Appendix A

Overview on Suprema and Limits

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A.1 Supremum and maximum

We herein review basic results on suprema and limits which are useful for the development of information theoretic coding theorems.

- Throughout, we work on subsets of $\mathbb{R}$, the set of real numbers.

Definition A.1 (Upper bound of a set) A real number u is called an *upper bound* of a **non-empty** subset $\mathcal{A}$ of $\mathbb{R}$ if every element of $\mathcal{A}$ is less than or equal to u; we say that $\mathcal{A}$ is *bounded above*. Symbolically, the definition becomes:

$$\mathcal{A} \subset \mathbb{R} \text{ is bounded above} \iff (\exists\, u \in \mathbb{R}) \text{ such that } (\forall\, a \in \mathcal{A}), a \le u.$$

Definition A.2 (Least upper bound or supremum) Suppose $\mathcal{A}$ is a **non-empty** subset of $\mathbb{R}$. Then we say that a real number s is a *least upper bound* or *supremum* of $\mathcal{A}$ if s is an upper bound of the set $\mathcal{A}$ and if $s \le s'$ for each upper bound s' of $\mathcal{A}$. In this case, we write $s = \sup \mathcal{A}$; other notations are $s = \sup_{x\in\mathcal{A}} x$ and $s = \sup\{x : x \in \mathcal{A}\}$.

Completeness Axiom: (Least upper bound property) Let $\mathcal{A}$ be a **non-empty** subset of $\mathbb{R}$ that is bounded above. Then $\mathcal{A}$ has a least upper bound (in $\mathbb{R}$).

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Completeness Axiom may not work for, say, $\mathbb{Q}$, the set of rational numbers.

Definition A.1′ (Upper bound of a set in $\mathbb{Q}$) A rational number u is called an *upper bound* of a **non-empty** subset $\mathcal{A}$ of $\mathbb{Q}$ if every element of $\mathcal{A}$ is less than or equal to u; we say that $\mathcal{A}$ is *bounded above*. Symbolically, the definition becomes:

$$\mathcal{A} \subset \mathbb{Q} \text{ is bounded above} \quad \Longleftrightarrow \quad (\exists\, u \in \mathbb{Q}) \text{ such that } (\forall\, a \in \mathcal{A}), a \leq u.$$

~~**Definition A.2′**~~ (Least upper bound or supremum in $\mathbb{Q}$) Suppose $\mathcal{A}$ is a **non-empty** subset of $\mathbb{Q}$. Then we say that a rational number s is a *least upper bound* or *supremum* of $\mathcal{A}$ if s is an upper bound of the set $\mathcal{A}$ and if $s \leq s'$ for each upper bound s' of $\mathcal{A}$. In this case, we write $s = \sup \mathcal{A}$; other notations are $s = \sup_{x \in \mathcal{A}} x$ and $s = \sup\{x \colon x \in \mathcal{A}\}$.

Example. $\mathcal{A} = \{x \in \mathbb{Q} : x^2 < 2\}$. Then, $\sup_{x \in \mathcal{A}} x$ is supposed to be "the largest rational number" less than $\sqrt{2}$!

A.1 Supremum and maximum

- It follows directly that if a **non-empty** set in $\mathbb{R}$ has a supremum, then this supremum is unique.
- By definition, the empty set ($\emptyset$) and any set not bounded above do not admit a supremum in $\mathbb{R}$.

Property A.4 (Properties of the supremum)

1. The supremum of any set in $\mathbb{R} \cup \{-\infty, \infty\}$ (the set of extended real numbers) always exits.

$$\sup \mathcal{A} := \begin{cases} -\infty, & \text{if } \mathcal{A} = \emptyset; \\ +\infty, & \text{if } \mathcal{A} \text{ is not bounded above.} \end{cases}$$

   These extended definitions will be adopted in this course.

