Introduction to Finite Element Method

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Chapter 12 Weighted Residual Methods

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12.1 Weighted Residual Method

- Consider the one-dimensional differential equation

  $Lu + g = 0, \quad a \le x \le b$

  – L: a differential operator
  – $u(x)$: unknown function
  – $g(x)$: known function
  – If L is chosen, it specifies the actual form of the differential equation

  e.g. $L = \frac{d}{dx} \Rightarrow \frac{du}{dx} + g = 0$ $\qquad$ $L = \frac{d^2}{dx^2} + 1 \Rightarrow \frac{d^2u}{dx^2} + u + g = 0$

- BCs

  $u(a) = u_a, \; u(b) = u_b$

  – u_a, u_b: known quantities

- Only certain problems can be solved analytically in terms of exact closed-form solutions

12.1 Weighted Residual Method

- To seek methods for solving the above differential equation for arbitrary expressions of L and g, consider the following approximation procedure:
  - weighted integral
  - $\int_a^b v(Lu + g)dx = 0$
    - where v is arbitrary
  - This is not a weak formulation unless an integration by parts is performed to reduce the order of differentiation of the unknown function

12.1 Weighted Residual Method

- Assume the approximation fulfilling the BCs
  - $u^{app} = \psi_1 a_1 + \psi_2 a_2 + \cdots + \psi_n a_n$
  - $a_1, a_2, \ldots, a_n$: unknown parameters
  - $\psi_1, \psi_2 \ldots, \psi_n$: trial functions, specified in advance
    - Trial functions are functions of x, may be taken as any approximation and this is why they are termed trial functions
  - Once $a_1, a_2, \ldots, a_n$ are known, the approximate solution is given by u^{app}
    - $u^{app} = \boldsymbol{\psi a}$
    - $\boldsymbol{\psi} = [\psi_1 \quad \psi_2 \quad \ldots \quad \psi_n]$, $\boldsymbol{a} = [a_1 \quad a_2 \quad \ldots \quad a_n]^T$

12.1 Weighted Residual Method

- Recall
  - $\int_a^b v(Lu + g)dx = 0$
- Substitute u by u^{app}
  - $\int_a^b v(Lu^{app} + g)dx = 0$
- u^{app} will not satisfy the equation exactly in general
  - $Lu^{app} + g = e$
    - e: residual, a measure for the error
  - $\int_a^b vedx = 0 \quad (1)$
    - The residual $e(x)$ is given a certain weight $v(x)$
    - The integral of the weighted residual $v(x)e(x)$ over the region of interest is required to be zero

12.1 Weighted Residual Method

- Consider the general form of the weight function
  - $v = V_1 c_1 + V_2 c_2 + \cdots + V_n c_n$
    - $c_1, c_2, \ldots, c_n$: certain parameters
    - $V_1, V_2 \ldots, V_n$: known functions of x, specified in advance
    - The numbers of terms in v and u^{app} are the same
  - $v = \boldsymbol{V}\boldsymbol{c} = (\boldsymbol{V}\boldsymbol{c})^T = \boldsymbol{c}^T\boldsymbol{V}^T \quad (2)$
    - $\boldsymbol{V} = [V_1 \quad V_2 \quad \ldots \quad V_n]$, $\boldsymbol{c} = [c_1 \quad c_2 \quad \ldots \quad c_n]^T$
    - As the weight function v is arbitrary and $\boldsymbol{V}$ is known, $\boldsymbol{c}$ is arbitrary
  - $(2) \Rightarrow (1)$: $\boldsymbol{c}^{\mathrm{T}} \int_a^b \boldsymbol{V}^{\mathrm{T}} e dx = 0$
    - which holds for arbitrary $\boldsymbol{c}^T$
  - $\int_a^b \boldsymbol{V}^{\mathrm{T}} e dx = 0 \quad (3)$

12.1 Weighted Residual Method

- $\int_a^b \boldsymbol{V}^{\mathrm{T}} e\,dx = 0 \quad (3)$
  - The column matrix $\boldsymbol{V}^T$ has the dimension $n \times 1$

$$\int_a^b V_1 e\,dx = 0$$
$$\int_a^b V_2 e\,dx = 0 \qquad (4)$$
$$\vdots$$
$$\int_a^b V_n e\,dx = 0$$

