $(s\perp x_0)$

$\Rightarrow \|x\|^2 = \|s\|^2 + \|x_0\|^2 \geq \|s\|^2$

$\Rightarrow$ s：最小解

II. 唯一性：

Let x_1 be another solution of $Ax=b$. Let s_1 be s.t.



$x_1 = s_1 + v_1$, where $s_1 \in R(A^*)$ and $v_1 \in N(A)$

we want $s = s_1$.

$\Rightarrow s - s_1 \in N(A)$ $(As = b,\ As_1 = b)$

$\Rightarrow s - s_1 \in R(A^*) \triangleq W$ $(N(A) = W^\perp)$

$\Rightarrow s - s_1 \in W^\perp \cap W = \{0\}$

$\Rightarrow s = s_1$

Thm 6.13. Let $A \in M_{m\times n}(F)$ and $b \in F^m$. Suppose that $Ax = b$ is consistent, then:

(1) $\exists!\ s \in R(L_{A^*})$ s.t. s only minimal solution of $Ax = b$ is the

(2) s is the only solution to $Ax = b$ that lies in $R(L_{A^*})$. That is, if u satisfies $(AA^*)u = b$, then $s = A^*u$.

Q: 实际上如何找 s? (④)

A: step1. 找任一 x_1 s.t. $AA^*x_1 = b$

step2. $s = A^*x_1$

Remark. $AA^* \in M_{m\times m}(F)$

① If $\operatorname{rank}(AA^*) = m \Rightarrow s = A^*(AA^*)^{-1}b$ (公式解!)

② If $\operatorname{rank}(A) = m$, then $\operatorname{rank}(AA^*) = m$

I am OKAY.

§6-4 Normal and Self-Adjoint Operators.

Def. An $n\times n$ matrix (real or complex) A (L.T. T) is normal if $AA^* = A^*A$ ($TT^* = T^*T$)

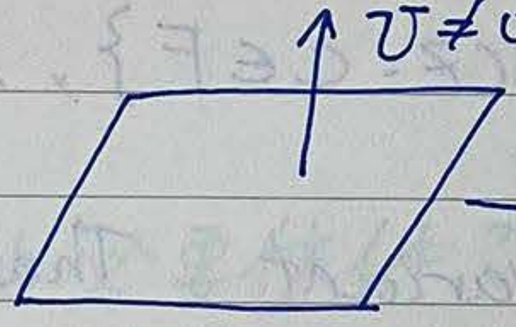
lemma. Let V be an inner product space, $\dim(V)<\infty$. If T has an eigenvector, then so does T^*

pf. $\because \exists\, v\neq 0$ s.t. $Tv=\lambda v$ or $(T-\lambda I)v=0$, $\lambda\in F$

$\therefore \langle (T-\lambda I)v, y\rangle = 0$, $\forall y\in V$.

$\Rightarrow \langle v, (T^*-(\lambda I)^*)y\rangle = 0$, "

$\Rightarrow \langle v, (T^*-\bar{\lambda} I)y\rangle = 0$, "



$\Rightarrow N(T^*-\bar{\lambda}I)\neq\emptyset$ by dim. Thm

$\Rightarrow \exists\, u\neq 0$ s.t. $(T^*-\bar{\lambda}I)u=0$

$\Rightarrow u$ is an eigenvector of T^*

Q.E.D.

Thm. (Schur)

Suppose the CP of T splits. Then T can be triangularized. More specifically, $\exists$ an orthonormal basis γ s.t. $[T]_\gamma$ is upper triangular.

I am OKAY.

Remark 1. 若 $F=\mathbb{C}$ ⇒ 任一定義在有限維內積空間的算子皆可三角形化

2. 即：若 $A\in M_{n\times n}(\mathbb{C})$，則 A 可三角化. 更正確的說：$\exists Q$: unitary matrix s.t. $U=Q^*AQ$ where $Q^*=Q^{-1}$，且 U 為上三角矩陣

pf. Let $\dim(V)=n$. We prove by the math induction

(1) $n=1 \Rightarrow$ V

(2) Suppose the Thm holds true for $n-1$

⇒ by lemma, $\exists z\neq 0$: unit vector s.t. $T^*z=\lambda z$

Let $W=\text{span}\{z\}=\{cz : c\in F\}$, $\dim W=1$.

Claim: $W^\perp$ is T-invariant. That is, we want $y\in W^\perp \Rightarrow T(y)\in W^\perp$

pf of claim: $\because y\in W^\perp \therefore \langle y, cz\rangle=0$

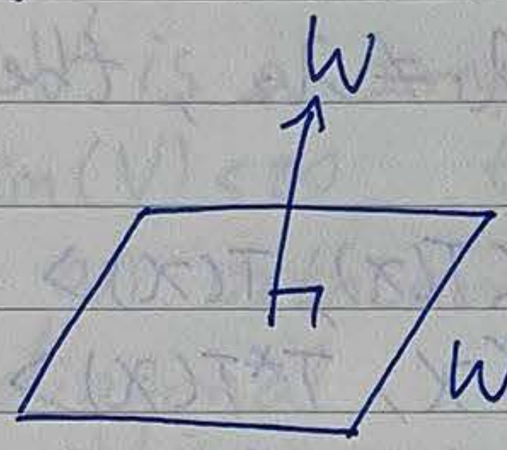
$$\begin{aligned}\Rightarrow \langle T(y), cz\rangle &= \langle y, T^*(cz)\rangle \\ &= \bar{c}\langle y, T^*(z)\rangle \\ &= \bar{c}\langle y, \lambda z\rangle \\ &= \overline{\lambda c}\langle y, z\rangle \\ &= 0.\end{aligned}$$

$\Rightarrow T(y)\in W^\perp$.

$\because$ The C.P. of T_V splits and C.P. of $T_{W^\perp}$ divide that of T_V

$\therefore \exists \beta$: orthonormal s.t. $[T_{W^\perp}]_\beta$ is upper

triangular.



$\therefore V = W \oplus W^{\perp}$, and $[T_{W^{\perp}}]_{\beta}$ is upper Δ

$\therefore$ Let $\gamma = \beta \cup \{z\}$, then

$$[T_V]_{\gamma} = \left(\begin{array}{c|c} [T_{W^{\perp}}]_{\beta} & \begin{matrix} \times \\ \times \\ \times \end{matrix} \\ \hline O & \times \end{array}\right)$$ is upper triangular

Q.E.D.

Q: T 对角化的必要條件?

A: If $[T]_{\gamma}$ is diagonal, then $TT^* = T^*T$

Q: If $T^*T = TT^* \overset{?}{\Rightarrow}$ T is diagnalizable?

Fact: T is normal $\Leftrightarrow$ For orthonormal basis γ, $[T]_{\gamma}$ is normal

Thm 6.15 Let T be a normal operator on V, then:

① $\|T(x)\| = \|T^*(x)\|$

② $T - cI$ is normal for any $c \in F$

③ If $T(x) = \lambda x$, $x \neq 0$, $\lambda \in F$, then $T^*(x) = \bar{\lambda} x$

I am OKAY.

④ If $T(x_1)=\lambda_1 x_1$, $T(x_2)=\lambda_2 x_2$, $x_1\neq 0$, $x_2\neq 0$, and $\lambda_1\neq\lambda_2$, then $x_1\perp x_2$

pf. ①: $\because \|T(x)\|^2=\langle T(x), T(x)\rangle$

$$\overset{\text{normal}}{\longrightarrow} = \langle x, T^*T(x)\rangle$$
$$= \langle x, TT^*(x)\rangle$$
$$= \langle T^*(x), T^*(x)\rangle$$
$$= \|T^*(x)\|^2$$

$\therefore \|T(x)\| = \|T^*(x)\|$

② It follows from that $(cI)^* = \bar{c}I$.

③ $T(x)=\lambda x \Rightarrow (T-\lambda I)(x)=0$

$$\Rightarrow 0=\|(T-\lambda I)(x)\|$$
$$\overset{(i)}{=} \|(T-\lambda I)^* x\|$$
$$= \|(T^*-\bar{\lambda} I)(x)\|$$
$$\Rightarrow (T^*-\lambda I)(x)=0$$
$$\Rightarrow T^*(x)=\bar{\lambda}x$$

④ $\because \lambda_1\langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle$

$$= \langle T(x_1), x_2\rangle$$
$$= \langle x_1, T^*(x_2)\rangle$$
$$= \langle x_1, \bar{\lambda}_2 x_2\rangle$$
$$= \lambda_2\langle x_1, x_2\rangle$$

$\therefore (\lambda_1-\lambda_2)\langle x_1, x_2\rangle = 0$

$\because \lambda_1-\lambda_2\neq 0$

$\therefore \langle x_1, x_2\rangle = 0$

$\Rightarrow x_1\perp x_2$

Q.E.D.

Thm 6.16 If V is a complex inner product space, $\dim(V)<\infty$, then T is normal $\Leftrightarrow \exists\ r$: an orthonormal basis s.t. $[T]_r$ is diagonal

Remark. (1) $\dim(V)<\infty$ → 必要.
反例: ex3 on 課本
(2) $F=\mathbb{C}$: 必要 (不然怎麼算 eigen value?)
(3) Matrix version of Thm 6.16: $A\in M_{n\times n}(\mathbb{C})$
$L_A: F^n \to F^m$

pf. ∵ 由 Thm 6.14, $\exists\ r$: orthonormal s.t. $[T]_r$: upper triangular.
∴ Let $A=[T]_r$, we have

$$\begin{cases} T(v_1)=A_{11}v_1 \\ T(v_2)=A_{12}v_1+A_{22}v_2 \\ \quad\vdots \\ T(v_n)=A_{1n}v_1+A_{2n}v_2+\cdots+A_{nn}v_n \end{cases}$$

where $v_1,\dots,v_n$ are eigenvectors

Claim: A is a diagonal matrix; that is,

$$A=\begin{pmatrix} \times & & O \\ & \times & \\ O & & \times \end{pmatrix}$$

pf of claim: First we prove $A_{12}=0$
∵ $A_{12}=\langle T(v_2), v_1\rangle$
$=\langle v_2, T^*(v_1)\rangle$
$=\langle v_2, \bar{A}_{11}v_1\rangle$

I am OKAY.

$= A_{11}\langle v_2, v_1\rangle$

$= 0 \quad (v_1 \perp v_2)$

同理, $T(v_3) = A_{13}v_1 + A_{23}v_2 + A_{33}v_3$,

we want $A_{13} = A_{23} = 0$

$\because A_{23} = \langle T(v_3), v_2\rangle$

$= \langle v_3, \overline{T^*(v_2)}\rangle$

$= \langle v_3, \bar{A}_{22}v_2\rangle$

$= A_{22}\langle v_3, v_2\rangle$

$= 0$

A_{13}, similarly, $= 0$

$\therefore$ We have $A_{13} = A_{23} = 0$

$\Rightarrow$ Repeat this procedure, we can show that $A_{jk} = 0$, for $j < k$.

$\Rightarrow$ A is diagonal

Q.E.D.

Remark 1. $A = \begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix}$, $A^*A \neq AA^*$

eigenvalue: $(\lambda-2)(\lambda-1) - 20 = 0 \Rightarrow \lambda = 6$ or -3

$\Rightarrow$ 可对角化 → 与 Thm 6.16 衝突?

→ no, 未必是 orthonormal basis

2. Matrix version: If $A \in M_{n\times n}(\mathbb{C})$, then A is normal $\Leftrightarrow \exists$ an orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$ s.t. $Q^{-1}AQ = D$, where $Q = (v_1 \cdots v_n)$

$\Rightarrow Q^{-1} = Q^*$, $Q^*Q = QQ^* = I$

3. $F=\mathbb{C}$, $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\in M_{2\times2}(\mathbb{R})$

$$A^*=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix}$$

$\Rightarrow AA^*=I=A^*A$

$\Rightarrow$ A is normal, 但A不可对角化

$\Rightarrow$ 但若說 $A\in M_{2\times2}(\mathbb{C})$, 則A可被对角化!

4. $\dim V<\infty$ 是大前提.

反例: 令 $S=\{f_n(t)\mid n\in\mathbb{Z}, f_n(t)=e^{int}\}$

$$\langle f_n,f_m\rangle=\frac{1}{2\pi}\int_0^{2\pi}f_n(t)\overline{f_m(t)}\,dt$$

$V\overset{\Delta}{=}\operatorname{span}S$, $\dim(V)=\infty$. Let $T:V\to V$

$T(f)=f_1f$, then we want to find T^*.