2. $(\forall\, a \in \mathcal{A} \subset \mathbb{R} \cup \{-\infty, \infty\})\ a \leq \sup \mathcal{A}$.
3. If $-\infty < \sup \mathcal{A} < \infty$, then $(\forall\, \varepsilon > 0)(\exists\, a_0 \in \mathcal{A})\ a_0 > \sup \mathcal{A} - \varepsilon$.
   (The existence of $a_0 \in (\sup \mathcal{A} - \varepsilon, \sup \mathcal{A}]$ for any $\varepsilon > 0$ under the condition of $|\sup \mathcal{A}| < \infty$ is called the *approximation property for the supremum.*)
4. If $\sup \mathcal{A} = \infty$, then $(\forall\, L \in \mathbb{R})(\exists\, B_0 \in \mathcal{A})\ B_0 > L$.
5. If $\sup \mathcal{A} = -\infty$, then $\mathcal{A}$ is empty.

A.1 Supremum and maximum

Definition A.3 (Maximum) If $\sup \mathcal{A} \in \mathcal{A}$, then $\sup \mathcal{A}$ is also called the *maximum* of $\mathcal{A}$, and is denoted by $\max \mathcal{A}$. However, if $\sup \mathcal{A} \notin \mathcal{A}$, then we say that the maximum of $\mathcal{A}$ does not exist.

E.g., if $\mathcal{A} = (0, 1]$, then $\max \mathcal{A} = \sup A = 1$.

E.g., if $\mathcal{A} = (0, 1)$, then $\sup \mathcal{A} = 1$ but $\max \mathcal{A}$ does not exist!

Observation A.5 (Supremum of a set and channel coding theorems) In information theory, a typical channel coding theorem establishes that a (finite) real number α is the supremum of a set $\mathcal{A}$. Thus, to prove such a theorem, one must show that α satisfies both properties 3 and 2 above, i.e.,

$$\text{Forward/Direct part: } \underline{(\alpha - \epsilon) \text{ is achievable in } \mathcal{A}} : (\forall\, \varepsilon > 0)(\exists\, a_0 \in \mathcal{A})\ a_0 > \alpha - \varepsilon \tag{A.1.1}$$

and

$$\text{Converse part: } \underline{\alpha \text{ is a bound for all achievable values in } \mathcal{A}} : (\forall\, a \in \mathcal{A})\ a \leq \alpha. \tag{A.1.2}$$

$$\boxed{(\forall \epsilon > 0)\ \alpha - \epsilon \leq \sup \mathcal{A} \leq \alpha}$$

A.1 Supremum and maximum

Property A.6 (Properties of the maximum)

1. $(\forall\ a \in \mathcal{A})\ a \le \max \mathcal{A}$, if $\max \mathcal{A}$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$.
2. $\max \mathcal{A} \in \mathcal{A}$.

- From the above property, in order to obtain $\boxed{\alpha = \max \mathcal{A}}$, one needs to show that α satisfies both

  Converse part: $\underline{\alpha \text{ is a bound for all achievable values in } \mathcal{A}}$: $(\forall\ a \in \mathcal{A})\ a \le \alpha$.

  and

  Achievability/Forward/Direct part: $\underline{\alpha \text{ is achievable in } \mathcal{A}}$: $\alpha \in \mathcal{A}$

  E.g. Computation of the channel capacity of a binary symmetric channel.

A.2 Infimum and minimum

The concepts of infimum and minimum are dual to those of supremum and maximum.

Definition A.7 (Lower bound of a set) A real number ℓ is called a *lower bound* of a non-empty subset $\mathcal{A}$ in $\mathbb{R}$ if every element of $\mathcal{A}$ is greater than or equal to ℓ; we say that $\mathcal{A}$ is *bounded below*. Symbolically, the definition becomes:

$$\mathcal{A} \subset \mathbb{R} \text{ is bounded below} \iff (\exists\, \ell \in \mathbb{R}) \text{ such that } (\forall\, a \in \mathcal{A})\ a \geq \ell.$$

Definition A.8 (Greatest lower bound or infimum) Suppose $\mathcal{A}$ is a non-empty subset of $\mathbb{R}$. Then we say that a real number ℓ is a *greatest lower bound* or *infimum* of $\mathcal{A}$ if ℓ is a lower bound of $\mathcal{A}$ and if $\ell \geq \ell'$ for each lower bound ℓ' of $\mathcal{A}$. In this case, we write $\ell = \inf \mathcal{A}$; other notations are $\ell = \inf_{x \in \mathcal{A}} x$ and $\ell = \inf\{x : x \in \mathcal{A}\}$.

