Introduction to Finite Element Method

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Dec. 20, 2016

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Chapter 13 Finite Element Formulation

- 13.1 FE Formulation of One Element
- 13.2 Axially Loaded Elastic Bar

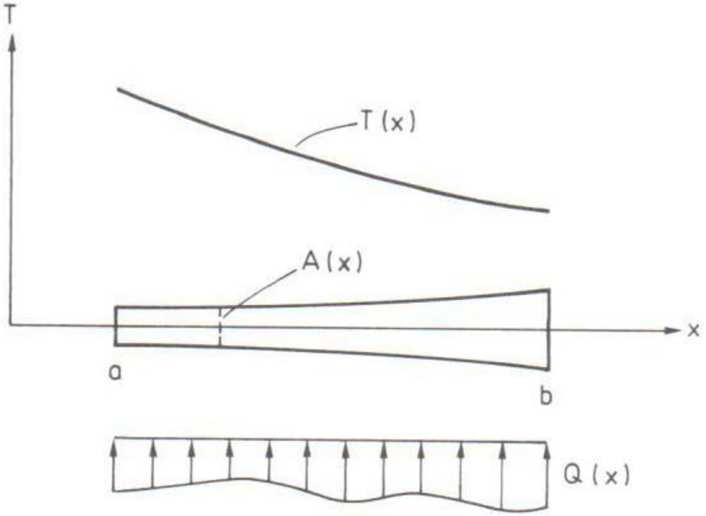
Chapter 13 Finite Element Formulation

- Basic steps in the FE formulation
  1. Establish the strong formulation of the problem
  2. Obtain the weak form of the problem
  3. The Galerkin method
  4. Make an elementwise approximation over the entire body of the unknown function

13.1 FE Formulation of One Element

- Consider the one-dimensional heat flow in the fin
  - The differential equation of heat conduction

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q=0, \qquad a\le x\le b$$



  - k: thermal conductivity
  - Q: heat supply per unit time and per unit length

13.1 FE Formulation of One Element

- Weighted integral

$$\int_a^b v\left[\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q\right]dx=0$$

- Integration by parts

$$\left[vAk\frac{dT}{dx}\right]_a^b-\int_a^b\frac{dv}{dx}Ak\frac{dT}{dx}dx+\int_a^b vQdx=0$$

$$\int_a^b\frac{dv}{dx}Ak\frac{dT}{dx}dx-\left[vAk\frac{dT}{dx}\right]_a^b-\int_a^b vQdx=0$$

  – which implies that the order of differentiation for T has decreased at the expense of the weight function v being differentiated

- Fourier's law: flux $q=-k\frac{dT}{dx}$

$$\int_a^b\frac{dv}{dx}Ak\frac{dT}{dx}dx+\left[vAq\right]_a^b-\int_a^b vQdx=0$$

13.1 FE Formulation of One Element

- Approximation over an element
  - $T = \boldsymbol{N}^e \boldsymbol{a}^e$
    - $\boldsymbol{N}^e$: element shape function matrix

      $\boldsymbol{N}^e = [N_1{}^e(x) \quad N_2{}^e(x) \quad \dots \quad N_{ND}{}^e(x)]$
    - $\boldsymbol{a}^e$: nodal temperature in an element

      $\boldsymbol{a}^e = [T_1{}^e \quad T_2{}^e \quad \dots \quad T_{ND}{}^e]^T$
      - ND: number of nodes in an element
  - $\frac{dT}{dx} = \frac{d\boldsymbol{N}^e}{dx}\boldsymbol{a}^e = \boldsymbol{B}^e\boldsymbol{a}^e, \boldsymbol{B}^e = \left[\frac{dN_1{}^e}{dx} \quad \frac{dN_2{}^e}{dx} \quad \dots \quad \frac{dN_{ND}{}^e}{dx}\right]$
  - Thus, $\int_{L_\alpha} \frac{dv}{dx} Ak\mathbf{B}^e dx\mathbf{a}^e + \left[vAq\right]_{L_\alpha} - \int_{L_\alpha} vQdx = 0$
    - where L_α is the region of the element α