$\because\langle T(f_n),f_m\rangle=\langle f_n,T^*(f_m)\rangle$

$=\langle T(f_n),f_m\rangle$

$=\langle f_{n+1},f_m\rangle$

$=\delta_{n+1,m}$, $\forall n,m\in\mathbb{Z}$.

$=\delta_{n,m-1}$, $\forall n,m\in\mathbb{Z}$

$=\langle f_n,f_{m-1}\rangle$

Define $U(f_m)=f_{m-1}$, and $U(f)=f_{-1}f$

$\therefore T^*=U$.

$\Rightarrow TT^*(f)=T^*T(f)=I(f)$

$\Rightarrow T$ is normal

Claim: Such T has no eigenvectors.

pf of claim: Suppose for $f\neq0$, $T(f)=\lambda f$.

$\because f\in V$

I am OKAY.

$\therefore f = \sum_{i=n}^{m} a_i f_i$ where $n, m \in \mathbb{Z}$ and

$a_m \neq 0$.

$$\Rightarrow \sum_{i=n}^{m} a_i T(f_i) = \lambda\left(\sum_{i=n}^{m} a_i f_i\right)$$

$$\Rightarrow \sum_{i=n}^{m} a_i f_{i+1} = \lambda\left(\sum_{i=n}^{m} a_i f_i\right)$$

$\because a_m f_{m+1}$ + 剩項 = 右边

$\therefore f_{m+1} = \frac{1}{a_m}$ (其它 f_i) $\rightarrow\leftarrow$

$\Rightarrow$ 与 S 为线性独立之集合矛盾!

$\Rightarrow$ T 沒有 eigenvalue

Q.E.D.

$\Rightarrow$ T無法被对角化.

Q: How about $F = R$?

Def. Let T be a linear operator on an inner product space V. We say that:

(1) T is self-adjoint (Hermitian) if $T = T^*$

(2) $A \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ is self-adjoint (Hermitian) if $A = A^*$

*. self adjoint $\Rightarrow$ Normal

real symmetric $\Rightarrow$ self adjoint

Thm 6.17. Let $F=\mathbb{R}$. Then T is self adjoint $\Leftrightarrow \exists$ an orthonormal basis $\beta = \{$ eigenvectors of $T\}$

Before proving, we need a lemma.

lemma. Let T be self-adjoint.
(1) If T has an e.v. λ, then $\lambda \in \mathbb{R}$
(2) Indeed, all eigenvalues exist and, hence, all real.

pf. (1) $\because$ T is normal
$\therefore T(x) = \lambda x = T^*(x) = \bar{\lambda} x$, $\forall x \neq 0$.
$\Rightarrow \lambda = \bar{\lambda}$
$\Rightarrow \lambda$ is real

(2) Let β: orthonormal basis. Let
$[T]_\beta = A \in M_{n\times n}(\mathbb{C})$
$\Rightarrow$ The C.P. of A splits, and so is T
$\because$ T is self adjoint
$\therefore$ A is "
$\Rightarrow$ T has real eigenvalues that are all real.

Q.E.D.

pf of Thm 6.17: $\Rightarrow$) from lemma, the C.P. of T splits. $\because$ Schur Thm
$\Rightarrow \exists$ β: orthonormal s.t. $[T]_\beta$ is an upper triangle matrix.

I am OKAY.

$\because [T^*]_\beta = [T]^*_\beta = [T]_\beta$

$\therefore [T]_\beta = D$, a diagonal matrix

$\Rightarrow \beta = \{$ eigenvectors of $T\}$

$\Leftarrow$) Let $[T]_\beta = D$, a diagonal matrix

$\Rightarrow D_{ii}$ = the eigenvalue $\in \mathbb{R}$

$\Rightarrow D$ is self-adjoint

$\Rightarrow T$ is self-adjoint

Q.E.D.

Remark. ① Let $A \in M_{n\times n}(\mathbb{R})$, $A^t = A \Rightarrow A$ is self-adjoint

② Matrix version of Thm 6.17:

§6.5 Unitary and Orthogonal Operators

Recall $Q=\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$, $\{v_i\}_{i=1}^n$: orthonormal basis.

Def. Let T be a linear operator on a finite-dimensional inner product space V over F.

((1) If $\|T(x)\|=\|x\|$, $\forall x\in V$, we call T a unitary operator)

(2) If $F=\mathbb{C}$, T is called unitary if $T^{-1}=T^*$, i.e., $TT^*=T^*T=I$

(3) " $\mathbb{R}$, " orthogonal"

可拆成矩阵A

Thm6.18. The following are equivalent.

(a) $T^*T=TT^*=I$

保角+保长

(b) $\langle T(x),T(y)\rangle=\langle x,y\rangle$, $\forall x,y\in V$

(c) $\forall\beta$: orthonormal basis $\Rightarrow T(\beta)$ = orthonormal basis

(d) $\exists\beta$: orthonormal basis $\Rightarrow T(\beta)$: orthonormal basis

(e) $\|T(x)\|=\|x\|$, $\forall x\in V$.

pf. (a)$\Rightarrow$(b). It follows from that

$\langle T(x),T(y)\rangle=\langle x,T^*T(y)\rangle=\langle x,y\rangle$, $\forall x,y\in V$.

(b)$\Rightarrow$(c): $\because\langle T(v_i),T(v_j)\rangle=\langle v_i,v_j\rangle=\delta_{ij}$

$\forall\beta=\{v_1,\dots,v_n\}$, $i,j=1,\dots,n$

I am OKAY.

∴ The result holds

(3) ⇒ (4) trivial

(4) ⇒ (5) Let $\beta = \{v_1, \dots, v_n\}$, let $x = \sum_{i=1}^{n} a_i v_i$

$\Rightarrow \|T(x)\|^2 = \langle T(x), T(x) \rangle$

$= \langle \sum_{i=1}^{n} a_i T(v_i), \sum_{i=1}^{n} a_i T(v_i) \rangle$

正交才可提出 Σ →

$\overset{\text{by (4)}}{=} \sum_{i=1}^{n} \langle a_i T(v_i), a_i T(v_i) \rangle$

$= \sum_{i=1}^{n} |a_i|^2 \langle T(v_i), T(v_i) \rangle$

$= \sum_{i=1}^{n} |a_i|^2$

$= \langle x, x \rangle$

$= \|x\|^2$

(5) ⇒ (1)

(思考

1. $T^*T = I \iff \langle x, T^*T(y) \rangle = \langle x, I(y) \rangle$, $\forall x, y$.
   $\iff \langle x, (I - T^*T)y \rangle = 0, \ \forall x, y$
2. $(I - T^*T)^* = I^* - T^*T^{**} = I - T^*T$
   ⇒ self adjoint

)

We need a lemma before proceeding.

lemma. Let U be self-adjoint and $\langle x, Ux\rangle = 0$
$\Rightarrow U = 0$ (零函数)

pf. Let U be self-adjoint, $\exists \beta = \{\text{eigenvectors}\}$
= orthonormal basis.
Let $v \in \beta$. $\because \langle v, U(v)\rangle = 0 = (v, \lambda v)$
$= \bar{\lambda}\langle v, v\rangle$
$\therefore \bar{\lambda} = 0 = \lambda$.
$\Rightarrow U(v) = \lambda v = 0$.
$\Rightarrow U = 0$

Q.E.D.

(續) pf of Thm 6.18:
Claim: $I - T^*T = 0$
pf of claim: $\because T$ is self adjoint
$\therefore \langle x, (T^*T - I)x\rangle = \langle x, T^*T(x)\rangle - \langle x, x\rangle$
$= \langle T(x), T(x)\rangle - \langle x, x\rangle$
$\overset{(5)}{=} 0$
$\Rightarrow$ by lemma, $I - T^*T = 0$

By claim, $T^*T = TT^* = I$

Q.E.D.

Def. Let $A \in M_{n\times n}(\mathbb{C})$, we say a is an
unitary matrix if $A^*A = AA^* = I$
" $\mathbb{R}$
orthogonal "

I am OKAY.

Cor. Let $F=\mathbb{R}$. Then T be an self-adjoint and orthogonal operator $\Leftrightarrow \exists \beta = \{\text{eigenvectors}\}$: orthonormal basis and $|\lambda|=1$. λ : any eigenvalue of T.

pf. $\Rightarrow$) $\because$ By Thm 6.18 (4), $\exists \beta = \{v_1, \dots, v_n\}$ is an orthonormal basis, where $v_1, \dots, v_n$ are eigenvecter of T.

$\therefore T(v_i) = \lambda_i v_i$

$\Rightarrow \|T(v)\| = \|\lambda_i v_i\| = |\lambda_i| \|v_i\|$

$\Rightarrow$ by (5), $|\lambda_i| = 1$.

$\Leftarrow$) exercise.

Q.E.D.

Cor. Let $F=\mathbb{C}$. Then T is an unitary operator $\Leftrightarrow \exists \beta = \{\text{eigenvectors}\}$: orthonormal basis and $|\lambda|=1$, λ : any e.v. of T.

Δ 三$\S$ Thm 的矩陣版

Thm 6.19 Let $A \in M_{n\times n}(\mathbb{C})$. Then A is normal $\Leftrightarrow$ A is unitarily equivalent to a diagonal matrix

Thm 6.20. Let $A \in M_{n\times n}(\mathbb{R})$ Then A is symmetric $\Leftrightarrow$ A is orthogonally equivalent to a real diagonal matrix.

Thm 6.21 (Schur). Let $A \in M_{n\times n}(F)$ be a matrix whose characteristic polynomial splits over F.

(a) If $F = \mathbb{C}$, then A is unitarily equivalent to a complex upper triangular matrix

(b) If $F = \mathbb{R}$, then A is orthogonally equivalent to a real upper triangular matrix.

△ Application: 2nd Derivative Test

* 2nd Derivative test of $f(x,y)$

Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth, a is a critical pt, i.e. $f_x(a) = 0$, $f_y(a) = 0$

方向導数: $f_u(a,b)$, $u = (u_1, u_2)$, $u_1^2 + u_2^2 = 1$

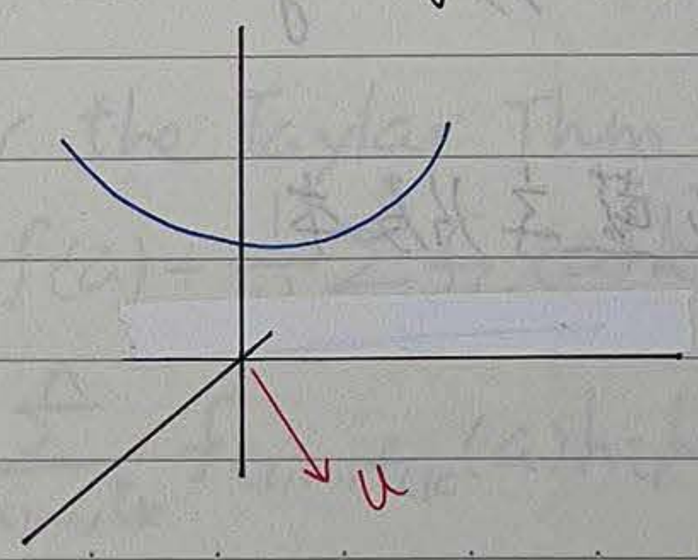
$f_u(a,b) = \nabla f \cdot u$, $\nabla f = (f_x(a,b), f_y(a,b))$

$\Rightarrow f_u(a,b) = f_x u_1 + f_y u_2 \triangleq g(x,y)$

$\Rightarrow g_u(a,b) = f_u(f_u(a,b)) = \nabla g \cdot u$

$= \langle f_{xx}u_1 + f_{yx}u_2, f_{xy}u_1 + f_{yy}u_2 \rangle \cdot u$

$= (u_1 \ u_2)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \triangleq x^T H x$



I am OKAY.

Suppose $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exists and anti.