Completeness Axiom: (Greatest lower bound property) Let $\mathcal{A}$ be a non-empty subset of $\mathbb{R}$ that is bounded below. Then $\mathcal{A}$ has a greatest lower bound.

A.2 Infimum and minimum

- It directly follows that if a **non-empty** set in $\mathbb{R}$ has an infimum, then this infimum is unique.
- By definition, the empty set ($\emptyset$) and any set not bounded below do not admit an infimum in $\mathbb{R}$.

Property A.10 (Properties of the infimum)

1. The infimum of any set in $\mathbb{R} \cup \{-\infty, \infty\}$ always exists.

$$\inf \mathcal{A} := \begin{cases} +\infty, & \text{if } \mathcal{A} = \emptyset; \\ -\infty, & \text{if } \mathcal{A} \text{ is not bounded below.} \end{cases}$$

   These extended definitions will be adopted in this course.

2. $(\forall\, a \in \mathcal{A} \subset \mathbb{R} \cup \{-\infty, \infty\})\; a \geq \inf \mathcal{A}$.
3. If $\infty > \inf \mathcal{A} > -\infty$, then $(\forall\, \varepsilon > 0)(\exists\, a_0 \in \mathcal{A})\; a_0 < \inf \mathcal{A} + \varepsilon$.
   (The existence of $a_0 \in [\inf \mathcal{A}, \inf \mathcal{A} + \varepsilon)$ for any $\varepsilon > 0$ under the assumption of $|\inf \mathcal{A}| < \infty$ is called the *approximation property for the infimum*.)
4. If $\inf \mathcal{A} = -\infty$, then $(\forall L \in \mathbb{R})(\exists\, B_0 \in \mathcal{A}) B_0 < L$.
5. If $\inf \mathcal{A} = \infty$, then $\mathcal{A}$ is empty.

A.2 Infimum and minimum

Definition A.9 (Minimum) If $\inf \mathcal{A} \in \mathcal{A}$, then $\inf \mathcal{A}$ is also called the *minimum* of $\mathcal{A}$, and is denoted by $\min \mathcal{A}$. However, if $\inf \mathcal{A} \notin \mathcal{A}$, we say that the minimum of $\mathcal{A}$ does not exist.

Observation A.11 (Infimum of a set and channel coding theorems) In information theory, a typical source coding theorem establishes that a (finite) real number α is the infimum of a set $\mathcal{A}$. Thus, to prove such a theorem, one must show that α satisfies both properties 3 and 2 above, i.e.,

$$\text{Forward/Direct part: } \underline{(\alpha+\epsilon) \text{ is achievable in } \mathcal{A}} : (\forall\, \varepsilon > 0)(\exists\, a_0 \in \mathcal{A})\ a_0 < \alpha + \varepsilon \tag{A.2.1}$$

and

$$\text{Converse part: } \underline{\alpha \text{ is a bound for all achievable values in } \mathcal{A}} : (\forall\, a \in \mathcal{A})\ a \geq \alpha. \tag{A.2.2}$$

A.2 Infimum and minimum

Property A.12 (Properties of the minimum)

1. $(\forall\ a \in \mathcal{A})\ a \geq \min \mathcal{A}$, if $\min \mathcal{A}$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$.
2. $\min \mathcal{A} \in \mathcal{A}$.

- From the above property, in order to obtain $\boxed{\alpha = \min \mathcal{A}}$, one needs to show that α satisfies both

  Converse part: $\underline{\alpha \text{ is a bound for all achievable values in } \mathcal{A}}$: $(\forall\ a \in \mathcal{A})\ a \geq \alpha$.

  and

  Achievability/Forward/Direct part: $\underline{\alpha \text{ is achievable in } \mathcal{A}}$: $\alpha \in \mathcal{A}$

  E.g. Computation of the rate-distortion function for binary DMS and Hamming distance measure (cf. Theorem 6.23).