13.1 FE Formulation of One Element

- The Galerkin method
  - weight function $v = \boldsymbol{N}^e \boldsymbol{c}^e$
    - Since v is arbitrary, $\boldsymbol{c}^e$ is arbitrary
    - The numbers of terms in v and T are the same
  - $v = \boldsymbol{N}^e \boldsymbol{c}^e = (\boldsymbol{N}^e \boldsymbol{c}^e)^T = \boldsymbol{c}^{e^T} \boldsymbol{N}^{e^T}$
  - $\frac{dv}{dx} = \boldsymbol{c}^{e^T} \frac{d\boldsymbol{N}^{e^T}}{dx} = \boldsymbol{c}^{e^T} \boldsymbol{B}^{e^T}$
  - Thus, $\mathbf{c}^{e\mathrm{T}} \left[\left(\int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx \right) \mathbf{a}^e + \left[Aq\mathbf{N}^{e\mathrm{T}} \right]_{L_\alpha} - \int_{L_\alpha} Q\mathbf{N}^{e\mathrm{T}} dx \right] = 0$
    - which holds for arbitrary $\boldsymbol{c}^{e^T}$

$$\left(\int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx \right) \mathbf{a}^e + \left[Aq\mathbf{N}^{e\mathrm{T}} \right]_{L_\alpha} - \int_{L_\alpha} Q\mathbf{N}^{e\mathrm{T}} dx = \mathbf{0}$$

13.1 FE Formulation of One Element

- Element stiffness equation

$$\left(\int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^{e}dx\right)\mathbf{a}^{e} = -\left[AqN^{e\mathrm{T}}\right]_{L_\alpha} + \int_{L_\alpha} Q\mathbf{N}^{e\mathrm{T}}dx$$

  - $\boldsymbol{k}\boldsymbol{a}^e = \boldsymbol{f}^e$
    - The equation contains only those degrees of freedom which belong to the element considered

$$\mathbf{k} = \int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^{e}dx$$

$$\mathbf{f}_q^e = -\left[Aq\mathbf{N}^{e\mathrm{T}}\right]_{L_\alpha},\ \mathbf{f}_Q^e = \int_{L_\alpha} Q\mathbf{N}^{e\mathrm{T}}dx$$

  - $\boldsymbol{k}$: element stiffness matrix for element α
  - ${\boldsymbol{f}_q}^e$: element boundary vector for element α
  - ${\boldsymbol{f}_Q}^e$: element load vector for element α
  - $\boldsymbol{f}^e = {\boldsymbol{f}_q}^e + {\boldsymbol{f}_Q}^e$: element force vector for element α

13.1 FE Formulation of One Element

- Property of $\boldsymbol{k}$

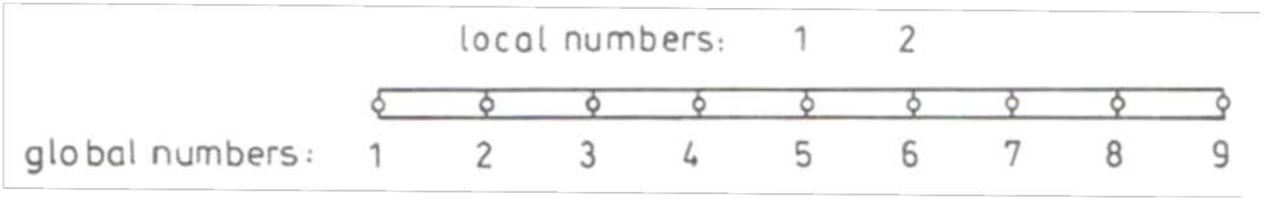
$$\mathbf{k}^{\mathrm{T}} = \left(\int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^{e} dx \right)^{\mathrm{T}} = \int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^{e} dx = \mathbf{k}$$

