Sample Problems for Quiz 7

1. For a time-varying multipath fading channel, the input-output relation is given by

$$s_o(t) = \int_{-\infty}^{\infty} h(\tau;t)s(t-\tau)d\tau, \tag{1}$$

where $s(t)$ is the input waveform, $s_o(t)$ is the output waveform and $h(\tau;t)$ is the impulse response of the time-varying multipath fading channel.

(a) Can we measure the impulse response $h(\tau;t)$ by transmitting a single impulse $s(t) = \delta(t)$ at the input? Justify your answer.

(b) Based on (1), show that

$$s_o(t) = \int_{-\infty}^{\infty} S(f)H(f;t)e^{j2\pi ft}df,$$

where $S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$ and $H(f;t) = \int_{-\infty}^{\infty} h(\tau;t)e^{-j2\pi f\tau}d\tau$.

(c) Give

$$s(t) = \text{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\},\ s_o(t) = \text{Re}\{\tilde{s}_o(t)e^{j2\pi f_c t}\} \text{ and } h(\tau;t) = \text{Re}\{\tilde{h}(\tau;t)e^{j2\pi f_c \tau}\}. \tag{2}$$

Prove that

$$\tilde{s}_o(t) = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{h}(\tau;t)\tilde{s}(t-\tau)d\tau,$$

provided that

$$\tilde{S}(f-f_c)\tilde{H}^*(-f-f_c;t) = \tilde{S}^*(-f-f_c)\tilde{H}(f-f_c;t) = 0.$$

Hint: From (2), we have

$$S(f) = \frac{1}{2}[\tilde{S}(f-f_c) + \tilde{S}^*(-f-f_c)]$$

and

$$H(f;t) = \frac{1}{2}[\tilde{H}(f-f_c;t) + \tilde{H}^*(-f-f_c;t)]. \tag{3}$$

Solution.

(a) With $s(t) = \delta(t)$, we can only have

$$s_o(t) = \int_{-\infty}^{\infty} h(\tau;t)\delta(t-\tau)d\tau = h(t;t).$$

Thus, we cannot identify $h(\tau;t)$ for any τ that is not equal to t. We shall send a series of input waveforms as $\{\delta(t-\tau_0)\}_{\tau_0=0}^{\infty}$ (therefore, we have

$$s_o(t) = \int_{-\infty}^{\infty} h(\tau;t)s((t-\tau)-\tau_0)d\tau = h(t-\tau_0;t)$$

for different τ_0) in order to have a complete knowledge of $h(\tau;t)$.

(b) See the last three lines on Slide IDC 5-55 and the first three lines on Slide IDC 5-56.

(c) Continue from (b) and use the derivations in Slide IDC 5-57.

2. (a) Below are typical delay spreads and Doppler spreads for several time-varying multi-path fading channels.

type of channels	delay spread σ_τ (in second)	Doppler spread σ_ν (in Hz)
1. Shortwave ionospheric propagation (HF)	10^{-2}	1
2. Ionospheric propagation under disturbed auroral condition (HF)	10^{-2}	100
3. Ionospheric forward scatter (VHF)	10^{-4}	10
4. Tropospheric scatter (SHF)	10^{-6}	10
5. Orbital scatter (X band)	10^{-4}	10^3

Which channel can be made nearly time-flat and frequency-flat? Justify your answer.

Hint: In order to obtain a nearly time-flat and frequency-flat channel, it often requires that the signal duration is (at least) ten times larger than the delay spread and the signal duration is (at least) ten times smaller than the coherent time.

(b) Below are delay spreads for several indoor environments.

Indoor space	typical delay spread
New York stock exchange	120 ns
Meeting room (5m×5m) with metal walls	55 ns
Single room with stone walls	35 ns
Indoor sports arena	120 ns
Factory	125 ns
Office building 1	130 ns
Office building 2	60 ns
Office buidling 3	65 ns
Office building 4	30 ns

Whether a nearly time-flat frequency-flat communication is feasible in these environments, if indoor walking speed is $v = 3$ m/sec and the carrier frequency is $f_c = 10$ GHz?

Hint: You may assume the Doppler spread is the maximum Doppler shift $\frac{v}{\lambda} = \frac{v}{c} f_c$, where $c = 3 \times 10^8$ m/sec is the wave speed.

Solution.