  - The residual e depends on the unknowns a_1, a_2,..., a_n. Thus, eq.(4) serves as a system of equations to determine the n unknowns
    - $e = L(\boldsymbol{\psi a}) + g = L(\boldsymbol{\psi})\boldsymbol{a} + g \quad (5)$
      - where $\boldsymbol{a}$ is independent of x

12.1 Weighted Residual Method

– (5) ⇒ (3): $\int_a^b \mathbf{V}^{\mathrm{T}} L(\boldsymbol{\psi}) dx \mathbf{a} = -\int_a^b \mathbf{V}^{\mathrm{T}} g dx$

– Define

$$\mathbf{K} = \int_a^b \mathbf{V}^{\mathrm{T}} L(\boldsymbol{\psi}) dx$$

$$\mathbf{f} = -\int_a^b \mathbf{V}^{\mathrm{T}} g dx \qquad \Rightarrow \mathbf{K}\mathbf{a} = \mathbf{f}$$

$$\int_a^b \mathbf{V}^{\mathrm{T}} e dx = 0 \quad (3)$$

$$e = L(\boldsymbol{\psi})\mathbf{a} + g \quad (5)$$

$$\mathbf{K} = \begin{bmatrix} \int_a^b V_1 L(\psi_1) dx & \int_a^b V_1 L(\psi_2) dx & \cdots & \int_a^b V_1 L(\psi_n) dx \\ \int_a^b V_2 L(\psi_1) dx & \int_a^b V_2 L(\psi_2) dx & \cdots & \int_a^b V_2 L(\psi_n) dx \\ \vdots & \vdots & & \vdots \\ \int_a^b V_n L(\psi_1) dx & \int_a^b V_n L(\psi_2) dx & \cdots & \int_a^b V_n L(\psi_n) dx \end{bmatrix}, \quad \mathbf{f} = -\begin{bmatrix} \int_a^b V_1 g dx \\ \int_a^b V_2 g dx \\ \vdots \\ \int_a^b V_n g dx \end{bmatrix}$$

– $\boldsymbol{K}$ is a square matrix with dimension $n \times n$

– The system consists of n linear equations from which the n unknowns $a_1, a_2, \ldots, a_n$, i.e. $\boldsymbol{a}$, can be determined

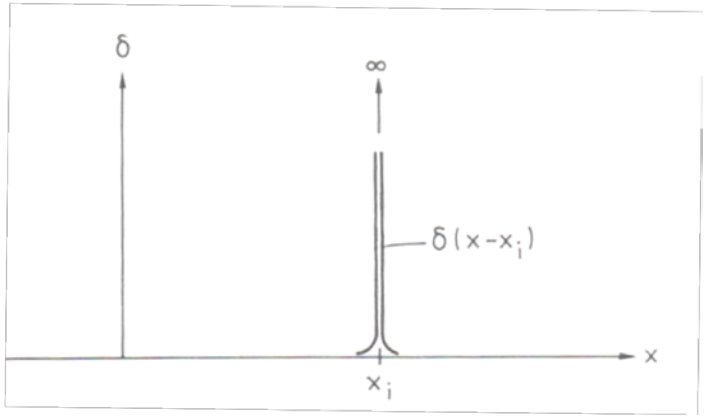
– When $\boldsymbol{a}$ is obtained, $u^{app} = \boldsymbol{\psi a}$ provides the approximate solution

12.1 Weighted Residual Method

- The procedure described above applies to all weighted residual methods
- A variety of different weighted residual methods is obtained depending on the choice of the weight function v, i.e. the choice for $\boldsymbol{V}$

12.2 Point Collocation Method

- Dirac delta function
  - $\delta(x - x_i) = \begin{cases} \infty & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$ (6)
  - $\int_{-\infty}^{\infty} \delta(x - x_i)dx = 1$
    - x_i: a given fixed value
  - Alternatively, $\int_{x_i^-}^{x_i^+} \delta(x - x_i)dx = 1$
    - x_i^+ and x_i^- denote x-values slightly larger than and smaller than x_i