  - $\boldsymbol{k}$ is symmetric as a consequence of the use of the Galerkin method
  - A component k_{ij} of $\boldsymbol{k}$: $k_{ij} = \int_{L_\alpha} Ak \frac{dN_i}{dx} \frac{dN_j}{dx} dx$
    - For an element with 2 nodes,

$$\mathbf{k} = \begin{bmatrix} \int_{L_\alpha} Ak \frac{dN_1}{dx}\frac{dN_1}{dx} dx & \int_{L_\alpha} Ak \frac{dN_1}{dx}\frac{dN_2}{dx} dx \\ \int_{L_\alpha} Ak \frac{dN_2}{dx}\frac{dN_1}{dx} dx & \int_{L_\alpha} Ak \frac{dN_2}{dx}\frac{dN_2}{dx} dx \end{bmatrix}$$

13.1 FE Formulation of One Element

- Assembly
  – Consider a problem discretized by eight 2-node elements



  - The local numbering relates to the FE equation

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} f_{q1}^{e} \\ f_{q2}^{e} \end{bmatrix} + \begin{bmatrix} f_{Q1}^{e} \\ f_{Q2}^{e} \end{bmatrix}$$

$$\begin{array}{c} \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 5 & 6 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \end{array} \Rightarrow \begin{array}{ll} k_{11} & \text{is added to } K_{55} \\ k_{12} & \text{is added to } K_{56} \\ k_{21} & \text{is added to } K_{65} \\ k_{22} & \text{is added to } K_{66} \end{array}$$

$$\begin{array}{c} 5 \\ 6 \end{array} \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix} \Rightarrow \begin{array}{ll} f_1^e & \text{is added to } f_1 \\ f_2^e & \text{is added to } f_2 \end{array}$$

13.1 FE Formulation of One Element

- To construct an efficient FE computer program, use the aforementioned “Element Formulation”
  - $\boldsymbol{k}$
  - $\boldsymbol{f}_Q{}^e$
  - $\boldsymbol{f}_q$

$$\mathbf{k} = \int_{L_\alpha} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx$$

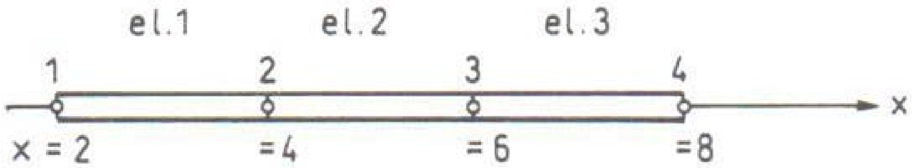
$$\mathbf{f}_Q^e = \int_{L_\alpha} Q\mathbf{N}^{e\mathrm{T}} dx$$

element boundary vector:

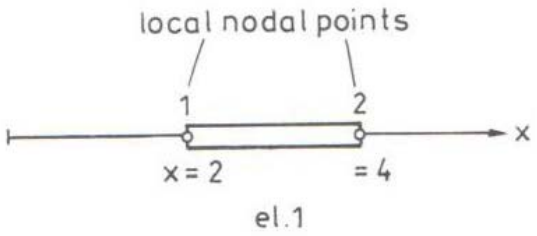
$$\mathbf{f}_q = -\left[Aq\mathbf{N}^\mathrm{T} \right]_a^b = \begin{bmatrix} (Aq)_{x=a} \\ 0 \\ \vdots \\ 0 \\ -(Aq)_{x=b} \end{bmatrix}$$

Example

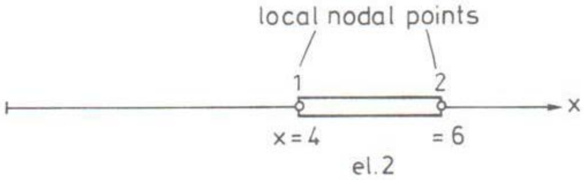
- Consider the following global nodal points and element numbering