(a) In math formulation, the requirement that the signal duration is ten times larger than the delay spread and is ten times smaller than the coherent time can be written as

$$\frac{1}{10\sigma_\nu} > T > 10\sigma_\tau,$$

which is equivalent to $\sigma_\tau \sigma_\nu < 10^{-2}$. Thus, only type-3 channel and type-4 channel can be made nearly time-flat and frequency-flat.

(b) The Doppler spread (Doppler shift) is equal to

$$\sigma_\nu = \frac{v}{c} f_c = \frac{3 \text{ m/sec}}{3 \times 10^8 \text{ m/sec}} 10 \text{ GHz} = 100 \text{ Hz}.$$

So, "near time-flat and frequency-flat" requires the delay spread being smaller than $10^{-2}/\sigma_\nu = 10^{-4} = 10^5$ ns. Apparently, all delay spreads listed in the table fulfill this requirement.

3. For a time-flat frequency-flat fading channel, the input signal suffers a multiplicative factor α. The signal-to-noise ratio of the system becomes $\gamma = \alpha^2 \frac{E_b}{N_0}$. Assume γ is Rayleigh distributed with probability density function

$$f_\gamma(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \quad \text{for } \gamma \geq 0,$$

where $\gamma_0 = E[\gamma]$.

(a) Show that the average bit error rate (BER) for coherent binary FSK is

$$\frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{2+\gamma_0}}\right),$$

where its BER without fading is equal to $\Phi(-\sqrt{\gamma})$. Show that $\frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)$ is approximately $1/(2\gamma_0)$ as γ_0 large.

(b) Show that the average bit error rate (BER) for binary DPSK is $\frac{1}{2(1+\gamma_0)}$, where its BER without fading is equal to $\frac{1}{2}e^{-\gamma}$.

(c) Show that the average bit error rate (BER) for non-coherent BFSK is $\frac{1}{2+\gamma_0}$, where its BER without fading is equal to $\frac{1}{2}e^{-\gamma/2}$.

(d) Show that subject to $\gamma_0 > 0$, we have

$$\underbrace{\frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)}_{\text{(a)}} < \underbrace{\frac{1}{2(1+\gamma_0)}}_{\text{(b)}} \quad \text{and} \quad \lim_{\gamma_0 \to \infty} \frac{\frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)}{\frac{1}{2\gamma_0}} = 1.$$

Solution.

(a)

$$\begin{aligned}
\text{BER} &= \int_0^\infty \Phi(-\sqrt{\gamma})\left(\frac{1}{\gamma_0}e^{-\gamma/\gamma_0}\right)\mathrm{d}\gamma \\
&= \Phi(-\sqrt{\gamma})\left(-e^{-\gamma/\gamma_0}\right)\Big|_0^\infty - \int_0^\infty\left(-\frac{1}{2\sqrt{2\pi\gamma}}e^{-\gamma/2}\right)\left(-e^{-\gamma/\gamma_0}\right)\mathrm{d}\gamma \\
&= \frac{1}{2} - \int_0^\infty\left(\frac{1}{2\sqrt{2\pi\gamma}}e^{-\gamma(1/2+1/\gamma_0)}\right)\mathrm{d}\gamma \quad (x=\gamma(1+2/\gamma_0)) \\
&= \frac{1}{2} + \frac{1}{\sqrt{1+2/\gamma_0}}\int_0^\infty\left(-\frac{1}{2\sqrt{2\pi x}}e^{-x/2}\right)\mathrm{d}\gamma \\
&= \frac{1}{2} + \frac{1}{\sqrt{1+2/\gamma_0}}\Phi\left(-\sqrt{x}\right)\Big|_0^\infty \\
&= \frac{1}{2} - \frac{1}{2\sqrt{1+2/\gamma_0}} = \frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right) &= \frac{1}{2}\left(\frac{\sqrt{2+\gamma_0}-\sqrt{\gamma_0}}{\sqrt{2+\gamma_0}}\right) \\
&= \frac{1}{2}\left(\frac{(\sqrt{2+\gamma_0}-\sqrt{\gamma_0})(\sqrt{2+\gamma_0}+\sqrt{\gamma_0})}{\sqrt{2+\gamma_0}(\sqrt{2+\gamma_0}+\sqrt{\gamma_0})}\right) \\
&= \frac{1}{2}\left(\frac{2}{2+\gamma_0+\sqrt{2+\gamma_0}\sqrt{\gamma_0}}\right) \quad (2+\gamma_0\approx\gamma_0 \text{ when } \gamma_0 \text{ large}) \\
&\approx \frac{1}{2}\left(\frac{2}{\gamma_0+\sqrt{\gamma_0}\sqrt{\gamma_0}}\right) \\
&= \frac{1}{2\gamma_0}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{BER} &= \int_0^\infty \frac{1}{2}e^{-\gamma}\left(\frac{1}{\gamma_0}e^{-\gamma/\gamma_0}\right)\mathrm{d}\gamma \\
&= \int_0^\infty \frac{1}{2\gamma_0}e^{-\gamma(1+1/\gamma_0)}\mathrm{d}\gamma \quad (x=\gamma(1+1/\gamma_0)) \\
&= \int_0^\infty \frac{1}{2(1+\gamma_0)}e^{-x}dx \\
&= \frac{1}{2(1+\gamma_0)}\left(-e^{-x}\right)\Big|_0^\infty \\
&= \frac{1}{2(1+\gamma_0)}
\end{aligned}$$