A.3 Boundedness and suprema operations

Definition A.13 (Boundedness) A subset $\mathcal{A}$ of $\mathbb{R}$ is said to be *bounded* if it is both bounded above and bounded below; otherwise it is called *unbounded*.

Lemma A.14 (Condition for boundedness) A subset $\mathcal{A}$ of $\mathbb{R}$ is bounded iff $(\exists\ k \in \mathbb{R})$ such that $(\forall\ a \in \mathcal{A})\ |a| \leq k$.

Lemma A.15 (Monotone property) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are **non-empty** subsets of $\mathbb{R}$ such that $\mathcal{A} \subset \mathcal{B}$. Then

1. $\sup \mathcal{A} \leq \sup \mathcal{B}$.
2. $\inf \mathcal{A} \geq \inf \mathcal{B}$.

A.3 Boundedness and suprema operations

Lemma A.16 (Supremum for set operations) Define the "addition" of two sets $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A} + \mathcal{B} := \{c \in \mathbb{R} \colon c = a + b \text{ for some } a \in \mathcal{A} \text{ and } b \in \mathcal{B}\}.$$

Define the "scaler multiplication" of a set $\mathcal{A}$ by a scalar $k \in \mathbb{R}$ as

$$k \cdot \mathcal{A} := \{c \in \mathbb{R} \ : \ c = k \cdot a \text{ for some } a \in \mathcal{A}\}.$$

Finally, define the "negation" of a set $\mathcal{A}$ as

$$-\mathcal{A} := \{c \in \mathbb{R} \ : \ c = -a \text{ for some } a \in \mathcal{A}\}.$$

Then the following hold.

1. If $\mathcal{A}$ and $\mathcal{B}$ are both bounded above, then $\mathcal{A} + \mathcal{B}$ is also bounded above and $\sup(\mathcal{A} + \mathcal{B}) = \sup \mathcal{A} + \sup \mathcal{B}$.
2. If $0 < k < \infty$ and $\mathcal{A}$ is bounded above, then $k \cdot \mathcal{A}$ is also bounded above and $\sup(k \cdot \mathcal{A}) = k \cdot \sup \mathcal{A}$.
3. $\sup \mathcal{A} = -\inf(-\mathcal{A}) \quad \text{and} \quad \inf \mathcal{A} = -\sup(-\mathcal{A})$.

A.3 Boundedness and suprema operations

- Property 1 does not hold for the "product" of two sets, where the "product" of sets $\mathcal{A}$ and $\mathcal{B}$ is defined as as

$$\mathcal{A} \cdot \mathcal{B} := \{c \in \mathbb{R} \colon c = ab \text{ for some } a \in \mathcal{A} \text{ and } b \in \mathcal{B}\}.$$

In this case, both of these two situations can occur:

$$\begin{aligned} \sup(\mathcal{A} \cdot \mathcal{B}) &> (\sup \mathcal{A}) \cdot (\sup \mathcal{B})) \\ \sup(\mathcal{A} \cdot \mathcal{B}) &= (\sup \mathcal{A}) \cdot (\sup \mathcal{B}). \end{aligned}$$

Example. $\mathcal{A} = [-1, 0)$ and $\mathcal{B} = [-1, 0)$. Then

$$\sup(\mathcal{A} \cdot \mathcal{B}) = 1 \text{ and } \sup \mathcal{A} = \sup \mathcal{B} = 0.$$

Example. $\mathcal{A} = [-1, 0)$ and $\mathcal{B} = [0, 1)$. Then

$$\sup(\mathcal{A} \cdot \mathcal{B}) = \sup \mathcal{A} = 0 \text{ and } \sup \mathcal{B} = 1.$$

A.3 Boundedness and suprema operations

Lemma A.17 (Supremum/infimum for monotone functions)

1. If $f\colon \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then

$$\sup\{x \in \mathbb{R} \colon f(x) < \varepsilon\} = \inf\{x \in \mathbb{R} \colon f(x) \geq \varepsilon\}$$

and

$$\sup\{x \in \mathbb{R} \colon f(x) \leq \varepsilon\} = \inf\{x \in \mathbb{R} \colon f(x) > \varepsilon\}.$$