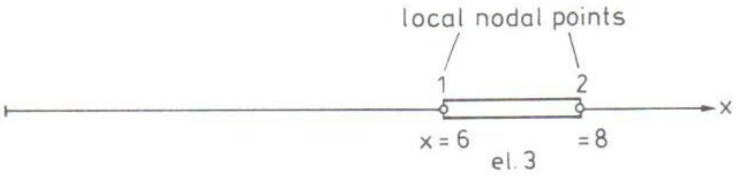


  - In element 1



  - In elements 2 and 3





Example

– In an arbitrary element with nodal coordinates x_1 and x_2

- $\boldsymbol{N}^e = \frac{1}{L^e}[-x + x_2 \quad x - x_1]$

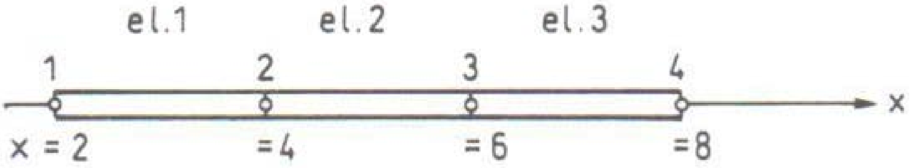
$$\mathbf{k} = \int_{x_1}^{x_2} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx = \int_{x_1}^{x_2} Ak \begin{bmatrix} -1/L^e \\ 1/L^e \end{bmatrix} \begin{bmatrix} -1/L^e & 1/L^e \end{bmatrix} dx = \frac{Ak}{L^{e2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underbrace{(x_2 - x_1)}_{L_\alpha}$$

$$\mathbf{f}_Q^e = \int_{x_1}^{x_2} Q\mathbf{N}^{e\mathrm{T}} dx = \frac{Q}{L^e} \int_{x_1}^{x_2} \begin{bmatrix} -x + x_2 \\ x - x_1 \end{bmatrix} dx = \frac{Q}{L^e} \left. \begin{bmatrix} -\frac{x^2}{2} + x_2 x \\ \frac{x^2}{2} - x_1 x \end{bmatrix} \right|_{x_1}^{x_2} = \frac{Q}{L^e} \begin{bmatrix} -\frac{(x_2^2 - x_1^2)}{2} + x_2 (x_2 - x_1) \\ \frac{(x_2^2 - x_1^2)}{2} - x_1 (x_2 - x_1) \end{bmatrix}$$

$$\Rightarrow \mathbf{k} = \frac{Ak}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{f}_Q^e = Q \begin{bmatrix} -\frac{x_2 + x_1}{2} + x_2 \\ \frac{x_2 + x_1}{2} - x_1 \end{bmatrix}$$

Example

– Stiffness matrix and force vector



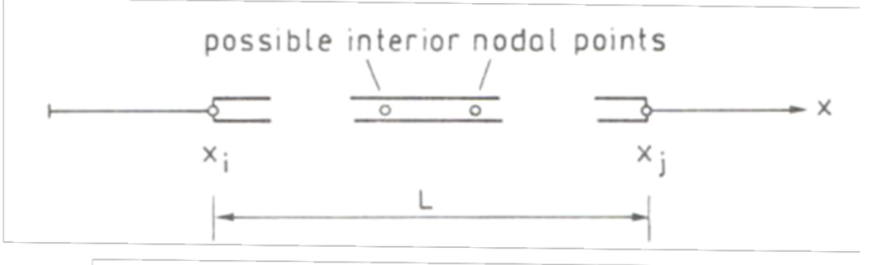
$$\mathbf{k}^1 = \mathbf{k}^2 = \mathbf{k}^3 = \begin{bmatrix} 25 & -25 \\ -25 & 25 \end{bmatrix}$$

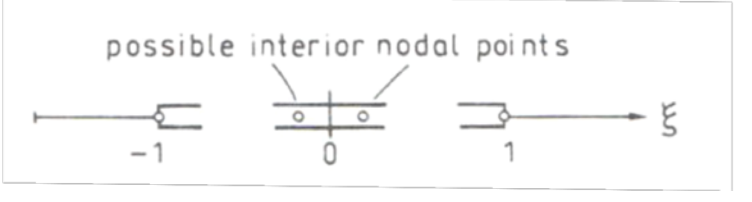
$$\mathbf{f}_Q^1 = \mathbf{f}_Q^2 = \mathbf{f}_Q^3 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 25 & -25 & 0 & 0 \\ -25 & 50 & -25 & 0 \\ 0 & -25 & 50 & -25 \\ 0 & 0 & -25 & 25 \end{bmatrix}, \quad \mathbf{f}_Q = \begin{bmatrix} 100 \\ 200 \\ 200 \\ 100 \end{bmatrix}, \quad \mathbf{f}_q = \begin{bmatrix} \left(Aq\right)_{x=a} \\ 0 \\ 0 \\ -\left(Aq\right)_{x=b} \end{bmatrix}$$