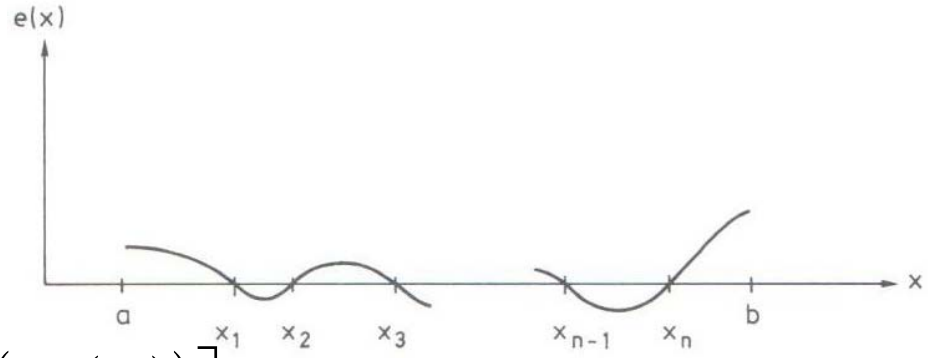
- The weight function v is chosen such that
  - $\boldsymbol{V} = [\delta(x - x_1) \quad \delta(x - x_2) \quad \ldots \quad \delta(x - x_n)]$
  - The fixed points $x_1, x_2, \ldots, x_n$ are collocation points chosen arbitrarily within the region $a \leq x \leq b$
- $\int_a^b V_i e(x)dx = \int_a^b \delta(x - x_i)e(x)dx = 0, \quad i = 1, 2, \ldots, n$

12.2 Point Collocation Method

- As Dirac delta function is zero unless $x = x_i$, we have

$$\int_a^b \delta(x-x_i)e(x)dx = \int_{x_i^-}^{x_i^+} \delta(x-x_i)e(x)dx = e(x_i)\int_{x_i^-}^{x_i^+} \delta(x-x_i)dx = e(x_i) \quad (7)$$

$$\int_a^b V_i e(x)dx = e(x_i) = 0, \quad i = 1,2,\ldots,n$$



- By analogy,

$$\mathbf{K} = \begin{bmatrix} L(\psi_1(x_1)) & L(\psi_2(x_1)) & \cdots & L(\psi_n(x_1)) \\ L(\psi_1(x_2)) & L(\psi_2(x_2)) & \cdots & L(\psi_n(x_2)) \\ \vdots & \vdots & & \vdots \\ L(\psi_1(x_n)) & L(\psi_2(x_n)) & \cdots & L(\psi_n(x_n)) \end{bmatrix}, \quad \mathbf{f} = -\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}$$

  - The differentiation indicated by the operator L, for instance $L(\psi_1(x_2))$, should be understood as follows: the differentiation of the function $\psi_1(x)$ is first carried out and then the value $x = x_2$ is inserted

Example

- Consider the following differential equation

  $\frac{d^2u}{dx^2} + u + x = 0, 0 \leq x \leq 1$

  BCs: $u(0) = u(1) = 0$

  The exact solution is $u = \frac{\sin x}{\sin 1} - x$

- Find the approximation u^{app} to the problem
  - Recall $Lu + g = 0$
  - $L = \frac{d^2}{dx^2} + 1, g = x$

Example

- Express the approximation in the following form:

$$u^{app} = \psi_1 a_1 + \psi_2 a_2 + \cdots + \psi_n a_n$$

- Consider

$$u^{app} = b_0 + (a_1 \sin cx + b_1 \cos cx) + (a_2 \sin 2cx + b_2 \cos 2cx) + \\ \ldots + (a_n \sin ncx + b_n \cos ncx)$$