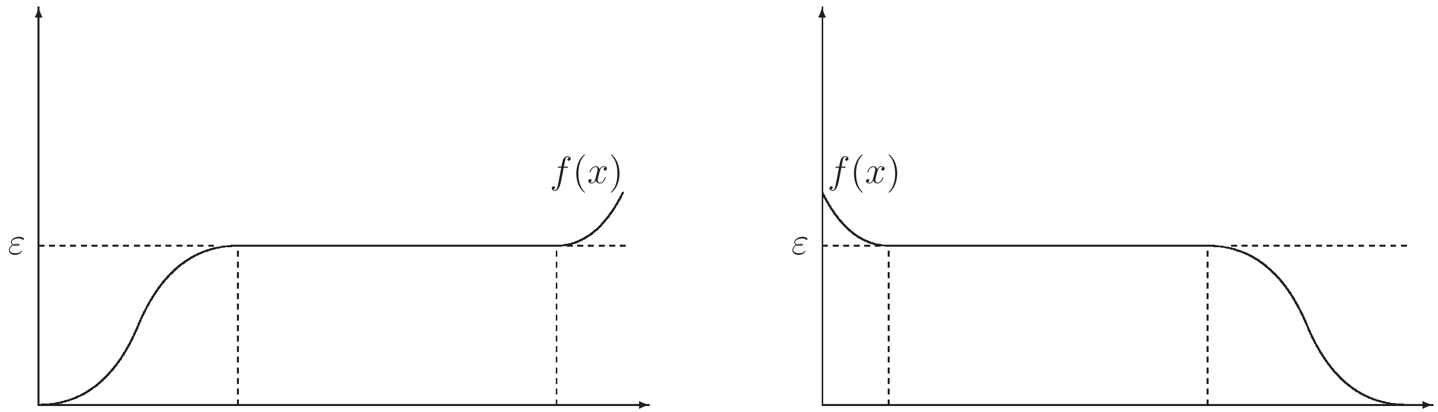
2. If $f\colon \mathbb{R} \to \mathbb{R}$ is a non-increasing function, then

$$\sup\{x \in \mathbb{R} \colon f(x) > \varepsilon\} = \inf\{x \in \mathbb{R} \colon f(x) \leq \varepsilon\}$$

and

$$\sup\{x \in \mathbb{R} \colon f(x) \geq \varepsilon\} = \inf\{x \in \mathbb{R} \colon f(x) < \varepsilon\}.$$

Illustration of Lemma A.17



$\sup\{x : f(x) < \varepsilon\}$
$= \inf\{x : f(x) \geq \varepsilon\}$

$\sup\{x : f(x) \leq \varepsilon\}$
$= \inf\{x : f(x) > \varepsilon\}$

$\sup\{x : f(x) > \varepsilon\}$
$= \inf\{x : f(x) \leq \varepsilon\}$

$\sup\{x : f(x) \geq \varepsilon\}$
$= \inf\{x : f(x) < \varepsilon\}$

A.4 Sequences and their limits

- Let $\mathbb{N}$ denote the set of "natural numbers" (positive integers) $1, 2, 3, \cdots$.

- A sequence drawn from a real-valued function is denoted by

$$f : \mathbb{N} \to \mathbb{R}.$$

  In other words, $f(n)$ is a real number for each $n = 1, 2, 3, \ldots$.

- It is usual to write $f(n) = a_n$, and we often indicate the sequence by any one of these notations

$$\{a_1, a_2, a_3, \cdots, a_n, \cdots\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

- One important question that arises with a sequence is what happens when n gets large. To be precise, we want to know that when n is large enough, whether or not every a_n is close to some fixed number L (which is the *limit* of a_n).

A.4 Sequences and their limits

Definition A.18 (Limit) The *limit* of $\{a_n\}_{n=1}^{\infty}$ is the real number L satisfying: $(\forall\ \varepsilon > 0)(\exists\ N)$ such that $(\forall\ n > N)$

$$|a_n - L| < \varepsilon.$$

In this case, we write $L = \lim_{n\to\infty} a_n$. If no such L satisfies the above statement, we say that the limit of $\{a_n\}_{n=1}^{\infty}$ does not exist.