13.1 FE Formulation of One Element

- To standardize the determination of the element stiffness matrix $\boldsymbol{k}$ and the element load vector ${\boldsymbol{f}_Q}^e$
  - Local (Natural) Coordinate Systems
  - Consider a linear transformation
    - $x = c\xi + d$
      - c, d: parameters
      - ξ: new variable
    - $\begin{cases} x_i = -c + d \\ x_j = \quad c + d \end{cases}$
    - $c = \frac{1}{2}(x_j - x_i),\ d = \frac{1}{2}(x_i + x_j)$
      - element length $L^e = x_j - x_i$
    - $x = \frac{L^e}{2}\xi + \frac{1}{2}(x_i + x_j)$
    - $dx = \frac{L^e}{2}d\xi$





13.1 FE Formulation of One Element

- Consider a simple linear element
  - $\boldsymbol{N}^e = \left[N_i{}^e \quad N_j{}^e \right]$
  - $N_i{}^e(x) = -\frac{1}{L^e}(x - x_j) = -\frac{1}{L^e}\left[\frac{L^e}{2}\xi + \frac{1}{2}(x_i + x_j) - x_j\right] = \frac{1}{2}(1 - \xi)$
  - $N_j{}^e(x) = \frac{1}{L^e}(x - x_i) = \frac{1}{L^e}\left[\frac{L^e}{2}\xi + \frac{1}{2}(x_i + x_j) - x_i\right] = \frac{1}{2}(1 + \xi)$
  - $\boldsymbol{B}^e = \frac{d\boldsymbol{N}^e}{dx} = \frac{d\boldsymbol{N}^e}{d\xi}\frac{d\xi}{dx} = \left[-\frac{1}{2} \quad \frac{1}{2}\right]\frac{2}{L^e} = \left[-\frac{1}{L^e} \quad \frac{1}{L^e}\right]$

$$\mathbf{k} = \int_{x_i}^{x_j} Ak\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx = \int_{-1}^{1} Ak \begin{bmatrix} -\frac{1}{L^e} \\ \frac{1}{L^e} \end{bmatrix} \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix} \frac{L^e}{2} d\xi = \frac{1}{2L^e}\int_{-1}^{1} Ak \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi$$

$$\mathbf{f}_Q^e = \int_{x_i}^{x_j} Q\mathbf{N}^{e\mathrm{T}} dx = \frac{1}{2}\int_{-1}^{1} Q \begin{bmatrix} 1-\xi \\ 1+\xi \end{bmatrix} \frac{L^e}{2} d\xi$$