  - BCs
    - $u^{app}(0) = 0 \Rightarrow b_0 = b_1 = b_2 = \cdots = b_n = 0$
    - $u^{app}(1) = 0 \Rightarrow c = \pi$
  - Thus,

$$u^{app} = a_1 \sin \pi x + a_2 \sin 2\pi x + \cdots + a_n \sin n\pi x$$

Example

- Consider one term in the series as a simple approximation
  - $u^{app} = a_1 \sin \pi x$
  - or $u^{app} = \psi_1 a_1 = \boldsymbol{\psi a}$
  - $\boldsymbol{\psi} = [\psi_1] = [\sin \pi x\ ]$
  - $\boldsymbol{a} = [a_1]$
- Use point collocation method to determine a_1

Example

- Recall
  - $L = \frac{d^2}{dx^2} + 1, g = x$
  - $\psi_1 = \sin \pi x$
  - $\boldsymbol{Ka} = \boldsymbol{f}$

$$\mathbf{K} = \left[L\left(\psi_1\left(x_1\right)\right)\right] = \left[-\pi^2 \sin \pi x_1 + \sin \pi x_1\right] = \left[\left(-\pi^2 + 1\right)\sin \pi x_1\right]$$

$$\mathbf{f} = -\left[g\left(x_1\right)\right] = -\left[x_1\right]$$

  - Choose the collocation point, for instance, the midpoint of the interval $x_1 = \frac{1}{2}$

$$\left(-\pi^2 + 1\right)\sin\frac{\pi}{2} a_1 = -\frac{1}{2} \Rightarrow a_1 = \frac{1}{2\left(\pi^2 - 1\right)} \approx 0.0564$$

12.3 Subdomain Collocation Method

- In the point collocation method, n points are chosen
- In the subdomain collocation method, the region is divided into n subregions
  - Each subregion is given by $x_i \leq x \leq x_{i+1}$, where both x_i and x_{i+1} are located in the region $a \leq x \leq b$
  - $\boldsymbol{V}$ is chosen such that
  - $V_i = \begin{cases} 1 & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad i = 1,2,\ldots,n$

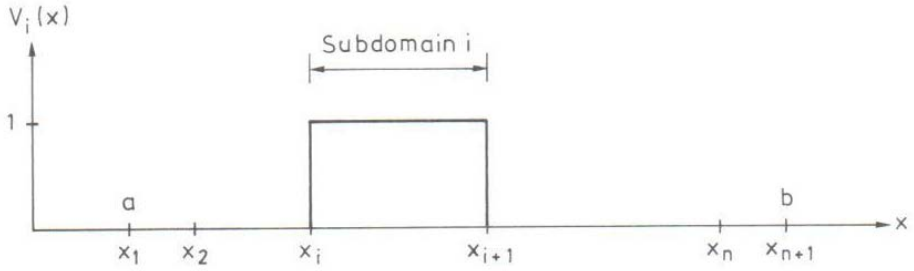
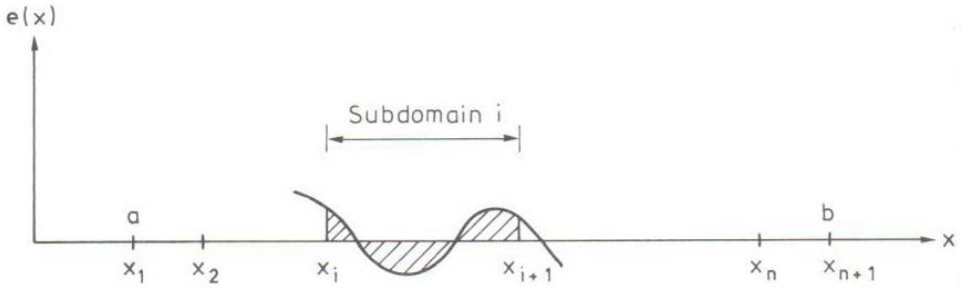


Figure 8.3 Weight function in subdomain collocation method

12.3 Subdomain Collocation Method

$$\int_a^b V_i e(x)\,dx = \int_{x_i}^{x_{i+1}} e(x)\,dx = 0, \quad i = 1, 2, \ldots, n$$