$\Rightarrow f_{xy} = f_{yx}$

$\Rightarrow H$ is symmetric

$\Rightarrow \exists Q$: orthogonal s.t. $Q^t H Q = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\Rightarrow H = QDQ^t$

$\Rightarrow f_{uu}(a,b) = x^t H x = x^t Q D Q^t x$

$= (Q^t x)^t D (Q^t x)$

$= (u_1' \; u_2') \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$

$= \lambda_1 (u_1')^2 + \lambda_2 (u_2')^2$

(1). If $\lambda_1, \lambda_2 > 0 \Rightarrow f_{uu}(a,b) > 0$

$\Rightarrow f(a,b)$ is a local min.

$(\Delta > 0, f_{xx} > 0)$

(2) If $\lambda_1, \lambda_2 < 0 \Rightarrow f(a,b)$ is a local max.

$(\Delta > 0, f_{xx} < 0)$

(3) If $\lambda_1 \lambda_2 < 0 \Rightarrow (a,b)$ is a saddle pt

$(\Delta < 0)$

(4) If $\lambda_1 \lambda_2 = 0 \Rightarrow$ it is inconclusive.

$(\Delta = 0)$

$\therefore \begin{cases} \lambda_1 + \lambda_2 = f_{xx} + f_{yy} \\ \lambda_1 \lambda_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2 \triangleq \Delta \end{cases}$

二微积分课本会写成蔓字版本

现在抵度到 n 级：

Thm. $f: \mathbb{R}^n \to \mathbb{R}$, f 為 smooth 以致於 H 為 symmetric.
$a=(a_1,\dots,a_n)$ 為 f 之 critical point, 即
$f_i(a)=0$, $f_i \triangleq \frac{\partial f}{\partial x_i}$

Let $H=(f_{ij}(a))_{n\times n}$, here $f_{ij}(a)=\frac{\partial^2 f}{\partial x_j \partial x_i}$
= Hessian matrix of f at a. Let eigenvalues of H be $\{\lambda_i\}$

(1) $\lambda_i>0$, for all $i \Rightarrow f(a)$ is a local min

(2) "$<$" max

(3) $\exists \lambda_i, \lambda_j$ s.t. $\lambda_i\lambda_j<0 \Rightarrow a$ is a saddle pt

(4) otherwise, the test fails.

pf. Let $u=(u_1,\dots,u_n)$, $\sum_{i=1}^{n} u_i^2=1$.
$\Rightarrow f_u=\langle f_1,\dots,f_n\rangle \cdot u = f_1u_1+\cdots+f_nu_n$
$\nabla f_u = \langle f_{11}u_1+f_{21}u_2+\cdots+f_{n1}u_n,$
$f_{12}u_1+f_{22}u_2+\cdots+f_{n2}u_n,$
$\cdots,$
$f_{1n}u_1+f_{2n}u_2+\cdots+f_{nn}u_n\rangle$
$\Rightarrow f_{uu}(a,b)=u^THu$.

Consider the Taylar Thm for smooth f,
$$f(a+h)=f(a)+\frac{1}{1!}\sum_{i_1=1}^{n} f_{i_1}(a)h_{i_1}+\frac{1}{2!}\sum_{i_1,i_2=1} f_{i_1i_2}(a)h_{i_1}h_{i_2}$$
$$+\cdots+\frac{1}{k!}\sum_{\substack{i_1,\dots,i_k\\=1}}^{n} f_{i_1,\dots,f_{ik}}(a)h_{i_1}h_{i_2}\cdots h_{ik}$$

I am OKAY.

$$+\frac{1}{(k+1)!}\sum_{\substack{i_1,\cdots,\\ i_{k+1}=1}}^{n} f_{i_1\cdots i_{k+1}}(c)\,h_{i_1},\cdots,h_{i_{k+1}}$$

$$\therefore \frac{1}{2!}\sum_{i_1,i_2=1}^{n} f_{i_1,i_2}(a)\,h_{i_1}h_{i_2} = \frac{1}{2!}(h_1\cdots h_n)\,H\begin{pmatrix}h_1\\ \vdots\\ h_n\end{pmatrix}$$

$$\triangleq \frac{1}{2!}\,h H h^{T}$$

$\therefore$ use the similar argument.

Q.E.D.

△ Application: The classification of Quadratic Family

2D: $ax^2+bxy+cy^2+dx+ey+f=0$

↓ 平移，消去一次項

$a'x^2+b'xy+c'y^2+f=0$

↓

$$(x\ y)\begin{pmatrix}a' & \frac{b'}{2}\\ \frac{b'}{2} & c'\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+f=0$$

3D: $ax^2+by^2+cz^2+dxy+eyz+fzx=g$

↕

$$(x\ y\ z)\,H\begin{pmatrix}x\\ y\\ z\end{pmatrix}=g$$

↕

$$H=\begin{pmatrix}a & \frac{d}{2} & \frac{f}{2}\\ \frac{d}{2} & b & \frac{e}{2}\\ \frac{f}{2} & \frac{e}{2} & c\end{pmatrix}$$

為方便起見，將 b 改寫 $2b \Rightarrow H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow (x\ y) H \begin{pmatrix} x \\ y \end{pmatrix} = f \Rightarrow \exists Q$: orthogonal s.t.
$Q^T H Q = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

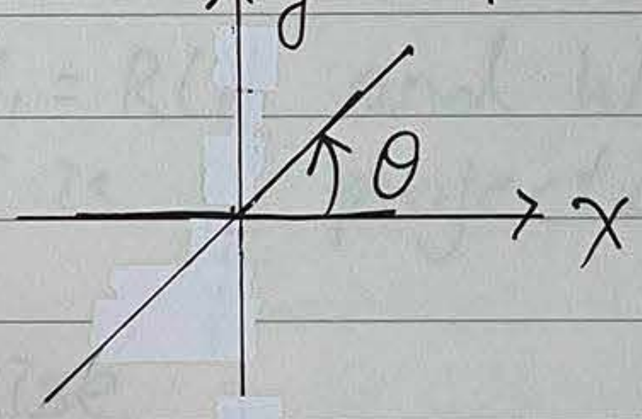
$\Rightarrow H = QDQ^T$

$\Rightarrow [(x\ y)Q] D [Q^T \begin{pmatrix} x \\ y \end{pmatrix}] = f$

$\Rightarrow (x'\ y') D \begin{pmatrix} x' \\ y' \end{pmatrix} = f$, where $Q^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$\Rightarrow \lambda_1 (x')^2 + \lambda_2 (y')^2 = f$

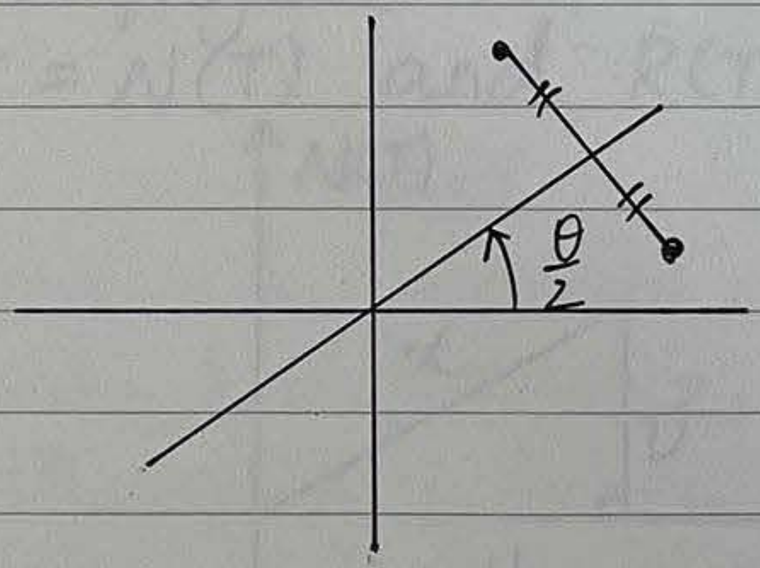
Recall 旋轉矩陣: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \triangleq A$



$AA^T = I$, A is orthogonal

Thm. $A \in M_{2\times 2}(\mathbb{R})$. A: orthogonal $\Rightarrow A$ 必為旋轉 or 鏡射矩陣.

鏡射:



I am OKAY.

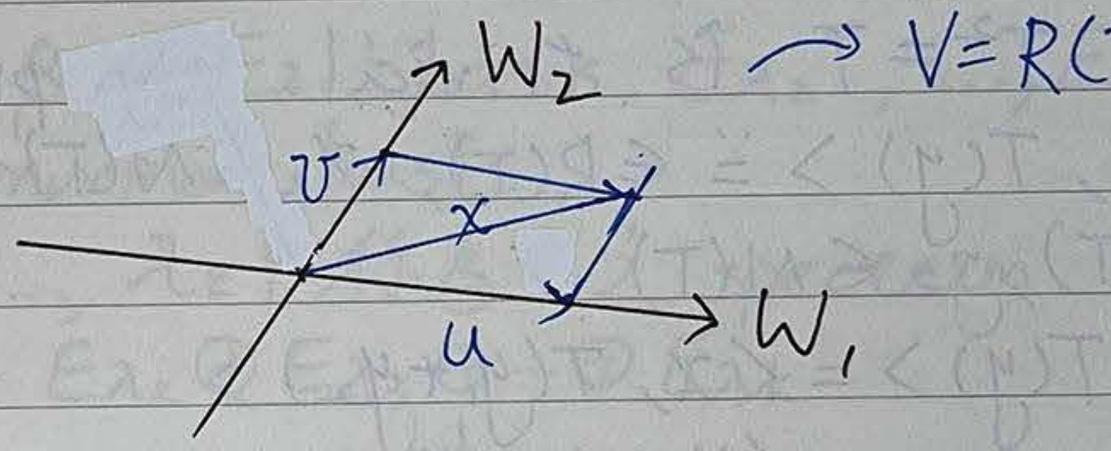
Note: 6.8节只是为来抛出刚才提的2个应用，
6.9, 6.11节为物理应用，但6.10很重要，
跳不在OCW笔围内，但会补充

§6.6 Orthogonal Projections and the Spectral Theorem

Def. Let V be an inner product space, then $T: V \to V$ is called a projection on W_1 along W_2 if $\exists\ W_1, W_2$: subspaces of V s.t.:

① $V = W_1 \oplus W_2$

② if $x = u + v$, $u \in W_1$, $v \in W_2$, then $T(x) = u$



$\rightsquigarrow V = R(T) \oplus N(T)$

Facts: (1) $W_1 = R(T)$ and $W_2 = N(T)$

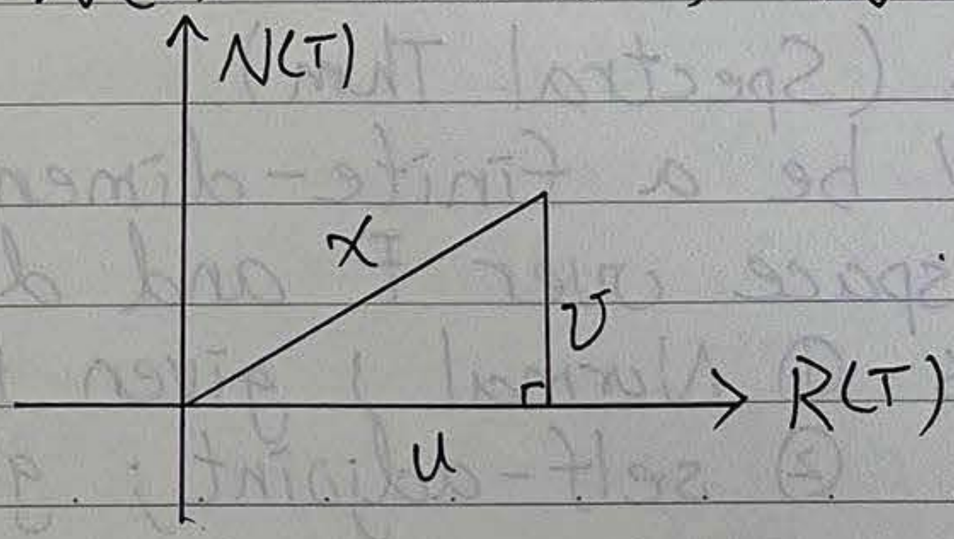
(2) T is a projection $\iff T^2 = T$

pf. exercise

Def. Let V be an inner product space, then $T: V \to V$ is an orthogonal projection if:

① T is a projection

② $R(T)^\perp = N(T)$ and $R(T) = N^\perp(T)$



I am OKAY.