13.1 FE Formulation of One Element

- For a simple linear element

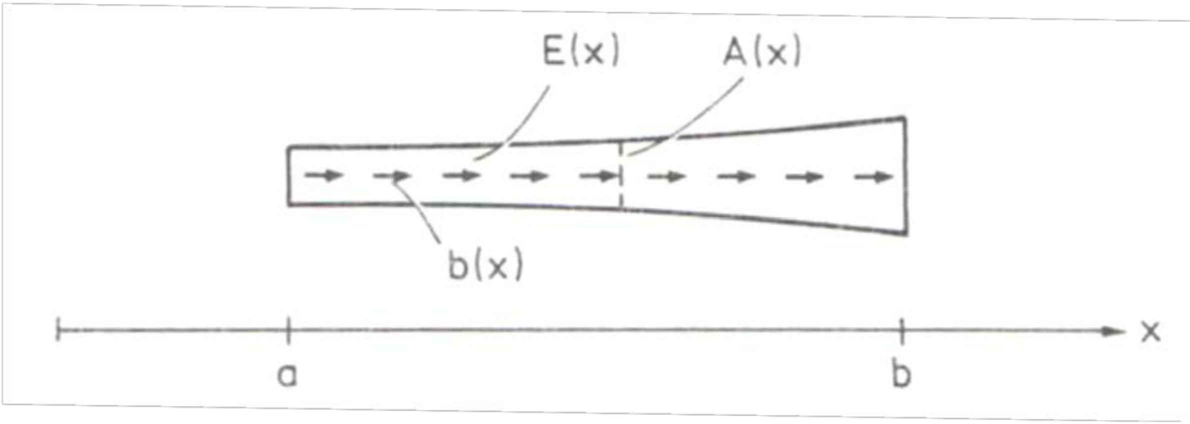
$$Ak = \text{constant} \Rightarrow \mathbf{k} = \frac{Ak}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Q = \text{constant} \Rightarrow \mathbf{f}_Q^e = \frac{QL^e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

  – Similar closed-form solutions may be established for a sequence of typical forms of Ak and Q for higher-order elements

13.2 Axially Loaded Elastic Bar

- Consider an axially loaded elastic bar



  - $E(x)$: Young's modulus
  - $b(x)$: $[N/m]$, body force per unit axial length
  - $A(x)$: the cross-sectional area of the bar
  - Let u be the displacement of the bar measured positive in the x direction

13.2 Axially Loaded Elastic Bar

- Recall the strong form of an axially loaded elastic bar

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\left(x\right)=0, \qquad a\le x\le b$$

- Weak form
  - Weighted integral

$$\int_a^b v\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b\right]dx=0$$

  - Integration by parts

$$\int_a^b \frac{dv}{dx}AE\frac{du}{dx}dx=\left[vAE\frac{du}{dx}\right]_a^b+\int_a^b vbdx=\left[vA\sigma\right]_a^b+\int_a^b vbdx$$

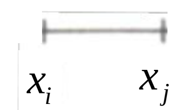
13.2 Axially Loaded Elastic Bar

- FE approximation
  - $u^e = \boldsymbol{N}^e \boldsymbol{a}^e$
  - $v^e = \boldsymbol{N}^e \boldsymbol{c}^e$

$$\int_{L_\alpha} AE\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx\mathbf{a}^e = \left[A\sigma\mathbf{N}^{e\mathrm{T}} \right]_{L_\alpha} + \int_{L_\alpha} b\mathbf{N}^{e\mathrm{T}} dx$$

$$\Rightarrow \mathbf{k}\mathbf{a}^e = \mathbf{f}_q^e + \mathbf{f}_Q^e = \mathbf{f}^e$$

- Define



$$\mathbf{k} = \int_{L_\alpha} AE\mathbf{B}^{e\mathrm{T}}\mathbf{B}^e dx$$

$$\mathbf{f}_q^e = \left[A\sigma\mathbf{N}^{e\mathrm{T}} \right]_{L_\alpha}, \ \mathbf{f}_Q^e = \int_{L_\alpha} b\mathbf{N}^{e\mathrm{T}} dx \text{ where } \mathbf{N}^{e\mathrm{T}} = \begin{bmatrix} -\frac{1}{L^e}\left(x - x_j\right) \\ \frac{1}{L^e}\left(x - x_i\right) \end{bmatrix}$$

13.2 Axially Loaded Elastic Bar

- FE approximation

$$AE = \text{constant}, \quad b = \text{constant}$$

$$\mathbf{k} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k = \frac{AE}{L^e} \ (k \text{ is the stiffness of the bar})$$

$$\mathbf{f}_Q^e = \frac{bL^e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