(c)

$$\begin{aligned}
\text{BER} &= \int_0^\infty \frac{1}{2}e^{-\gamma/2}\left(\frac{1}{\gamma_0}e^{-\gamma/\gamma_0}\right)\mathrm{d}\gamma \\
&= \int_0^\infty \frac{1}{2\gamma_0}e^{-\gamma(1/2+1/\gamma_0)}\mathrm{d}\gamma \quad (x=\gamma(1/2+1/\gamma_0)) \\
&= \int_0^\infty \frac{1}{(2+\gamma_0)}e^{-x}dx \\
&= \frac{1}{(2+\gamma_0)}\left(-e^{-x}\right)\Big|_0^\infty \\
&= \frac{1}{2+\gamma_0}
\end{aligned}$$

(d) Subject to $\gamma_0 > 0$, we derive

$$\begin{aligned}
&\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right) < \frac{1}{2(1+\gamma_0)} \\
\iff\ & \frac{\gamma_0}{1+\gamma_0} < \sqrt{\frac{\gamma_0}{2+\gamma_0}} \\
\iff\ & \frac{\gamma_0^2}{(1+\gamma_0)^2} < \frac{\gamma_0}{2+\gamma_0} \\
\iff\ & \gamma_0(2+\gamma_0) < (1+\gamma_0)^2 \\
\iff\ & 0 < 1.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\gamma_0\to\infty}\frac{\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)}{\frac{1}{2\gamma_0}} &= \lim_{\gamma_0\to\infty}\gamma_0\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right) \\
&= \lim_{\gamma_0\to\infty}\gamma_0\left(\frac{\sqrt{2+\gamma_0}-\sqrt{\gamma_0}}{\sqrt{2+\gamma_0}}\right) \\
&= \lim_{\gamma_0\to\infty}\gamma_0\left(\frac{2}{\sqrt{2+\gamma_0}(\sqrt{2+\gamma_0}+\sqrt{\gamma_0})}\right) \\
&= 1.
\end{aligned}$$

Note: On Slide IDC 5-69, we derive

$$\begin{aligned}
&\int_0^\infty \Phi\left(-\sqrt{2\frac{(\alpha^2 E_b)}{N_0}}\right) f_\alpha(\alpha)\mathrm{d}\alpha \quad (\text{Let } \gamma = \alpha^2\frac{E_b}{N_0}) \\
=\ &\int_0^\infty \Phi(-\sqrt{2\gamma})f_\gamma(\gamma)\mathrm{d}\gamma
\end{aligned}$$

Here, we give the detail of how this derivation is done. Formally, we should examine the relation between the cumulative distribution functions (cdfs) of α and γ (in order to get the relation between their probability density functions (pdfs)). For $a \geq 0$,