Property A.19 If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both have a limit in $\mathbb{R}$, then the following hold.

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}(\alpha \cdot a_n) = \alpha \cdot \lim_{n\to\infty} a_n$.
3. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$.

A.4 Sequences and their limits

- Note that in the above definition, $-\infty$ and ∞ cannot be a legitimate limit for any sequence.
- In fact, if $(\forall\ L)(\exists\ N)$ such that $(\forall\ n > N)\ a_n > L$, then we say that a_n *diverges* to ∞ and write $a_n \to \infty$. A similar argument applies to a_n diverging to $-\infty$.
- For convenience, we will work in the set of extended real numbers and thus state that a sequence $\{a_n\}_{n=1}^{\infty}$ that diverges to either ∞ or $-\infty$ has a limit in $\mathbb{R} \cup \{-\infty, \infty\}$.

Lemma A.20 (Convergence of monotone sequences) If $\{a_n\}_{n=1}^{\infty}$ is non-decreasing in n, then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}\cup\{-\infty,\infty\}$. If $\{a_n\}_{n=1}^{\infty}$ is also bounded from above – i.e., $a_n \leq L\ \forall n$ for some L in $\mathbb{R}$ – then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$. Likewise, if $\{a_n\}_{n=1}^{\infty}$ is non-increasing in n, then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}\cup\{-\infty,\infty\}$. If $\{a_n\}_{n=1}^{\infty}$ is also bounded from below – i.e., $a_n \geq L\ \forall n$ for some L in $\mathbb{R}$ – then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$.

A.4 Sequences and their limits

- The limit of a sequence may not exist.

  Example. $a_n = (-1)^n$.

  Then a_n will be close to either -1 or 1 for n large.

- Hence, more generalized definitions that can describe the general limiting behavior of a sequence is required.

Definition A.21 (limsup and liminf) The *limit supremum* of $\{a_n\}_{n=1}^{\infty}$ is the extended real number in $\mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \left(\sup_{k\geq n} a_k\right),$$

and the *limit infimum* of $\{a_n\}_{n=1}^{\infty}$ is the extended real number defined by

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \left(\inf_{k\geq n} a_k\right).$$

Some also use the notations $\overline{\lim}$ and $\underline{\lim}$ to denote limsup and liminf, respectively.

- Note that the limit supremum and the limit infimum of a sequence is always defined in $\mathbb{R} \cup \{-\infty, \infty\}$, since the sequences $\sup_{k\geq n} a_k = \sup\{a_k \colon k \geq n\}$ and $\inf_{k\geq n} a_k = \inf\{a_k \colon k \geq n\}$ are monotone in n (cf. Lemma A.20).

A.4 Sequences and their limits

Lemma A.22 (Limit) For a sequence $\{a_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} a_n = L \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L.$$

A.4 Sequences and their limits

Property A.23 (Properties of the limit supremum)

1. The limit supremum always exists in $\mathbb{R} \cup \{-\infty, \infty\}$.
2. If $|\limsup_{m\to\infty} a_m| < \infty$, then $(\forall\ \varepsilon > 0)(\exists\ N)$ such that $(\forall\ n > N)\ a_n < \limsup_{m\to\infty} a_m + \varepsilon$. (Note that this holds for every $n > N$.)
3. If $|\limsup_{m\to\infty} a_m| < \infty$, then $(\forall\ \varepsilon > 0$ and integer $K)(\exists\ N > K)$ such that $a_N > \limsup_{m\to\infty} a_m - \varepsilon$. (Note that this holds *only* for *one* N, which is larger than K.)

Property A.24 (Properties of the limit infimum)

1. The limit infimum always exists in $\mathbb{R} \cup \{-\infty, \infty\}$.
2. If $|\liminf_{m\to\infty} a_m| < \infty$, then $(\forall\ \varepsilon > 0$ and $K)(\exists\ N > K)$ such that $a_N < \liminf_{m\to\infty} a_m + \varepsilon$. (Note that this holds *only* for *one* N, which is larger than K.)
3. If $|\liminf_{m\to\infty} a_m| < \infty$, then $(\forall\ \varepsilon > 0)(\exists\ N)$ such that $(\forall\ n > N)\ a_n > \liminf_{m\to\infty} a_m - \varepsilon$. (Note that this holds for every $n > N$.)