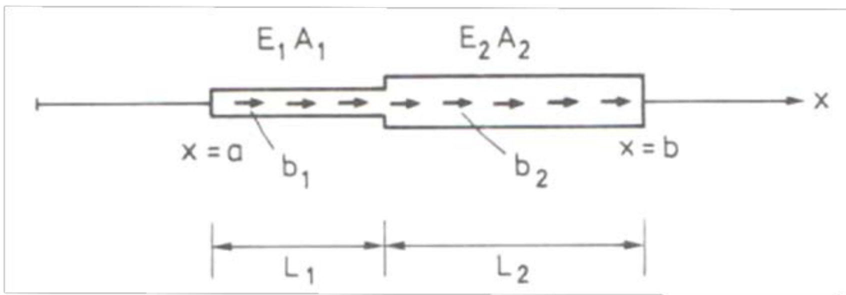
- Stress

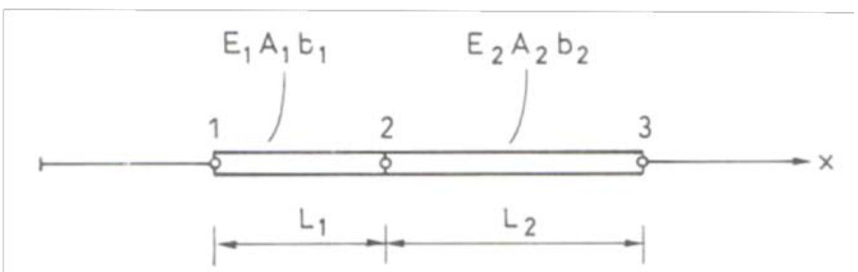
– $\varepsilon^e = \frac{du^e}{dx} = \boldsymbol{B}^e \boldsymbol{a}^e$

$$\sigma^e = E\varepsilon^e = E\mathbf{B}^e\mathbf{a}^e = E\begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix}\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}$$

Example

- Consider an axially loaded bar analyzed by using two simple linear elements





$$\mathbf{k}^1 = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{f}_Q^1 = \frac{b_1 L_1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{k}^2 = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{f}_Q^2 = \frac{b_2 L_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}, \quad \mathbf{f}_Q = \frac{1}{2} \begin{bmatrix} b_1 L_1 \\ b_1 L_1 + b_2 L_2 \\ b_2 L_2 \end{bmatrix}$$

Example

$$\mathbf{f}_q^e = \left[A\sigma \mathbf{N}^{e\mathrm{T}} \right]_{L_\alpha} = \begin{bmatrix} A\sigma N_1^e \\ A\sigma N_2^e \end{bmatrix}_{L_\alpha}$$

- Consider element 1 at local node 1, i.e. $x = x_1 = a$
  - $N_1{}^{e=1}(x = x_1) = 1$

$$f_{q1}^1 = \left(A\sigma N_1\right)_{x=b} - \left(A\sigma N_1\right)_{x=a} = -\left(A\sigma\right)_{x=a}$$

- Consider element 2 at local node 2, i.e. $x = x_2 = b$
  - $N_2{}^{e=2}(x = x_2) = 1$

$$f_{q2}^2 = \left(A\sigma N_2\right)_{x=b} - \left(A\sigma N_2\right)_{x=a} = \left(A\sigma\right)_{x=b}$$

- BCs

$$\mathbf{f}_q = \begin{bmatrix} -\left(A\sigma\right)_{x=a} \\ 0 \\ \left(A\sigma\right)_{x=b} \end{bmatrix}$$

Example

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\left(A\sigma\right)_{x=a} \\ 0 \\ \left(A\sigma\right)_{x=b} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} b_1L_1 \\ b_1L_1+b_2L_2 \\ b_2L_2 \end{bmatrix}$$

- BCs

$$u_1 = u\left(x=a\right) = 0$$

$$\sigma\left(x=b\right) = 0$$

- Thus, $\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\left(A\sigma\right)_{x=a} = F_1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} b_1L_1 \\ b_1L_1+b_2L_2 \\ b_2L_2 \end{bmatrix}$

$$u_2 = \frac{1}{2k_1}\left(b_1L_1 + 2b_2L_2\right),\ u_3 = \frac{1}{2k_1k_2}\left[k_2b_1L_1 + \left(k_1+2k_2\right)b_2L_2\right]$$

$$F_1 = -\left(A\sigma\right)_{x=a} = -\left(b_1L_1 + b_2L_2\right)$$