$$
\begin{aligned}
F_\alpha(a) &= \Pr[\alpha \leq a] \\
&= \Pr[\alpha^2 \leq a^2] \quad (\text{Since } \Pr[\alpha \geq 0] = 1) \\
&= \Pr\left[\alpha^2 \frac{E_b}{N_0} \leq a^2 \frac{E_b}{N_0}\right] \\
&= \Pr\left[\gamma \leq a^2 \frac{E_b}{N_0}\right] \\
&= F_\gamma\left(a^2 \frac{E_b}{N_0}\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
f_\alpha(\alpha) &= \frac{\mathrm{d}F_\alpha(\alpha)}{\mathrm{d}\alpha} \\
&= \frac{\mathrm{d}F_\gamma\left(\alpha^2 \frac{E_b}{N_0}\right)}{\mathrm{d}\alpha} \\
&= \frac{\mathrm{d}\left(\alpha^2 \frac{E_b}{N_0}\right)}{\mathrm{d}\alpha} \cdot f_\gamma\left(\alpha^2 \frac{E_b}{N_0}\right) \\
&= 2\alpha \frac{E_b}{N_0} \cdot f_\gamma\left(\alpha^2 \frac{E_b}{N_0}\right) \\
&= 2\alpha \frac{E_b}{N_0} \cdot f_\gamma(\gamma).
\end{aligned}
$$

Together with $\mathrm{d}\gamma = 2\alpha \frac{E_b}{N_0} \mathrm{d}\alpha$, we obtain

$$
f_\alpha(\alpha)\, \mathrm{d}\alpha = \left(2\alpha \frac{E_b}{N_0} \cdot f_\gamma(\gamma)\right)\left(\frac{\mathrm{d}\gamma}{2\alpha \frac{E_b}{N_0}}\right) = f_\gamma(\gamma)\mathrm{d}\gamma.
$$

4. Let $h(\tau;t) = \mathrm{Re}\{\tilde{h}(\tau;t)e^{j2\pi f_c \tau}\}$ denote the impulse response of a time-varying multipath fading channel.

   (a) Suppose $\tilde{h}(\tau;t)$ is a stationary, zero-mean, complex-valued Gaussian random process (in t). Is $h(\tau;t)$ stationary in t? Does $h(\tau;t)$ have zero-mean? Justify your answer.

   (b) Suppose

   $$\tilde{h}(\tau;t) = \tilde{g}(\tau;t) + \mu(\tau),$$

   where $\tilde{g}(\tau;t)$ is a stationary, zero-mean, complex-valued Gaussian random process (in t) and $\mu(\tau)$ is a deterministic function of τ (in other words, $\tilde{h}(\tau;t)$ is not necessarily a zero-mean process for each τ). If $\tilde{g}(\tau;t)$ satisfies "uncorrelated scattering" condition, i.e.,

   $$R_{\tilde{g}}(\tau_1, \tau_2; \Delta t) = E\left[\tilde{g}^*(\tau_1;t)\tilde{g}(\tau_2;t+\Delta t)\right] = r_{\tilde{g}}(\tau_1;\Delta t)\delta(\tau_1 - \tau_2).$$

   Determine

   $$R_{\tilde{h}}(\tau_1, \tau_2; \Delta t) = E\big[\tilde{h}^*(\tau_1;t)\tilde{h}(\tau_2;t+\Delta t)\big].$$

(c) Continue from (b). Does $R_{\tilde{H}}(f_1, f_2; \Delta t) = E[\tilde{H}^*(f_1; t)\tilde{H}(f_2; t + \Delta t)]$ depend only on time difference and frequency difference? Justify your answer.
Hint: Denote $M(f) = \int_{-\infty}^{\infty} \mu(\tau)e^{-j2\pi f\tau}d\tau$ as the Fourier transform of $\mu(\tau)$ and use it in your derivation.

Solution.

(a) Since (by definition of stationarity for complex-valued random processes) $\text{Re}\{\tilde{h}(\tau; t)\}$ and $\text{Im}\{\tilde{h}(\tau; t)\}$ are jointly stationary and

$$h(\tau; t) = \text{Re}\{\tilde{h}(\tau; t)\}\cos(2\pi f_c\tau) - \text{Im}\{\tilde{h}(\tau; t)\}\sin(2\pi f_c\tau)$$

is a linear combination of $\text{Re}\{\tilde{h}(\tau; t)\}$ and $\text{Im}\{\tilde{h}(\tau; t)\}$, $h(\tau; t)$ is stationary in t. In addition,

$$\begin{aligned} E[h(\tau; t)] &= E[\text{Re}\{\tilde{h}(\tau; t)\}]\cos(2\pi f_c\tau) + E[\text{Im}\{\tilde{h}(\tau; t)\}]\sin(2\pi f_c\tau) \\ &= 0 \cdot \cos(2\pi f_c\tau) + 0 \cdot \sin(2\pi f_c\tau) = 0; \end{aligned}$$

hence $h(\tau; t)$ is a zero-mean random process.
Note: By the zero-mean assumption, $h(\tau; t)$ could be negative with certain probability, which means that without the additive noise (i.e., noiseless), the multipath fadings may reverse the sign of the transmission waveform.