- The average of the residual over each subdomain is forced to be equal to zero

$$\mathbf{K} = \begin{bmatrix} \int_{x_1}^{x_2} L(\psi_1)\,dx & \int_{x_1}^{x_2} L(\psi_2)\,dx & \cdots & \int_{x_1}^{x_2} L(\psi_n)\,dx \\ \int_{x_2}^{x_3} L(\psi_1)\,dx & \int_{x_2}^{x_3} L(\psi_2)\,dx & \cdots & \int_{x_2}^{x_3} L(\psi_n)\,dx \\ \vdots & \vdots & & \vdots \\ \int_{x_n}^{x_{n+1}} L(\psi_1)\,dx & \int_{x_n}^{x_{n+1}} L(\psi_2)\,dx & \cdots & \int_{x_n}^{x_{n+1}} L(\psi_n)\,dx \end{bmatrix}, \quad \mathbf{f} = -\begin{bmatrix} \int_{x_1}^{x_2} g\,dx \\ \int_{x_2}^{x_3} g\,dx \\ \vdots \\ \int_{x_n}^{x_{n+1}} g\,dx \end{bmatrix}$$

Example

- Consider the differential equation $\frac{d^2u}{dx^2} + u + x = 0$, $0 \leq x \leq 1$
- BCs: $u(0) = u(1) = 0$
- Use one term in the series as a simple approximation
  - $u^{app} = a_1 \sin\pi x = \boldsymbol{\psi a}$
  - $\boldsymbol{\psi} = [\psi_1] = [\sin\pi x\,]$, $\boldsymbol{a} = [a_1]$
- Use subdomain collocation method to determine a_1

Example

- In this case, only one subdomain is involved and this is chosen as the entire region of interest
  - $L = \frac{d^2}{dx^2} + 1, g = x$
  - $\psi_1 = \sin\pi x$
  - $\boldsymbol{Ka} = \boldsymbol{f}$

$$\mathbf{K} = \left[\int_0^1 L(\psi_1)dx\right] = \left[\int_0^1 \left(-\pi^2 + 1\right)\sin \pi x dx\right] = \left[\frac{2}{\pi}\left(-\pi^2 + 1\right)\right]$$

$$\mathbf{f} = -\left[\int_0^1 g dx\right] = -\left[\int_0^1 x dx\right] = -\left[\frac{1}{2}\right]$$

$$\frac{2}{\pi}\left(-\pi^2 + 1\right)a_1 = -\frac{1}{2} \Rightarrow a_1 = \frac{\pi}{4\left(\pi^2 - 1\right)} \approx 0.0885$$

12.4 Least-Squares Method

- Recall the residual $e = L(\boldsymbol{\psi})\boldsymbol{a} + g$ (5) $\Rightarrow e = e(x, a_1, a_2, \ldots, a_n)$
- In the least-squares method, $\boldsymbol{V}$ is chosen such that

$$V_i = \frac{\partial e}{\partial a_i}, \quad i = 1, 2, \ldots, n \quad (8)$$

$$\int_a^b V_i e(x)\, dx = \int_a^b \frac{\partial e}{\partial a_i} e(x)\, dx = 0, \quad i = 1, 2, \ldots, n \quad (9)$$

- To evaluate this choice, consider $I = \int_a^b e^2(x, a_1, a_2, \ldots, a_n)dx$
- As the integration is carried out over x, $I = I(a_1, a_2, \ldots, a_n)$

$$\frac{\partial I}{\partial a_i} = 2\int_a^b e \frac{\partial e}{\partial a_i} dx = 0, \quad i = 1, 2, \ldots, n$$

  - The weight function in eq.(8) implies that I is stationary. The stationary of I is a minimum.
  - Consequently, the square of the error is a minimum, which gives the terminology of the least-squares method

12.4 Least-Squares Method

- $e = L(\boldsymbol{\psi})\boldsymbol{a} + g$ (5)