Thm 6.24 $\dim(V) < \infty$, V: inner product space
then T is an orthogonal projection $\Leftrightarrow$
$T^2 = T = T^*$
(If $\dim(V) = \infty$, the Thm holds if
T^* exists.)

pf. $\Leftarrow$) exercise

$\Rightarrow$) $T^2 = T$ is trivial. We prove $T = T^*$.
Let $x = x_1 + x_2$, $x_1 \in R(T)$, $x_2 \in N(T)$; $y = y_1 + y_2$,
$y_1 \in R(T)$, $y_2 \in N(T)$.

$$\begin{cases} \langle x, T(y) \rangle = \langle x, T(y_1 + y_2) \rangle \\ \qquad = \langle x_1 + x_2, y_1 \rangle \\ \qquad = \langle x_1, y_1 \rangle \\ \langle T(x), y \rangle = \langle T(x_1 + x_2), y_1 + y_2 \rangle \\ \qquad = \langle x_1, y_1 + y_2 \rangle \\ \qquad = \langle x_1, y_1 \rangle \end{cases}$$

$\Rightarrow T = T^*$

Q.E.D.

Q: can T: orthogonal projection be diagonized?
How about T: projection?

Thm 6.25. (Spectral Thm).
Let V be a finite-dimensional inner product space over F and $\dim V < \infty$.
If T is ① Normal; given $F = \mathbb{C}$
② self-adjoint; given $F = \mathbb{R}$

(that is, $\exists$ orthonormal basis = {eigenvectors}),
Let λ_i, $i=1,2,\dots,k$ be all distinct eigenvalues of T, then:

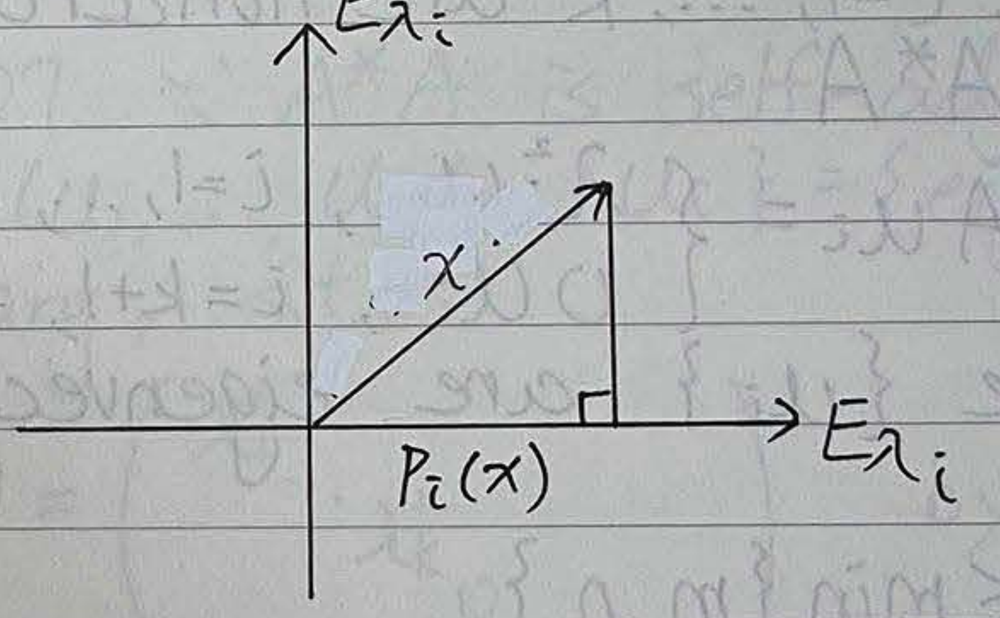
(1) $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$ and $E_{\lambda_i} \perp E_{\lambda_j}$ if $i \neq j$

(2) $T = \sum_{i=1}^{k} \lambda_i P_i$, where P_i are orthogonal projections on E_{λ_i} along $E_{\lambda_i}^{\perp}$

Remark. ① $\beta = \{E_{\lambda_1}\text{'s basis}, E_{\lambda_2}'', \dots, E_{\lambda_k}''\}$ 为 orthonormal basis

② $\{\lambda_1, \dots, \lambda_k\}$ 称为 T 之 spectrum

③ $E_{\lambda_1}^{\perp} = E_{\lambda_2} \oplus E_{\lambda_3} \oplus \cdots \oplus E_{\lambda_k}$



pf. Let $x \in V$, $x = u_1 + \cdots + u_k$, where $u_i \in E_{\lambda_i}$

$$\Rightarrow T(x) = \sum_{i=1}^{k} T(u_i) = \sum_{i=1}^{k} \lambda_i u_i = \sum_{i=1}^{k} \lambda_i P_i(x)$$

Q.E.D.

I am OKAY.

§6.7 The Singular Value Decomposition

Thm 6.27 (Singular Value Decomposition Thm for Matrices)

Let $A \in M_{m\times n}(F)$ and $\operatorname{rank}(A)=k$, then

① $\exists\ U, V$: unitary matrices,

$U=(u_1 \cdots u_n)$, $V=(v_1 \cdots v_m)$.

s.t. $V^*_{m\times m} A_{m\times n} U_{n\times n} = \Sigma_{m\times n}$, where

$$\Sigma = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \\ & \cdots & 0 \end{pmatrix},$$

and $\lambda_i > 0$, $i=1, 2, \dots, k$

② λ_i^2, $i=1, \dots, k$ are nonzero eigenvalues of A^*A

③ $A^*A u_i = \begin{cases} \lambda_i^2 u_i, & i=1, \dots, k \\ 0\, u_i, & i=k+1, \dots, n \end{cases}$,

where $\{u_i\}$ are eigenvector of A^*A

Remark. ① $k \le \min\{m, n\}$

② λ_i are called singular values of A

③ Σ is called a SVD of A

④ In the proof, we can see how to construct v_i

⑤ $\beta = \{u_1, \dots, u_n\}$: orthonormal basis for F^n

$\gamma = \{v_1, \dots, v_m\}$: " F^m

⑥ $\operatorname{rank}(A)=k \Rightarrow \operatorname{rank}(A^*A)=k$

and the Pseudoinverse

Fact 矩陣A^*A之 eigenvalue 有k個是正數, n-k 個是0...

pf. Claim: if λ_i are eigenvalue of A^*A, then $\lambda_i \geq 0$

pf of claim: let x be eigenvector of A^*A, then $\lambda_i \langle x, x \rangle > 0$

$\because \lambda_i \langle x, x \rangle = \langle \lambda_i x, x \rangle$

$= \langle A^*A x, x \rangle$

$= \langle Ax, Ax \rangle$

≥ 0

$\therefore \lambda_i > 0$.

Q.E.D.

pf of Thm6.27. $\because A^*A$ is self-adjoint

$\therefore \exists U = (u_1, \dots, u_n)$, $\{u_i\} = \{\text{eigenvectors}\}$,

U: unitary, s.t.

$$U^*A^*AU = \begin{pmatrix} \lambda_1^2 & & & \\ & \ddots & & \\ & & \lambda_k^2 & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \text{ by Fact.}$$

Let $W_{m\times n} = A_{m\times n} U_{n\times n} = \begin{pmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_n \\ | & | & & | \end{pmatrix}$

$$\begin{cases} \langle w_i, w_i \rangle = w_i^* w_i = \begin{cases} \lambda_i^2, & 1 \leq i \leq k \\ 0, & k < i \leq n \end{cases} \\ \langle w_i, w_j \rangle = 0, & i \neq j. \end{cases}$$

I am OKAY.

$$\Rightarrow W_{m\times n} = \begin{pmatrix} | & | & & | & | & | \\ w_1 & w_2 & \cdots & w_k & 0 & 0 & \cdots & 0 \\ | & | & & | & | & | \end{pmatrix}$$

(w_{k+1} = 0, w_{k+2} = 0, w_n = 0)

思考

如何找V?

$V^*AU = V^*W = \Sigma$ ← 希望得到

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -v_1^*- \\ \vdots \\ -v_m^*- \end{pmatrix} \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

$\Rightarrow$ 找 v_1 s.t. $\langle \frac{w_1}{\lambda_1}, w_1 \rangle = \lambda_1$;

$\langle w_1, w_1 \rangle = \lambda_1^2$

Let $v_i = \frac{w_i}{\lambda_i}$, $1 \le i \le k$. Extend $\{v_1, \ldots, v_k\}$ so that $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$ is an orthonormal basis for F^m. It is easy to check

$$\Rightarrow \begin{cases} \langle v_i, w_i \rangle = \begin{cases} \lambda_i, & i = 1, \ldots, k \\ 0, & k < i \le n. \end{cases} \\ \langle v_i, w_j \rangle = 0, \quad i \ne j \end{cases}$$

$\Rightarrow V^*(AU) = \Sigma$

Q.E.D.

Thm 6.26 (SVD for LT)

Let V and W be finite dimensional inner product space, and $A: V \to W$ be a L.T. of rank k. Then, there exist orthonormal basis $\{v_1, \ldots, v_n\}$ for V

and $\{u_1, \dots, u_m\}$ for W, and positive scalars $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ s.t.

$$A_{1\times 1}(u_1, u_2, \dots, u_n)_{1\times n} = (v_1, \dots, v_n)_{1\times m}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \\ & \cdots\ O & \end{pmatrix}_{m\times n}$$

↑ Block multiplication

$$\Rightarrow (Au_1, Au_2, \dots, Au_n) = (\lambda_1 v_1, \dots, \lambda_k v_k, 0, \dots, 0)$$

or, equivalently,

$$Au_i = \begin{cases} \lambda_i v_i, & 1 \leq i \leq k \\ 0 v_i, & k \leq i \leq n \end{cases}$$

Remark. (1) $v_j \perp v_i$, $k+1 \leq j \leq m$, $1 \leq i \leq k$;
$\|v_j\| = 1$, $\forall j$

(2) $\operatorname{span}\{u_{k+1}, \dots, u_k\} = N(A)$
$\Rightarrow F^n = N(A) \oplus N(A)^\perp$, where
$N(A)^\perp = \operatorname{span}\{u_1, \dots, u_k\}$

(3) $\operatorname{Range}(A) = \operatorname{span}\{v_1, \dots, v_k\}$
$\operatorname{Range}(A)^\perp = \operatorname{span}\{v_{k+1}, \dots, v_m\}$

ex. Let $A \in M_{2\times 2}(\mathbb{R})$ be invertible, $S = \{(x, y) \mid x^2 + y^2 = 1\}$. Find $A(S)$.