(b)

$$\begin{aligned} R_{\tilde{h}}(\tau_1, \tau_2; \Delta t) &= E\left[\tilde{h}^*(\tau_1; t)\tilde{h}(\tau_2; t + \Delta t)\right] \\ &= E\left[(\tilde{g}^*(\tau_1; t) + \mu^*(\tau_1))(\tilde{g}(\tau_2; t + \Delta t) + \mu(\tau_2))\right] \\ &= E\left[\tilde{g}^*(\tau_1; t)\tilde{g}(\tau_2; t + \Delta t)\right] + \underbrace{E\left[\tilde{g}^*(\tau_1; t)\right]}_{=0}\mu(\tau_2) \\ &\quad + \mu^*(\tau_1)\underbrace{E\left[\tilde{g}(\tau_2; t + \Delta t)\right]}_{=0} + \mu^*(\tau_1)\mu(\tau_2) \\ &= r_{\tilde{g}}(\tau_1; \Delta t)\delta(\tau_1 - \tau_2) + \mu^*(\tau_1)\mu(\tau_2) \end{aligned}$$

(c)

$$\begin{aligned} R_{\tilde{H}}(f_1, f_2; \Delta t) &= E[\tilde{H}^*(f_1; t)\tilde{H}(f_2; t + \Delta t)] \\ &= E\left[\left(\int_{-\infty}^{\infty} \tilde{h}(\tau_1; t)e^{-j2\pi f_1\tau_1}d\tau_1\right)^* \left(\int_{-\infty}^{\infty} \tilde{h}(\tau_2; t + \Delta t)e^{-j2\pi f_2\tau_2}d\tau_2\right)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{\tilde{h}}(\tau_1, \tau_2; \Delta t)e^{j2\pi(f_1\tau_1 - f_2\tau_2)}d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[r_{\tilde{g}}(\tau_1; \Delta t)\delta(\tau_1 - \tau_2) + \mu^*(\tau_1)\mu(\tau_2)\right] e^{j2\pi(f_1\tau_1 - f_2\tau_2)}d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_{\tilde{g}}(\tau_1; \Delta t)\delta(\tau_1 - \tau_2)e^{j2\pi(f_1\tau_1 - f_2\tau_2)}d\tau_1 d\tau_2 \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(\tau_1)\mu(\tau_2)e^{j2\pi(f_1\tau_1 - f_2\tau_2)}d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} r_{\tilde{g}}(\tau_1; \Delta t)e^{j2\pi(\Delta f)\tau_1}d\tau_1 + M^*(f_1)M(f_2) \end{aligned}$$

Therefore, $R_{\tilde{H}}(f_1, f_2; \Delta t)$ may not be a function of $\Delta f = f_1 - f_2$ only.

<u>Note 1</u>: If $\mu(\tau) = \mu_0$ is a constant function, then $M(f) = \mu_0 \delta(f)$ and

$$R_{\tilde{H}}(f_1, f_2; \Delta t) = \int_{-\infty}^{\infty} r_{\tilde{g}}(\tau_1; \Delta t) e^{j2\pi(\Delta f)\tau_1} d\tau_1 + |\mu_0|^2 \delta(f_1)\delta(f_2).$$

Thus, for $f_1 \neq 0$ and $f_2 \neq 0$,

$$R_{\tilde{H}}(f_1, f_2; \Delta t) = \int_{-\infty}^{\infty} r_{\tilde{g}}(\tau_1; \Delta t) e^{j2\pi(\Delta f)\tau_1} d\tau_1$$

is still a function of frequency difference $\Delta f = f_1 - f_2$ only.

<u>Note 2</u>: Alternatively, if $\mu(\tau) = \mu_0 \delta(\tau)$, i.e., a "mean" exists only when $\tau = 0$ (which denotes the direct path with zero excess delay), then $M(f) = \mu_0$, and

$$R_{\tilde{H}}(f_1, f_2; \Delta t) = \int_{-\infty}^{\infty} r_{\tilde{g}}(\tau_1; \Delta t) e^{j2\pi(\Delta f)\tau_1} d\tau_1 + |\mu_0|^2$$

is still a function of frequency difference $\Delta f = f_1 - f_2$ only.