  $V_i = \dfrac{\partial e}{\partial a_i}, \quad i = 1, 2, \ldots, n \quad (8)$

  – Insert eq.(5) into eq.(8) yields $V_i = L(\psi_i), \quad i = 1, 2, \ldots, n$

$$\mathbf{K} = \begin{bmatrix} \int_a^b L(\psi_1)L(\psi_1)dx & \int_a^b L(\psi_1)L(\psi_2)dx & \cdots & \int_a^b L(\psi_1)L(\psi_n)dx \\ \int_a^b L(\psi_2)L(\psi_1)dx & \int_a^b L(\psi_2)L(\psi_2)dx & \cdots & \int_a^b L(\psi_2)L(\psi_n)dx \\ \vdots & \vdots & & \vdots \\ \int_a^b L(\psi_n)L(\psi_1)dx & \int_a^b L(\psi_n)L(\psi_2)dx & \cdots & \int_a^b L(\psi_n)L(\psi_n)dx \end{bmatrix}, \mathbf{f} = -\begin{bmatrix} \int_a^b L(\psi_1)gdx \\ \int_a^b L(\psi_2)gdx \\ \vdots \\ \int_a^b L(\psi_n)gdx \end{bmatrix}$$

  – The coefficient matrix $\boldsymbol{K}$ is symmetric

Example

- Consider the differential equation $\frac{d^2u}{dx^2} + u + x = 0$, $0 \leq x \leq 1$
- BCs: $u(0) = u(1) = 0$
- Use one term in the series as a simple approximation
  - $u^{app} = a_1 \sin\pi x = \boldsymbol{\psi a}$
  - $\boldsymbol{\psi} = [\psi_1] = [\sin\pi x\,]$, $\boldsymbol{a} = [a_1]$
- Use least-squares method to determine a_1

Example

- Recall
  - $L = \frac{d^2}{dx^2} + 1, g = x, \psi_1 = \sin \pi x$
  - $\boldsymbol{Ka} = \boldsymbol{f}$

$$\mathbf{K} = \left[\int_0^1 L(\psi_1) L(\psi_1) dx \right] = \left[\int_0^1 \left(-\pi^2 + 1\right)^2 \sin^2(\pi x) dx \right] = \left[\frac{1}{2} \left(-\pi^2 + 1\right)^2 \right]$$

$$\mathbf{f} = -\left[\int_0^1 L(\psi_1) g dx \right] = -\left[\int_0^1 \left(-\pi^2 + 1\right) x \sin \pi x dx \right] = -\left[\frac{1}{\pi} \left(-\pi^2 + 1\right) \right]$$

  - where $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin 2x$
  - Thus, $\int x \sin x dx = \sin x - x \cos x$

$$\frac{1}{2}\left(-\pi^2 + 1\right)^2 a_1 = -\frac{1}{\pi}\left(-\pi^2 + 1\right) \Rightarrow a_1 = \frac{2}{\pi\left(\pi^2 - 1\right)} \approx 0.0718$$

12.5 The Galerkin Method

- $\boldsymbol{V}$ is chosen as $V_i = \psi_i, \quad i = 1, 2, \ldots, n$
  - i.e., weight (test) functions = trial functions

$$\mathbf{K} = \begin{bmatrix} \int_a^b \psi_1 L(\psi_1)dx & \int_a^b \psi_1 L(\psi_2)dx & \cdots & \int_a^b \psi_1 L(\psi_n)dx \\ \int_a^b \psi_2 L(\psi_1)dx & \int_a^b \psi_2 L(\psi_2)dx & \cdots & \int_a^b \psi_2 L(\psi_n)dx \\ \vdots & \vdots & & \vdots \\ \int_a^b \psi_n L(\psi_1)dx & \int_a^b \psi_n L(\psi_2)dx & \cdots & \int_a^b \psi_n L(\psi_n)dx \end{bmatrix}, \quad \mathbf{f} = -\begin{bmatrix} \int_a^b \psi_1 g dx \\ \int_a^b \psi_2 g dx \\ \vdots \\ \int_a^b \psi_n g dx \end{bmatrix}$$