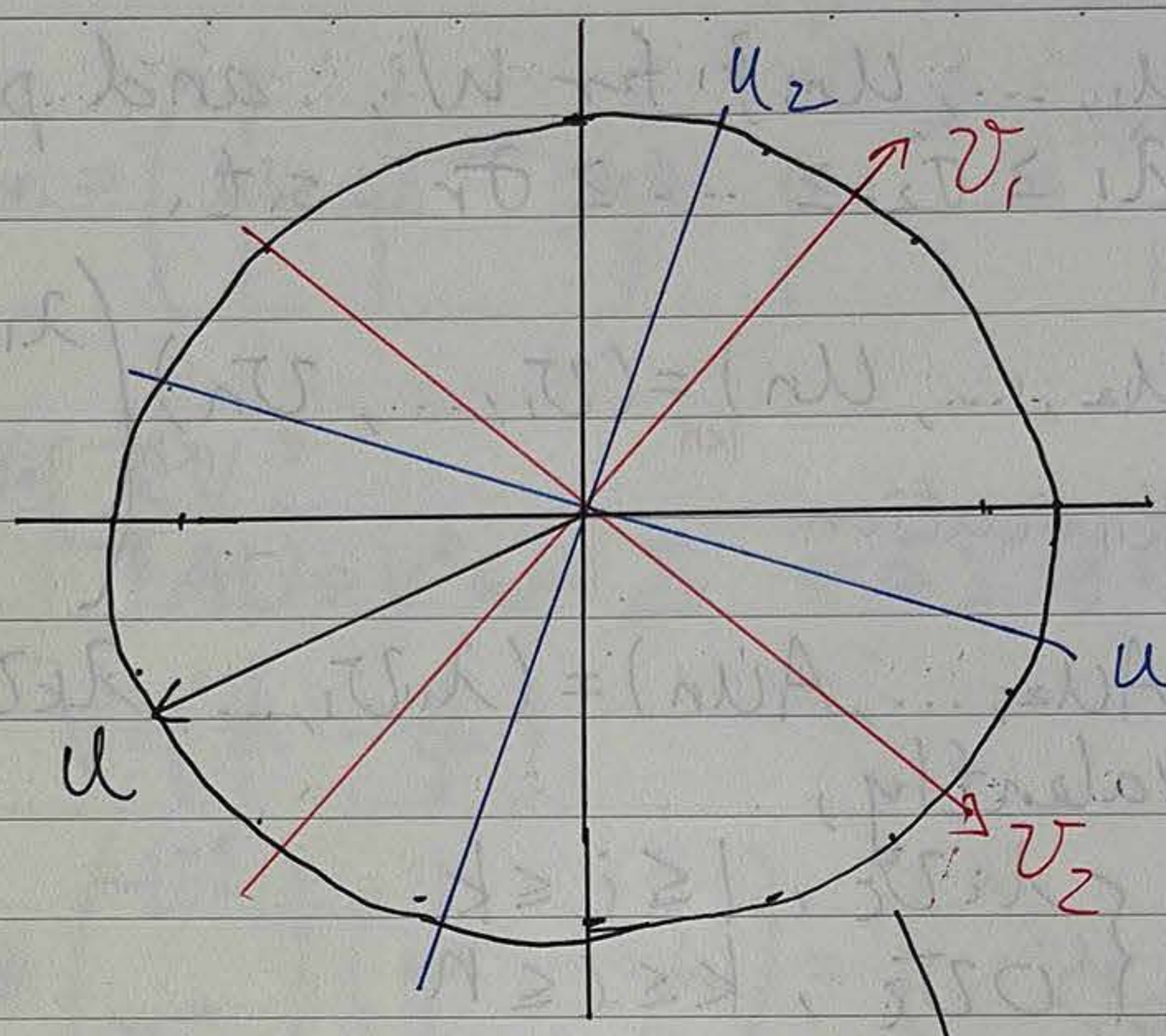
-sol- $\because \operatorname{Rank}(A) = 2$

$\therefore$ Let $\{u_i\}, \{v_i\}$ be two basis of R^2 s.t. $Au_i = \lambda_i v_i$

Let $u \in S$, then $u = \langle u, u_1 \rangle u_1 + \langle u, u_2 \rangle u_2 \triangleq a_1 u_1 + a_2 u_2$

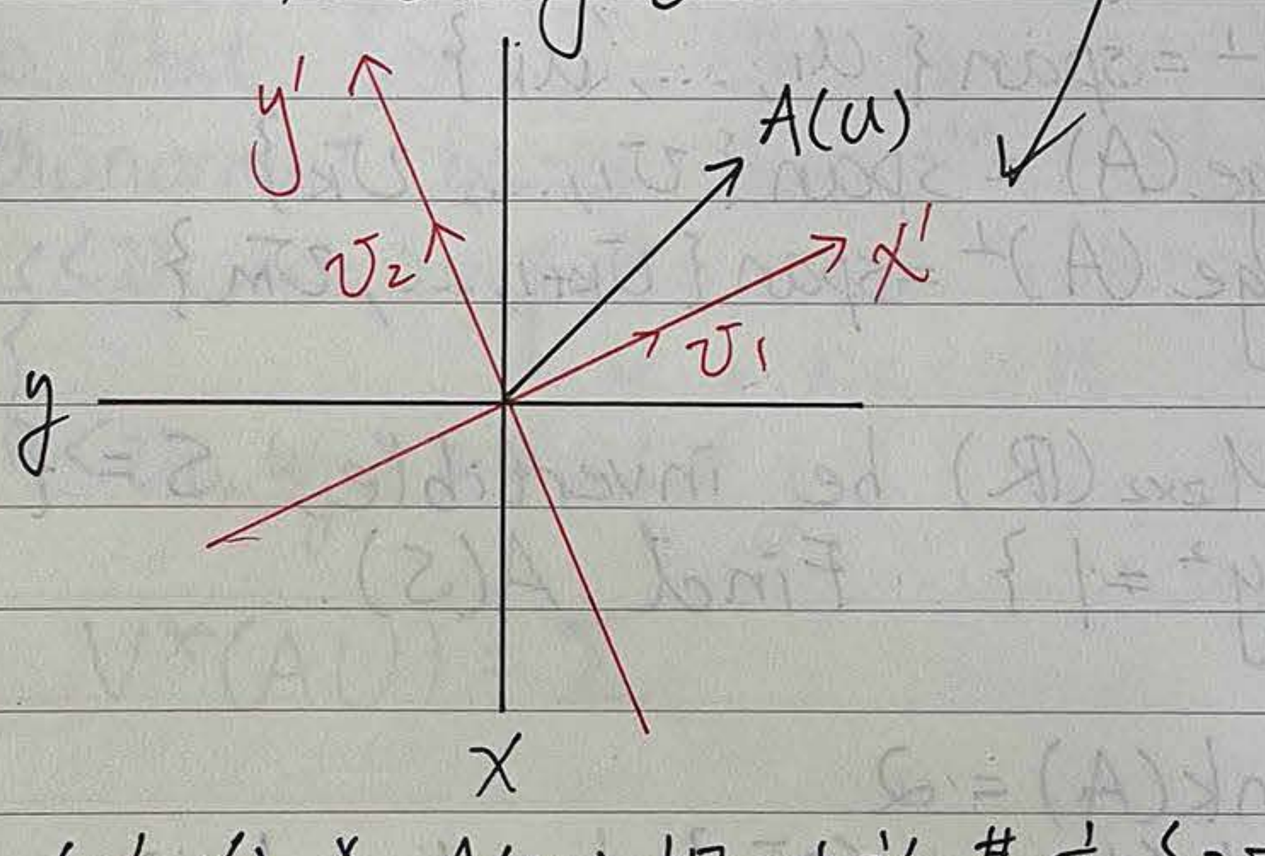
相对於 $\{u_1, u_2\}$ 之坐標

$\Rightarrow a_1^2 + a_2^2 = 1$



$$\Rightarrow A(u) = A(a_1u_1 + a_2u_2)$$

SVD $\rightarrow$
$$= a_1Au_1 + a_2Au_2$$
$$= a_1\lambda_1v_1 + a_2\lambda_2v_2$$
$$\triangleq x'v_1 + y'v_2$$



$\Rightarrow (x', y')$ 为 $A(u)$ 相对于基底 $\{v_1, v_2\}$ 之坐标.

$x' = a_1\lambda_1$, $y' = a_2\lambda_2$

$\because \lambda_1, \lambda_2 > 0$

$\therefore \left(\frac{x'}{\lambda_1}\right)^2 + \left(\frac{y'}{\lambda_2}\right)^2 = 1$. $\rightarrow$ 椭圆

一般L.T.之SVD分解，可先轉為矩陣之SVD後，再轉回去.

△ Pseudo inverse

想法：$T: V \to W$, $N(T)^{\perp} \oplus N(T) = V$. Consider $T: N(T)^{\perp} \to R(T)$, Let $L = T|_{N(T)^{\perp}}$

$T^{\dagger}$ (dagger) : paseudo inverse : $W \to V$ is defined by:

$$T^{\dagger}(y) = \begin{cases} L^{-1}(y), & y \in R(T) \\ 0, & y \in R(T)^{\perp}, \end{cases}$$

given that $R(T) \oplus R(T)^{\perp}$.

Def. Let $T: V \to W$ be a L.T. The L.T. $T^{+}: W \to V$ defined by

$$T^{+}(y) = \begin{cases} L^{-1}(y), & y \in R(T) \\ 0, & y \in R(T)^{\perp} \end{cases}$$

is called the pseudo inverse of T, given that $L = T|_{N(T)^{\perp}}$

Q: How to compute T^{+}?

Remarks. (1) T^{+} is unique.

(2) If $V = W$, $Ker(T) = 0$, then $T^{+} = T^{-1}$

(3) $V = N(T) \oplus N(T)^{\perp}$

(4) $W = R(T) \oplus R(T)^{\perp}$

I am OKAY.

SVD $\Rightarrow N(T)$, $N(T)^{\perp}$, $R(T)$, $R(T)^{\perp}$ 一清二楚,

設 $\operatorname{rank}(T)=k$.

$\Rightarrow \exists\ \beta=\{u_1,\dots,u_n\}$, $\gamma=\{v_1,\dots,v_m\}$:
orthonormal basis s.t. $T(u_i)=\lambda_i v_i$,
$\lambda_i>0$, $i=1,\dots,k$; $\lambda_i=0$, $k+1\le i\le n$

其中 λ_i^2 為 T^*T 之 eigenvalue
λ_i 為 T 的 singularvalue
u_i : T^*T 相對於 λ_i^2 之 eigenvector
v_i : ?

$\Rightarrow N(T)=\operatorname{span}\{u_{k+1},\dots,u_n\}$
$N(T)^{\perp}=\operatorname{span}\{u_1,\dots,u_k\}$
$R(T)=\operatorname{span}\{v_1,\dots,v_k\}$
$R(T)^{\perp}=\operatorname{span}\{v_{k+1},\dots,v_m\}$

$$\Rightarrow T^{+}(v_i)=\begin{cases}\frac{1}{\lambda_i}u_i, & 1\le i\le k\\ 0, & k+1\le i\le m\end{cases}$$

由 SVD $\Leftrightarrow V^*AU=\Sigma$,

$$U=\begin{pmatrix}| & & |\\ u_1 & \cdots & u_n\\ | & & |\end{pmatrix},\quad V=\begin{pmatrix}| & & |\\ v_1 & \cdots & v_m\\ | & & |\end{pmatrix},$$

$$\Sigma=\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \lambda_k & \\ & & & \ddots 0\end{pmatrix},\quad V^*AU=\Sigma.$$

$$\Rightarrow U^*T^{+}V\triangleq\Lambda=\begin{pmatrix}\frac{1}{\lambda_1} & & & \\ & \ddots & & \\ & & \frac{1}{\lambda_k} & \\ & & & \ddots 0\end{pmatrix}$$

$$\Rightarrow T^+ = U\Lambda V^*$$

Thm. (6.29) Let $A \in M_{m\times n}(F)$, $\operatorname{rank}(A)=r$ with SVD: $A=V\Sigma U^*$ and nonzero values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Define

$$\Sigma^+ = \begin{cases} \frac{1}{\lambda_i}, & i=j\leq r \\ 0, & \text{otherwise} \end{cases}$$

Then $A^+ = V\Sigma^+ U^*$, and this is SVD of A^+.

ex. $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}_{4\times 2}$, Find $A^+=?$

-sol-

$$\because A^*A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda-5)(\lambda-1) = 0 \Rightarrow \lambda = 5, 1$$

$$\therefore A^*A - 5I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\Rightarrow u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore A^*A - I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\therefore u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

I am OKAY.

$$\Rightarrow U=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

To find V, recall $U^*A^*AU=\begin{pmatrix}\lambda_1^2 & & & \\ & \ddots & & \\ & & \lambda_k^2 & \\ & & & 0\end{pmatrix}\triangleq W^*W$

$$\Rightarrow W=AU \Rightarrow VW=V^*AU=\Sigma$$

$$\Rightarrow \begin{cases}\langle w_i, w_i\rangle=\lambda_i^2\\ \langle w_j, w_i\rangle=0, \quad i\neq j\end{cases}$$

We want $\langle w_i, v_i\rangle=\lambda_i$, $i=1,2,\dots,k$,

then $v_i=\dfrac{w_i}{\lambda_i}$

*Summary: 如何找 v_i

$v_i=\dfrac{w_i}{\lambda_i}$, $i=1,2,\dots,k$

只要確保 $v_{k+1},\dots,v_m$ 使得 $\{v_1,\dots,v_m\}$ 是 orthonormal basis.

$$W=AU=\begin{pmatrix}1 & 1\\ 0 & 1\\ 1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\sqrt{2} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \sqrt{2} & 0\end{pmatrix}=\begin{pmatrix}| & |\\ w_1 & w_2\\ | & |\end{pmatrix}$$

$\because \operatorname{rank}(A)=2 \therefore w_1, w_2$ 都要

$$\Rightarrow v_1=\frac{w_1}{\lambda_1}=\frac{1}{\sqrt{5}}\begin{pmatrix}\sqrt{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ \sqrt{2}\end{pmatrix}, \quad v_2=\frac{w_2}{\lambda_2}=\frac{1}{1}\begin{pmatrix}0\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{5}} & 0 \end{pmatrix}$$

$$\Rightarrow A^+ = U \Lambda V^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \sqrt{\frac{2}{5}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$A = U \Sigma V^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \sqrt{\frac{2}{5}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Thm 6.30 Let $A \in M_{m \times n}(F)$ and the system $Ax = b$:

(1) If $Ax = b$ is consistent, then $z = A^+ b$ is the unique minimum solution, i.e., if $Ay = b$, then $\|z\| \leq \|y\|$

(2) If $Ax = b$ is inconsistent (無解), then $z = A^+ b$ is the unique solution of the minimum error.
(i.e., $\|Az - b\| \leq \|Ay - b\|, \forall y \in \mathbb{R}^n$)

I am OKAY.