A.4 Sequences and their limits

Definition A.25 (Sufficiently large) We say that a property holds for a sequence $\{a_n\}_{n=1}^{\infty}$ *almost always* or for *all sufficiently large* n if the property holds for every $n > N$ for some N.

Definition A.26 (Infinitely often) We say that a property holds for a sequence $\{a_n\}_{n=1}^{\infty}$ *infinitely often* or for *infinitely many* n if for every K, the property holds for *one* (specific) N with $N > K$.

- Then Properties 2 and 3 of Property A.23 can be respectively re-phrased as: if $|\limsup_{m\to\infty} a_m| < \infty$, then $(\forall\ \varepsilon > 0)$

$$a_n < \limsup_{m\to\infty} a_m + \varepsilon \quad \text{for all sufficiently large } n$$

and

$$a_n > \limsup_{m\to\infty} a_m - \varepsilon \quad \text{for infinitely many } n.$$

- Similarly, Properties 2 and 3 of Property A.24 becomes: if $|\liminf_{m\to\infty} a_m| < \infty$, then $(\forall\ \varepsilon > 0)$

$$a_n < \liminf_{m\to\infty} a_m + \varepsilon \quad \text{for infinitely many } n$$

and

$$a_n > \liminf_{m\to\infty} a_m - \varepsilon \quad \text{for all sufficiently large } n.$$

A.4 Sequences and their limits

Lemma A.27

1. $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$.

2. If $a_n \le b_n$ for all sufficiently large n, then

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \quad \text{and} \quad \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n.$$

3. $\limsup_{n\to\infty} a_n < r \;\Rightarrow\; a_n < r$ for all sufficiently large n.

4. $\limsup_{n\to\infty} a_n > r \;\Rightarrow\; a_n > r$ for infinitely many n.

5.
$$\begin{aligned}\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n &\le \liminf_{n\to\infty}(a_n + b_n)\\ &\le \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n\\ &\le \limsup_{n\to\infty}(a_n + b_n)\\ &\le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

6. If $\lim_{n\to\infty} a_n$ exists, then

$$\liminf_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

and

$$\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Concept behind limsup and liminf

- Limsup = largest clustering point
- Liminf = smallest clustering point
- A clustering point is a point that the sequence a_n hits close for infinitely many times.

  E.g., $a_n = \sin(n\pi/2)$

  $\Rightarrow \{a_n\}_{n\geq 1} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

  There are three clustering points in this sequence, which are -1, 0 and 1.

  Consequently,

$$\begin{aligned}\limsup_{n\to\infty} a_n &= 1 = \text{the largest clustering point}\\ \liminf_{n\to\infty} a_n &= -1 = \text{the smallest clustering pint}\end{aligned}$$

  E.g., $a_n = -n$. Then $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

  E.g., $a_n = n$. Then $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty$.

A.5 Equivalence

- We close this appendix by providing some equivalent statements that are often used to simplify proofs.
- For example, instead of directly showing that quantity x is less than or equal to quantity y, one can take an arbitrary constant $\varepsilon > 0$ and prove that $x < y + \varepsilon$.
- Since $y + \varepsilon$ is a larger quantity than y, in some cases it might be easier to show $x < y + \varepsilon$ than proving $x \leq y$.

Theorem A.28 For any x, y and a in $\mathbb{R}$,

1. $x < y + \varepsilon$ for all $\varepsilon > 0$ iff $x \leq y$;
2. $x < y - \varepsilon$ for some $\varepsilon > 0$ iff $x < y$;
3. $x > y - \varepsilon$ for all $\varepsilon > 0$ iff $x \geq y$;
4. $x > y + \varepsilon$ for some $\varepsilon > 0$ iff $x > y$;
5. $|a| < \varepsilon$ for all $\varepsilon > 0$ iff $a = 0$.

Key Notes

- Supremum and Infimum over a subset of real line
- Limsup and Liminf (and their properties)
- Sufficiently large and infinitely often
- Equivalence