  - In general, the coefficient matrix $\boldsymbol{K}$ is not symmetric

Example

- Consider the differential equation $\frac{d^2u}{dx^2} + u + x = 0$, $0 \leq x \leq 1$
- BCs: $u(0) = u(1) = 0$
- Use one term in the series as a simple approximation
  - $u^{app} = a_1 \sin\pi x = \boldsymbol{\psi a}$
  - $\boldsymbol{\psi} = [\psi_1] = [\sin\pi x\,]$, $\boldsymbol{a} = [a_1]$
- Use Galerkin method to determine a_1

Example

- Recall
  - $L = \frac{d^2}{dx^2} + 1, g = x, \psi_1 = \sin\pi x$
  - $\boldsymbol{Ka} = \boldsymbol{f}$

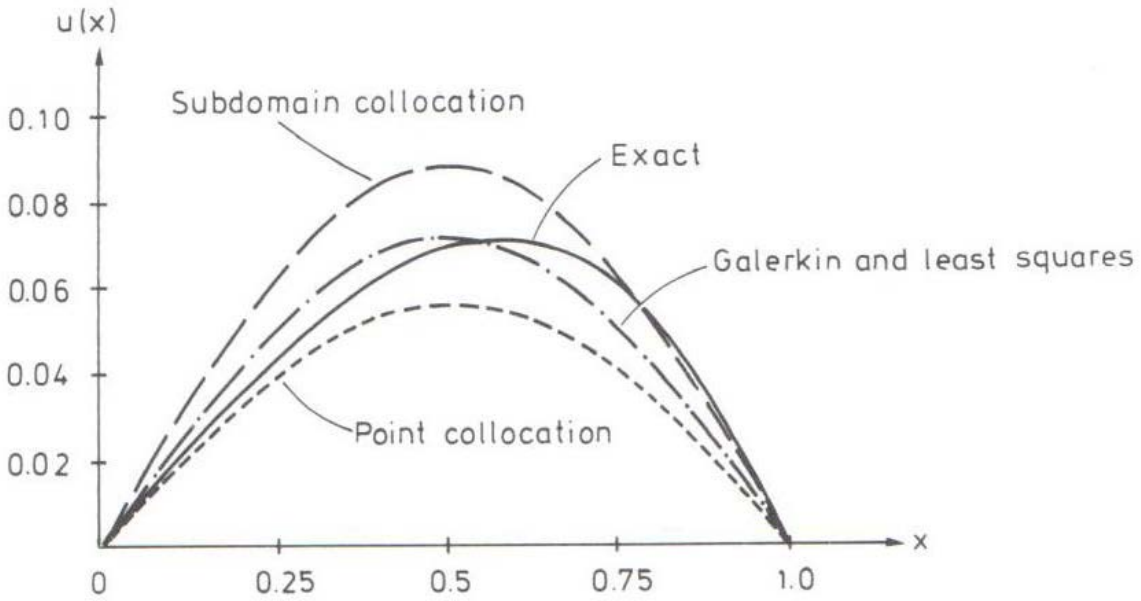
$$\mathbf{K} = \left[\int_0^1 \psi_1 L(\psi_1) dx\right] = \left[\int_0^1 \left(-\pi^2 + 1\right)\sin^2(\pi x) dx\right] = \frac{1}{2}\left(-\pi^2 + 1\right)$$

$$\mathbf{f} = -\left[\int_0^1 \psi_1 g dx\right] = -\left[\int_0^1 x \sin \pi x dx\right] = -\frac{1}{\pi}$$

$$\frac{1}{2}\left(-\pi^2 + 1\right)a_1 = -\frac{1}{\pi} \Rightarrow a_1 = \frac{2}{\pi\left(\pi^2 - 1\right)} \approx 0.0718$$

Example

- Comparison of different weighted residual methods
  - In this example, the Galerkin and least-squares methods provide the same results, and the superiority of the two methods is obvious for the example considered
  - The subdomain method is more accurate than the point collocation method



Example

- Comparison of different weighted residual methods
  - The least-squares method and the Galerkin method turn out to be very efficient
  - The least-squares method always results in a symmetric coefficient matrix, which is an advantage in numerical calculations
  - When the Galerkin method is used in combination with the weak formulation (in which an integration by parts is performed), a symmetric coefficient matrix arises