* 結論：如何做SVD分解？（設 $A \in M_{m\times n}(\mathbb{R})$, $\operatorname{rank}(A)=k$）

1. 計算 A^*A 和其 eigenvalue $\lambda_1^2, \dots, \lambda_k^2$

$\Rightarrow$ 得 $\Sigma = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_k & \\ & & & \ddots \end{pmatrix}_{m\times n}$，和奇異值 $\lambda_1, \dots, \lambda_k$

2. 求出 A^*A 各特徵值對應之特徵向量 $\{u_1, \dots, u_n\}$

$\Rightarrow U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}_{n\times n}$

3. 計算 $W = AU = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix}_{m\times n}$

4. 計算 $v_i = \frac{w_i}{\lambda_i}$, $i=1,\dots,k$，並找一組 $\{v_{k+1}, \dots, v_m\}$ 使得 $\{v_1, \dots, v_m\}$ 為 $\mathbb{R}^m$ 之 orthonormal basis

$\Rightarrow V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix}_{m\times m}$

5. $A_{m\times n} = V_{m\times m}\,\Sigma_{m\times n}\,U^*_{n\times n}$

* SVD分解不唯一！
* 老師用的符號和課本相反！老師的 V 是課本的 U；U 是課本的 V

§6.8 Bilinear and Quadratic Forms

△ Bilinear Forms

Def. Let V be a vector space over F. A function $H: V \times V \to F$ is called a bilinear form on V if $\forall x, y, x_1, x_2, y_1, y_2 \in V$ and $a \in F$:

(1) $H(ax_1 + x_2, y) = aH(x_1, y) + H(x_2, y)$

(2) $H(x, ay_1 + y_2) = aH(x, y_1) + H(x, y_2)$

We denote the set of all bilinear forms on V by $B(V)$.

ex. The inner product on $\mathbb{R}^n$ is a bilinear form, but it is not if on $\mathbb{C}^n$.

Def. Let V be a vector space, let H_1 and H_2 be bilinear forms on V, and let a be a scalar. We define the sum $H_1 + H_2$ and the scalar product aH_1 by the equations

$$\begin{cases} (H_1 + H_2)(x, y) = H_1(x, y) + H_2(x, y) \\ (aH_1)(x, y) = a(H_1(x, y)), \quad \forall x, y \in V. \end{cases}$$

ex. Let $H: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = 2a_1b_1 + 3a_1b_2 + 4a_2b_1 - a_2b_2,$$

$$\forall \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{x}, \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{y} \in \mathbb{R}^2.$$

I am OKAY.

$\Rightarrow H(x,y) = x^T A y$, $A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$

$\because H(ax_1 + x_2, y) = (ax_1 + x_2)^T A y$
$= (ax_1^T + x_2^T) A y$
$= ax_1^T A y + x_2^T A y$
$= aH(x_1, y) + H(x_2, y)$, and
$H(x, ay_1 + y_2) = aH(x, y_1) + H(x, y_2)$
by a similar argument
$\therefore H$ is bilinear.

Prop. For any bilinear form H on a vector space V over a field F:
(1) If $\forall x \in V$, the functions L_x, R_x: $V \to F$ are defined by
$\begin{cases} L_x(y) = H(x,y) \\ R_x(y) = H(y,x) \end{cases}$, $\forall y \in V$
then L_x, R_x are linear
(2) $H(0,x) = H(x,0) = 0$, $\forall x \in V$
(3). $\forall x, y, z, w \in V$, $H(x+y, z+w)$
$= H(x,z) + H(x,w) + H(y,z) + H(y,w)$
(4) $J: V \times V \to F : J(x,y) \triangleq H(y,x)$ is bilinear.

Def. Let $A \in M_{n \times n}(F)$, $A_{ij} = H(v_i, v_j)$, $\forall i, j = 1, \cdots, n$. Then A is called the matrix representation of H w.r.t. the ordered basis $\beta \overset{\wedge}{=} \{v_i\}$ denoted by $\psi_\beta(H)$

ex. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, $B = \psi_\beta(H)$.

$$\Rightarrow B_{11} = H\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 2+3+4-1 = 8$$

$$B_{12} = H\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = 2-3+4+1 = 4$$

$$B_{21} = H\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 2+3-4+1 = 2$$

$$B_{22} = H\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = 2-3-4-1 = -6$$

$$\Rightarrow \psi_\beta(H) = \begin{pmatrix}8 & 4\\ 2 & -6\end{pmatrix} \quad \#$$

Thm. ① $B(V)$ is a vector space of dim. n^2 (6.31, Cor 1)
② $\psi_\beta : B(V) \to M_{n\times n}(F)$ is an isomurphism (6.32)
③ $\exists!\ A = \psi_\beta(H) \in M_{n\times n}(F)$ s.t. $H(x,y) = X^T A\, y$ (Cor 3)
④ $\psi_\beta(H) = A \iff H(x,y) = [\phi_\beta(x)]^T A\, [\phi_\beta(y)]$. (Cor 2)

pf: skip

Def. A bilinear form H is symmetric if
$H(x,y) = H(y,x)$, $\forall\, x, y \in V$.

Thm 6.34. Let $\dim(V) < \infty$, H is bilinear on V.
let β be an ordered basis for V. Then
H is symmetric $\iff$ $\psi_\beta(H)$ is symmetric.

pf: skip. 這2了定理都很好說, 很obvious.

I am OKAY.

△ Quadratic Forms.

Def. Let V be a vector space over F.
A function $K: V \to F$ is called a quadratic form if $\exists$ a symmetric bilinear form $H \in B(V)$ s.t. $K(x) = H(x, x)$, $\forall x \in V$.

Remark. We can recover H from K by

$$H(x,y) = \frac{1}{2}\left[K(x+y) - K(x) - K(y)\right]$$

ex. Given the variables $t_1, \dots, t_n$, the polynomial $f(t_1, \dots, t_n) = \sum_{i \le j} a_{ij} t_i t_j$

$\Rightarrow \psi_\beta(H) \triangleq A$, and $A_{ij} = A_{ji} = \begin{cases} a_{ii}, & i = j \\ \frac{1}{2}a_{ij}, & i \neq j \end{cases}$

ex2. $f(t_1, t_2, t_3) = 2t_1^2 - t_2^2 + 6t_1t_2 - 4t_2t_3$

$\Rightarrow A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$.

$\Rightarrow f(t_1, t_2, t_3) = t^T A t$, $t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$

Thm 6.33 ↓

lemma. Let V be a finite-dimensional vector space with ordered bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_2\}$. Let Q be the change of coordinate matrix changing γ into β.
Then, $\forall H \in B(V)$, $\psi_\gamma(H) = Q^T \psi_\beta(H) Q$.

pf. 麻煩, skip.

Thm 6.36. Let V be an real inner product space, $\dim V < \infty$. Then $\exists$ an orthonormal basis β for V s.t. $\Psi_\beta(H)$ is a diagonal matrix, $\forall H \in B(V)$.

pf. Let $\gamma = \{v_1, \dots, v_n\}$ be any orthonormal basis for V, and $A = \Psi_\gamma^{(}(H)$

$\because$ A is symmetric

$\therefore$ $\exists$ an orthogonal matrix Q and a diagonal matrix D s.t. $D = Q^T A Q$ by Thm 6.20

Let $\beta = \{w_1, \dots, w_n\}$, $w_j \triangleq \sum_{i=1}^{n} Q_{ij} v_i$, $1 \le j \le n$

$\Rightarrow$ By Thm 6.33, $\Psi_\beta(H) = D$.

$\because$ Q is orthogonal and γ is orthonormal

$\therefore$ β is orthonormal by direct inner product.

Q.E.D.

我們的目標是把所有沒有平移过的二次曲面做分類: $f(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz = 0$. 並用一矩陣表式之.

先來建立方程式与二次型之关係.

I am OKAY.

Cor. Let K be a quadratic form on V, $\dim V < \infty$, then $\exists$ an orthonormal basis $\beta = \{v_1, \dots, v_n\}$ and scalars $\lambda_1, \dots, \lambda_n$ (not necessarily distinct) s.t. $x = \sum_{i=1}^{n} s_i v_i$, $s_i \in \mathbb{R}$

坐标 basis. ex. e_1, e_2, e_3

and $K(x) = \sum_{i=1}^{n} \lambda_i s_i^2$ ex. (x, y, z)

pf. Let H be the symmetric bilinear form for which $K(x) = H(x, x)$, $\forall x \in V$.

$\Rightarrow$ By Thm 6.36, $\exists\ \beta = \{v_1, \dots, v_n\}$ = orthonormal basis for V s.t. $\psi_\beta(H)$ is the diagnal matrix $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Rightarrow$ Let $x \in V$ and if $x = \sum_{i=1}^{n} s_i v_i$,

then $K(x) = H(x)$

$= [\phi_\beta(x)]^T D [\phi_\beta(x)]$

$= (s_1, \dots, s_n) D \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$

$= \sum_{i=1}^{n} \lambda_i s_i^2$

Q.E.D.

$\Rightarrow$ 意義：basis 找得好，$f(x) = ax^2 + by^2 + cz^2$
不是多想、(x, y, z) 為关於此 basis 之坐標。
→ 転回正的！

先用一个例子：

ex. $f(t_1, t_2) = 5t_1^2 + 2t_2^2 + 4t_1t_2$ 試找一basis $\gamma = \{x_1, x_2\}$ for $\mathbb{R}^2$ 与 $\lambda_1, \lambda_2 \in \mathbb{R}$ s.t.
if $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2$ and $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = s_1x_1 + s_2x_2$
then $f(t_1, t_2) = K(s_1, s_2) = \lambda_1 s_1^2 + \lambda_2 s_2^2$

-sol-. Let H be the symmetric bilinear form corresponding to the quadratic form by f. $\beta = \{(1, 0), (0, 1)\}$
$\Rightarrow A = \psi_\beta(H) = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$

Compute the eigenvalue of A. $\Rightarrow \cdots$
$\Rightarrow \lambda_1 = 6, \lambda_2 = 1$. Also, $v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow$ 取 $Q = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$. 則 $Q^TAQ = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$.

故 $\{x_1, x_2\}$ 即 $\{v_1, v_2\}$，且 $f(t_1, t_2)$
$= K(s_1, s_2) = 6s_1^2 + s_2^2$ $\left(= (s_1\ s_2)\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\right)$ #

$f(t_1, t_2, t_3)$
ex. $2t_1^2 + 6t_1t_2 + 5t_2^2 - 2t_2t_3 + 2t_3^2 = 0$ 是什麼曲面？

-sol-. 令 H 為 f 之 symmetric bilinear form,
$A = \psi_\beta(H)$, $\beta = \{e_1, e_2, e_3\}$
$\Rightarrow A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

I am OKAY.

計算 A 之 eigenvalue 与 eigenvector, 得
$\lambda_1=2,\ \lambda_2=7,\ \lambda_3=0$, 且

$$v_1=\frac{1}{\sqrt{10}}\begin{pmatrix}1\\0\\3\end{pmatrix},\quad v_2=\frac{1}{\sqrt{35}}\begin{pmatrix}3\\5\\-1\end{pmatrix},\quad v_3=\frac{1}{\sqrt{14}}\begin{pmatrix}-3\\2\\1\end{pmatrix}$$

$\Rightarrow$ 取 $\gamma=\{v_1,v_2,v_3\}$, $Q=\begin{pmatrix}\frac{1}{v_1}&\frac{1}{v_2}&\frac{1}{v_3}\\1&1&1\end{pmatrix}$

則 $Q^TAQ=\begin{pmatrix}2&0&0\\0&7&0\\0&0&0\end{pmatrix}=\psi_\gamma(H)$

故由 Cor, 此時若更換 bases: 令 $x=s_1v_1+s_2v_2+s_3v_3$, (s_1,s_2,s_3) 為 (t_1,t_2,t_3) 相對於 γ 之新坐標,

則 $K(x)\triangleq(s_1\ s_2\ s_3)\begin{pmatrix}2&0&0\\0&7&0\\0&0&0\end{pmatrix}\begin{pmatrix}s_1\\s_2\\s_3\end{pmatrix}$

$=2s_1^2+7s_2^2$

$\Rightarrow$ 是 $\varnothing$!!!

ex3. $g(t_1,t_2,t_3)=2t_1^2+6t_1t_2+5t_2^2-2t_2t_3+2t_3^2+3t_1-2t_2-t_3+14=0$ 是什麼曲面?

-sol- 承 ex2. 進行坐標代換 (新舊关係代回已知).

$$(*)\begin{cases}t_1=\frac{s_1}{\sqrt{10}}+\frac{3s_2}{\sqrt{35}}-\frac{3s_3}{\sqrt{14}}\\ t_2=\frac{5}{\sqrt{35}}s_2+\frac{2}{\sqrt{14}}s_3\\ t_3=\frac{3}{\sqrt{10}}s_1-\frac{s_2}{\sqrt{35}}+\frac{s_3}{\sqrt{14}}\end{cases}\quad\left(\begin{pmatrix}t_1\\t_2\\t_3\end{pmatrix}=Q\begin{pmatrix}s_1\\s_2\\s_3\end{pmatrix}\right)$$

前面已證 $f(t_1,t_2,t_3)=2t_1^2+6t_1t_2+5t_2^2-2t_2t_3+2t_3^2$ 為 $2s_1^2+7s_2^2$

代入 (*)

後面的 $3t_1 - 2t_2 - t_3 = \cdots\cdots = -\sqrt{14}s_3$

故 t_1, t_2, t_3 之方程為 $2s_1^2 + 7s_2^2 - \sqrt{14}s_3 + 14 = 0$

後

$\Rightarrow$ 以 x, y, z 替換 s_1, s_2, s_3，得

$$z = \frac{\sqrt{14}}{7}x^2 + \frac{\sqrt{14}}{2}y^2 + \sqrt{14}$$

$\Rightarrow g(t_1, t_2, t_3) = 0$ 為 elliptic paraboloid #

§6.10 Conditioning and the Rayleigh Quotient

△ Conditioning

Def. A system having the property that small relative changes in the coeffcients of the system cause small relative errors in the solution is called well-conditioned, otherwise, the system is called ill-conditioned.

Consider the system $Ax=b$. We concentrat on change of b.

ex. $\begin{cases} x_1+x_2=5 \\ x_1-x_2=1 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\begin{cases} x_1+x_2=5 \\ x_1-x_2=1.0001 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3.00005 \\ 1.99995 \end{pmatrix}$

Def. The system $Ax=b$. The relative change in b (x) is the scalar $\|\delta b\|/\|b\|$ ($\|\delta x\|/\|x\|$)

ex. $\begin{cases} x_1+x_2=5 \\ x_1-x_2=1+\delta \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3+\frac{\delta}{2} \\ 2-\frac{\delta}{2} \end{pmatrix}$

$\Rightarrow \delta b = \begin{pmatrix} 5 \\ 1+h \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ h \end{pmatrix}$

$\Rightarrow \dfrac{\|\delta b\|}{\|b\|} = \dfrac{|h|}{\sqrt{26}}$

$\delta x = \begin{pmatrix} 3+\frac{h}{2} \\ 2-\frac{h}{2} \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} h/2 \\ -h/2 \end{pmatrix}$

$$\therefore \frac{\|\delta x\|}{\|x\|} = \frac{|h/\sqrt{2}|}{\sqrt{13}} = \frac{|h|}{\sqrt{26}}$$

$$\Rightarrow \frac{\|\delta b\|}{\|b\|} = \frac{\|\delta x\|}{\|x\|}$$

$\Rightarrow$ The system is well-conditioned.

ex. Consider the system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 + 1.0000\,x_2 = 3.0000 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The related system

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 + 1.0000\,x_2 = 3.0001 + \delta \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 - 10^5 h \\ 1 + 10^5 h \end{pmatrix}$$

$$\Rightarrow \begin{cases} \dfrac{\|\delta x\|}{\|x\|} = 10^5 \sqrt{\dfrac{2}{5}}\,|h| \geq 10^4 |h| \\ \dfrac{\|\delta b\|}{\|b\|} \doteq \dfrac{|h|}{3\sqrt{2}} \end{cases}$$ ← 差3~4个数量级!

$\Rightarrow$ The system is very ill-conditioned!

△ Rayleigh Quotient

Def. Let $A \in M_{n\times n}(\mathbb{C})$ or $M_{n\times n}(\mathbb{R})$. The (Euclidean) norm of A is defined by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}, \quad x \in \mathbb{C}^n \text{ or } \mathbb{R}^n.$$

Def. Let $B \in M_{n\times n}(\mathbb{C})$ or $M_{n\times n}(\mathbb{R})$ be self-adjoint, the Rayleigh quotient for $x \neq 0$ is defined to be the scalar $R(x) = \dfrac{\langle Bx, x\rangle}{\|x\|^2}$

I am OKAY.

Thm 6.43 For a self-adjoint matrix B, we have that

$$\begin{cases} \max_{x \neq 0} R(x) = \text{The largest eigenvalue of } B \\ \min_{x \neq 0} R(x) = \text{" smallest "} \end{cases}$$

pf. By Thm 6.19 and 6.20, we may choose an orthonormal basis $\{v_1, \dots, v_n\}$ of eigenvectors of B s.t. $Bv_i = \lambda_i v_i$ $(1 \leq i \leq n)$, where $\lambda_1 \geq \dots \geq \lambda_n$

Also, by lemma of Thm 6.17, the eigenvalues of B are real

$\therefore$ for $x \in F^n$, $\exists\ a_1, \dots, a_n$ s.t. $x = \sum_{i=1}^{n} a_i v_i$

$$\therefore R(x) = \frac{\langle Bx, x \rangle}{\|x\|^2}$$

$$= \frac{\left\langle \sum_{i=1}^{n} a_i \lambda_i v_i, \sum_{j=1}^{n} a_j v_j \right\rangle}{\|x\|^2}$$

$$= \frac{\sum_{i=1}^{n} \lambda_i |a_i|^2}{\|x\|^2}$$

$$\leq \frac{\lambda_1 \sum_{i=1}^{n} |a_i|^2}{\|x\|^2}$$

$$= \frac{\lambda_1 \|x\|^2}{\|x\|^2}$$

$$= \lambda_1$$

$\Rightarrow$ it is easy to see that $R(v_1)=\lambda_1$

$\Rightarrow$ By a similar argument, one can prove that $R(x) \geq \lambda_n$ and $R(v_n)=\lambda_n$

Q.E.D.

Cor1. For any square matrix A, $\|A\|$ is finite and $\|A\|=\sqrt{\lambda}$, where λ is the largest eigenvalue of A^*A.

pf. $\because$ B is self adjoint

$\therefore \exists A$ s.t. $B=A^*A$.

Let λ be the largest eigenvalue of B,

$\because x \neq 0$

$$\therefore 0 \leq \frac{\|Ax\|^2}{\|x\|^2} = \frac{\langle Ax, Ax\rangle}{\|x\|^2}$$

$$= \frac{\langle A^*Ax, x\rangle}{\|x\|^2}$$

$$= \frac{\langle Bx, x\rangle}{\|x\|^2}$$

$$= R(x)$$

$\Rightarrow$ by Thm 6.43, $\|A\|^2=\lambda$

Q.E.D.

lemma. For any square matrix A, λ is an eigenvalue of $A^*A \Leftrightarrow \lambda$ is an eigenvalue of AA^*

pf. If $\lambda=0$:

$\Rightarrow$) $\because A^*A$ is not invertible

I am OKAY.

$\therefore A$ and A^* are not invertible
$\Rightarrow \lambda$ is also an eigenvalue of AA^*

$\Leftarrow$) Similar (把所有 A^* 和 A 換位置)

If $\lambda \neq 0$:

$\Rightarrow$) $\because \exists\, x \neq 0$ s.t. $A^*Ax = \lambda x$
$\therefore (AA^*)Ax = \lambda(Ax)$
$\because Ax \neq 0$ (lest $\lambda x = 0$)
$\therefore \lambda$ is an eigenvalue of AA^*

$\Leftarrow$) Similar (一樣換位置)

Q.E.D.

Cor. Let A be invertible matrix. Then $\|A^{-1}\| = \frac{1}{\sqrt{\lambda}}$, where λ is the smallest eigenvalue of A^*A.

pf. recall that λ is an e.v. of an invertible matrix $\Leftrightarrow \lambda^{-1}$ is an e.v. of its inverse
Let $\lambda_1 \geq \cdots \geq \lambda_n$ be e.v. of A^*A.
$\because$ by lemma, $\lambda_1, \ldots, \lambda_n$ are also the e.v. of AA^*
$\therefore \|A^{-1}\|^2 =$ the largest e.v. $(A^{-1})^*A^{-1} = (AA^*)^{-1}$
$\Rightarrow \|A^{-1}\| = \frac{1}{\sqrt{\lambda}}$

Q.E.D.

Def. The number $\|A\| \cdot \|A^{-1}\|$ is called the condition number of A, denoted by $\text{cond}(A)$.

Thm 6.44. For the system $Ax=b$, where A is invertible and $b \neq 0$, the following holds:

(1) For any norm $\|\cdot\|$, we have

$$\frac{1}{\text{cond}(A)} \frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\delta b\|}{\|b\|}$$

(2). If $\|\cdot\|$ is the Euclidean norm, then

$$\text{cond}(A) = \sqrt{\frac{\lambda_1}{\lambda_n}}, \text{ where}$$

$\begin{cases} \lambda_1 = \text{the largest e.v. of } A^*A \\ \lambda_n \text{ " smallest " } \end{cases}$

pf. (1). It is obvious that $\|Ax\| \leq \|A\| \cdot \|x\|$, $\forall x$.

For a given δb, let δx be the vector that satisfies $A(x+\delta x) = b + \delta b$, Then

$A(\delta x) = \delta b$

$\Rightarrow \delta x = A^{-1}(\delta b)$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \frac{\|x\|^2}{\|b\| / \|A\|}$$

$$\leq \frac{\|A^{-1}\| \cdot \|\delta b\| \cdot \|A\|}{\|b\|}$$

$$= \|A\| \cdot \|A^{-1}\| \cdot \left(\frac{\|\delta b\|}{\|b\|}\right)$$

$\Rightarrow$ The ineq. holds

I am OKAY.

(2) Follows from Cor 1, 2.

Q.E.D.

ex. Let $A=\begin{pmatrix}1&0&1\\-1&1&0\\0&1&1\end{pmatrix}$, Compute $R(x)$ for $B=A^*A$, $x=\begin{pmatrix}a\\b\\c\end{pmatrix}$.

-sol-

$\because B=A^*A=\begin{pmatrix}2&-1&1\\-1&2&1\\1&1&2\end{pmatrix}$, and its e.v.

are 3, 3, 0

$\therefore \|A\|=\sqrt{3}$

$\Rightarrow \forall x\in\mathbb{R}^3, x\neq 0,$

$$3\geq R(x)=\frac{\langle Bx,x\rangle}{\|x\|^2}$$

$$=\frac{2(a^2+b^2+c^2-ab+ac+bc)}{a^2+b^2+c^2}$$

#

Remark. 由於要算 A^{-1}, 一般不会由定義去算 cond(A). 会用数值逼近的方法.

Ch7 Canonical Forms

§7.1 Jordon Canonical Forms

Q: 可对角线化之矩阵可藉矩阵分解得到一矩阵 $\Lambda=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$，那不可对角线化矩阵有無辦法「化簡」到「簡單的形式」呢？

ex. $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $CP=(\lambda-1)^2$, $am(\lambda=1)=2$

$\Rightarrow A-1I=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $rank(A-I)=1$

$\Rightarrow$ by dim Thm, $N(A-I)$ 之 dim $=1$

$\Rightarrow gm(\lambda=1)=1$

ex. Let

$$A=\begin{pmatrix}2 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & 2\end{pmatrix}_{100\times 100}$$

$\Rightarrow C.P.=(\lambda-2)^{100}$, $am(2)=100$.

$rank(A-2I)=99$, where

$$A-2I=\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0\end{pmatrix},$$

$Nullity(A-2I)=1=gm(2)$

I am OKAY.

Def. The matrix

$$A_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

is called a Jordan block
The matrix A of the form

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$$

is called a Jordon Canonical Form (JCF), where A_i are Jordon blocks.

Goal: prove that: 「若 C.P. of A 可一次分解, 則 $\exists \beta$: a basis s.t. $[L_A]_\beta$ = a J.C.F.」

ex. T: $\exists \beta = \{v_1, \dots, v_8\}$ a basis

$$[T]_\beta = \left(\begin{array}{cccc|cc|cc} v_1 & v_2 & \cdots & & & & & v_8 \\ \hline 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)_{8\times 8} = \left(\begin{array}{c|c|c|c} A_1 & 0 & 0 & 0 \\ \hline 0 & A_2 & 0 & 0 \\ \hline 0 & 0 & A_3 & 0 \\ \hline 0 & 0 & 0 & A_4 \end{array}\right)$$

C.P. of $T=(\lambda-2)^4(\lambda-3)^2\lambda^2$

Q: (1) β 中，哪些 v_i 是 eigenvectors?
(2) 不是 eigenvectors 之 v_i 又有那些性質

v_1 ~~v_2~~ ~~v_3~~ v_4 v_5 ~~v_6~~ v_7 ~~v_8~~

(由 $Tv_i = 2v_1 + 0v_2 + \cdots + 0v_8$ 可看出)

A: Consider $T(v_2) = v_1 + 2v_2$

$\Rightarrow (T-2I)v_2 = v_1 \Rightarrow (T-2I)^2 v_2 = 0$. —(*)

$T(v_3) = v_2 + 2v_3 \Rightarrow (T-2I)v_3 = v_2$

$\Rightarrow$ by (*), $(T-2I)^3 = 0$.

$(T-2I)v_4 = 0$

Def. (1) $v \neq 0$ is a generalized eigenvector of T if $\exists \lambda \in F$, and a $p \in \mathbb{N}$ s.t. $(T-\lambda I)^p(v) = 0$ → generalized eigenspace

(2) Let $K_\lambda = \{v \in V : \exists m \in \mathbb{N} \text{ s.t. } (T-\lambda I)^m v = 0\}$
$= \{$ generalized eigenvector correspond to $\lambda \}$

Remarks. (1) λ is an eigenvalue of T since for k being the smallest positive integer s.t. $(T-\lambda I)^k(v) = 0$, then $(T-\lambda I)^{k-1}(v)$ is an eigenvector of T corresponding to eigenvalue of λ

(2) If $p=1$, then v is an eigenvectors of T.

I am OKAY.

(1) 說明: $(T-\lambda I)^k v = 0 \Rightarrow (T-\lambda I)\underbrace{\left((T-\lambda I)^{k-1}\right)(v)}_{u} = 0$

$\Rightarrow (T-\lambda I)u = 0$

$\Leftrightarrow T(u) = \lambda u.$

目的: 1. $\dim(K_\lambda) = am(\lambda)$

$\beta = \{\text{generalized eigenvectors}\}$

ex.

$$A_1 = \left(\begin{array}{ccc|cc} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ \hline & & & 2 & \\ & & & & 2 \end{array}\right)_{5\times 5} \qquad A_2 = \left(\begin{array}{ccc|cc} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & 2 & 1 \\ & & & 0 & 2 \end{array}\right)_{5\times 5}$$

$C.P. = (2-t)^5 = \det(A_2 - tI)$

$5 = \underline{4+1} = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1+1$

↳ block 之 dim 和

$$A_3 = \left(\begin{array}{ccc|c|c} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ \hline & & & 2 & \\ \hline & & & & 2 \end{array}\right), \quad A_4 = \left(\begin{array}{cc|ccc} 2 & 1 & & & \\ & 2 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & \\ \hline & & & & 2 \end{array}\right)$$

$$A_5 = \left(\begin{array}{cc|c|c} 2 & 1 & & \\ & 2 & & \\ \hline & & 2 & \\ \hline & & & 2 \end{array}\right)$$

設 $A_1 = [T_1]_{\beta_1}$, $\beta_1 = \{v_1, v_2, \ldots, v_5\}$

則 v_1, v_5 爲 eigenvector, $v_2\ v_3, v_4$ 爲 generalized eigenvector

$$\begin{cases} T_1(v_4) = v_3 + 2v_4 \\ (T_1 - 2I)v_4 = v_3 \\ (T_1 - 2I)v_3 = v_2 \\ (T_1 - 2I)v_2 = v_1 \end{cases}$$

$$\Rightarrow \begin{cases} (T_1 - 2I)^2 v_2 = (T - 2I)v_1 = 0 \\ (T_1 - 2I)^3 v_3 = 0 \\ (T_1 - 2I)^4 v_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} v_4 \to v_3 \to v_2 \to v_1 \\ v_5 \end{cases}$$

Def. Let v be a generalized eigenvector with corresponding eigenvalue λ.
Let p be the smallest positive integer s.t. $(T - \lambda I)^p(v) = 0$. Then
$\gamma \triangleq \{(T - \lambda I)^{p-1}(v), (T - \lambda I)^{p-2}(v), \ldots, (T - \lambda I)(v), v\}$ is called a cycle of a generalized eigenvectors corresponding to λ.
We say that the length of the cycle is p.

ex. 如上例中的 $\{v_4, v_3, v_2, v_1\}$ 為 length = 4 之 cycle
($\lambda = 2$)
$\{v_5\}$ = length 為 1 之 cycle

I am OKAY.

② $(T-\lambda I)^{p-1}(v)$ and v are called the initial vector and each vector of the cycle γ respectively.

* $\beta = \{\text{cycles of generalized ev's}\}$

Fact: (1) γ is linearly indep.

pf. Let $\sum_{i=0}^{p-1} a_i (T-\lambda I)^i(x) = 0$. We want $a_i = 0, \forall i$.

$$\Rightarrow a_{p-1}(T-\lambda I)^{p-1}(x) + \cdots + a_1(T-\lambda I)^1(x) + a_0 x = 0 \quad (*)$$

$$\Rightarrow a_0 \underbrace{(T-\lambda I)^{p-1}(x)}_{\neq 0} = 0$$

↑ $\times (T-\lambda I)^{p-1}$

$$\Rightarrow a_0 = 0$$

Similarly, Apply $(T-\lambda I)^{p-2}, (T-\lambda I)^{p-3}, \ldots$ on $(*)$, we can prove that $a_i = 0, \forall i = 0, \ldots, p-1$

(2) Let γ_i be "disjoint" cycles, then $\bigcup_i \gamma_i$ is linearly indep.

(此處之"disjoint"是指將不同的cycle之initial vectors收集所成的集合是l.i.)
又此處γ_i所对应之eigenvalue皆為λ.

pf: 同(1)

(3) Let $\beta = \{$ disjoint cycle of generalized eigenvectors w.r.t. $\lambda\}$, then β is a basis for K_λ.

(4) Let λ_i, $i=1,\dots,k$ be distinct eigenvalues of T. Then We can do (1)-(3) for each K_λ

(5) $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$
$\Rightarrow \beta = \bigcup_{i=1}^{k} \beta_i$ is a Jordan Canonical basis for V.

Summary: Let $A \in M_{n\times n}(\mathbb{C})$. Then $\exists\ \beta =$ a Jordan Canonical basis $= \bigcup \beta_i$ = g.e.'s $= \{$ cycles of generalized eigenvectors $\}$ s.t. $[L_A]_\beta = Q^{-1}AQ =$ a JCF, where $Q = [\beta]$
= 由 β 排出來的矩陣.

ex. Let

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 & \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}_{6\times 6}$$

Find the JCF (Q, β) of A
that is,

I am OKAY.

-sol- step 1. C.P. $= \det(A-\lambda I) = (\lambda-2)^5\lambda$

$\Rightarrow \lambda=2$ or $\lambda=0$ are eigenvalues of A

$\Rightarrow V = K_2 \oplus K_0$, $K_0 = E_0$, and

$\beta_1 = \{$ disjoint cycles of g.e.'s w.r.t. $\lambda=2\}$

step2. $\lambda=2$ 有几个 disjoint cycles?

$\Rightarrow$ 有 $gm(\lambda=2)$ 个

$$\therefore A-2I = \begin{array}{c} \begin{matrix} v & v & \triangle & \triangle & \star & \heartsuit \end{matrix} \\ \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array} \quad \begin{matrix} \rightarrow 4 \text{ 个 l.i. column} \end{matrix}$$

$\therefore \operatorname{rank}(A-2I) = 4$

$\Rightarrow \text{Nullity}(A-2I) = 2$

$\Rightarrow \beta$ 含 2 个不同的 cycle.

cycle 长度为 4 / " / 1

Fact: $\dim K_2 = 5 = 4+1 = 3+2$

step3.

$$\therefore (A-2I)^2 = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix} & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix} 2 & -2 \\ -2 & 2 \end{matrix} \end{array}\right)$$

$\therefore \operatorname{rank}(A-2I)^2 = 2$

$$(A-2I)^3=\begin{pmatrix}0 & 0\\ 0 & \begin{matrix}-4 & 4\\ 4 & -4\end{matrix}\end{pmatrix}$$

$\therefore$ $\operatorname{rank}(A-2I)^3=1$

$\because$ $(A-2I)^4=\begin{pmatrix}0 & 0\\ 0 & \begin{matrix}8 & -8\\ -8 & 8\end{matrix}\end{pmatrix}$

$\therefore$ $\operatorname{rank}(A-2I)^4=1$ $\Rightarrow$ 可作在 $(A-2I)^3$ $(p=3)$

$\Rightarrow$ A 只可做「3+2」的拆角羊!

step4. $\because 5=3+2$

$\therefore$ 找 - cycle 長度 $=3$, say $\{v_1, v_2, v_3\}$

$=\{(T-2I)^2 v_3,\ (T-2I)v_3,\ v_3\}$

先挑 - v_3, 候选:

✓ ✓ ✓

$$\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\quad\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\quad\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\quad\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\quad\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

↓

挑此主 v_3

I am OKAY.

$$\Rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Similarly, take

$$v_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow v_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \{v_1, v_2, v_3, v_4, v_5\}$ correspond to $\lambda = 2$

As for v_6 ($\lambda = 0$'s eigenvetor), it is obvious that we can take

$$v_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$\Rightarrow \beta = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

step5. Let

$$Q = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_6 \\ | & & | \end{pmatrix}$$

$$\Rightarrow Q^{-1}AQ = \left(\begin{array}{cc|c|c} \begin{matrix}2&1\\&2\end{matrix} & & & \\ & \begin{matrix}2&1\\&2\end{matrix} & & \\ \hline & & \begin{matrix}2&1\\&2\end{matrix} & \\ \hline & & & 0 \end{array}\right) \triangleq J$$

where

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

ex. Find the JCF of

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$$

-sol- step 1. $\det(A-\lambda I) = (\lambda-2)^3(\lambda-3)$

$$\Rightarrow Q^{-1}AQ = \left(\begin{array}{ccc|c} 2 & & & \\ & 2 & & \\ & & 2 & \\ \hline & & & 3 \end{array}\right)$$

I am OKAY.

上面的block,

$\Rightarrow 3 = 1+1+1 = 2+1$，先判断是哪一种

step2.

$$\because A-2I = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$\therefore \text{rank}(A-2I) = 2$, $\text{Nullity}(A-2I) = 2$

$\Rightarrow$ 取 $4 = 2+1+1$，共有2个 cycle

$$\Rightarrow Q^{-1}AQ = \left(\begin{array}{cc|c} 2 & 1 & \\ & 2 & \\ \hline & & 2 \\ \hline & & & 3 \end{array}\right), \text{ and}$$

$$Q = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}$$，v_4为3之 eigenvector.

再去找 v_2，后找 v_1.