Sample Problems for Quiz 5

For the preparation of Quiz 5, please pay special attention on Problem 2.

1. (a) From Slide IDC3-49, we obtain that the best estimate of θ for a given τ is

$$\hat{\theta} = \arg \min_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k - \boldsymbol{s}_k(\theta)\|^2,$$

where

$$\boldsymbol{x}_k = \begin{bmatrix} \int_\tau^{T+\tau} x_k(t)\cdot\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \\ \int_\tau^{T+\tau} x_k(t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_c t)dt \end{bmatrix},$$

$$\boldsymbol{s}_k(\theta) = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix} = \begin{bmatrix} \sqrt{E}\cos(\alpha_k+\theta) \\ -\sqrt{E}\sin(\alpha_k+\theta) \end{bmatrix},$$

and $\alpha_k \in \{0, \frac{2\pi}{M}, \ldots, (M-1)\frac{2\pi}{M}\}$. Can we rewrite $\hat{\theta}$ as

$$\hat{\theta} = \arg \max_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T \boldsymbol{s}_k(\theta)?$$

Justify your answer.

Hint: For the receiver, $x(t)$ is a waveform that has been received and known, and so is $\boldsymbol{x}_k$, provided τ is given.

(b) From Slide IDC3-55, we obtain that the best estimate of τ is

$$\hat{\tau} = \arg\min_{\tau} \sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k(\tau) - \boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\|^2,$$

where

$$\boldsymbol{x}_k(\tau) = \begin{bmatrix} \int_\tau^{T+\tau} x_k(t)\cdot\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \\ \int_\tau^{T+\tau} x_k(t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_c t)dt \end{bmatrix},$$

$$\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0) = \begin{bmatrix} \sqrt{E}\left(\frac{T-|\tau_0-\tau|}{T}\right)\cos(\alpha_k+\theta) \\ -\sqrt{E}\left(\frac{T-|\tau_0-\tau|}{T}\right)\sin(\alpha_k+\theta) \end{bmatrix},$$

and $\alpha_k \in \{0, \frac{2\pi}{M}, \ldots, (M-1)\frac{2\pi}{M}\}$. Can we rewrite $\hat{\tau}$ as

$$\hat{\tau} = \arg \max_{\substack{\cancel{\theta\in[0,\pi)} \\ \tau}} \sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T(\tau)\, \boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)?$$

Justify your answer.

(c) Continue from (b). Since θ is unknown, we take the expected value of the probability quantities with respect to it over $[0, 2\pi)$. In other words,

$$\hat{\tau} = \arg\max_{\tau} E_\theta\left[e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k(\tau)-\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\|^2}\right].$$

Show that

$$\hat{\tau} \;=\; \arg\max_{\tau} e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k(\tau)\|^2} e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}E\left(\frac{T-|\tau-\tau_0|}{T}\right)^2} E_\theta\left[e^{\frac{2}{N_0}\sum_{k=0}^{L_0-1}\boldsymbol{x}_k^T(\tau)\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)}\right].$$

Solution.

(a) Yes, we can rewrite $\hat{\theta}$ as

$$\hat{\theta} \;=\; \arg\max_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T\boldsymbol{s}_k(\theta)$$

because

$$\begin{aligned}
\hat{\theta} &= \arg\min_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k - \boldsymbol{s}_k(\theta)\|^2 \\
&= \arg\min_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \left(\|\boldsymbol{x}_k\|^2 - 2\boldsymbol{x}_k^T\boldsymbol{s}_k(\theta) + \|\boldsymbol{s}_k(\theta)\|^2\right) \\
&= \arg\min_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \left(\|\boldsymbol{x}_k\|^2 - 2\boldsymbol{x}_k^T\boldsymbol{s}_k(\theta) + E\right) \quad \text{(From the formula of } \boldsymbol{s}_k(\theta).\text{)} \\
&= \arg\min_{\theta\in[0,\pi)} \left(\underbrace{\cancel{\sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k\|^2}}_{\text{irrelevant to }\theta} - 2\sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T\boldsymbol{s}_k(\theta) + \underbrace{\cancel{\sum_{k=0}^{L_0-1} E}}_{\text{irrelevant to }\theta} \right) \\
&= \arg\min_{\theta\in[0,\pi)} \left(-2\sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T\boldsymbol{s}_k(\theta) \right) \\
&= \arg\max_{\theta\in[0,\pi)} \sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T\boldsymbol{s}_k(\theta)
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{\tau} &= \arg\min_{\tau} \sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k(\tau) - \boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\|^2 \\
&= \arg\min_{\tau} \sum_{k=0}^{L_0-1} \left(\|\boldsymbol{x}_k(\tau)\|^2 - 2\boldsymbol{x}_k^T(\tau)\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0) + \|\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\|^2\right) \\
&= \arg\min_{\tau} \sum_{k=0}^{L_0-1} \left(\|\boldsymbol{x}_k(\tau)\|^2 - 2\boldsymbol{x}_k^T(\tau)\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0) + E\left(\frac{T-|\tau_0-\tau|}{T}\right)^2\right)
\end{aligned}$$

Hence, even if the received waveform $x_k(t)$ has nothing to do with τ, $\boldsymbol{x}_k(\tau)$ does. The third term is also functionally dependent on τ. Therefore, in principle, we cannot remove the first and the third terms in the determination of $\hat{\tau}$.

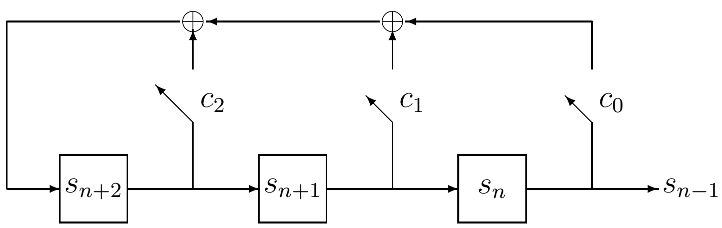
Note: The receiver actually cannot calculate $(\frac{T-|\tau_0-\tau|}{T})^2$ because the receiver does not know τ_0, which is the actual delay. If $|\tau-\tau_0|$ is much smaller than T (i.e., τ is not much deviated from τ_0), then $(\frac{T-|\tau_0-\tau|}{T})^2 \approx 1$. In such case, $\|\boldsymbol{x}_k(\tau)\|^2$ is almost invariant with respect to τ. As a consequence, we can omit these two terms in the decision rule.

However, if $|\tau-\tau_0|$ is not much smaller than T, then the third term cannot be omitted. But still, the inclusion of the third term will cause a problem in its implementation; thus, we may still need to find a way to compensate for its effect.

(c)

$$
\begin{aligned}
\hat{\tau} &= \arg\max_{\tau} E_\theta\left[e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k(\tau)-\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\|^2}\right] \\
&= \arg\max_{\tau} E_\theta\left[e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k(\tau)\|^2} e^{\frac{2}{N_0}\sum_{k=0}^{L_0-1}\boldsymbol{x}_k^T(\tau)\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)} e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\|^2}\right] \\
&= \arg\max_{\tau} E_\theta\left[e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k(\tau)\|^2} e^{\frac{2}{N_0}\sum_{k=0}^{L_0-1}\boldsymbol{x}_k^T(\tau)\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)} e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}E\left(\frac{T-|\tau-\tau_0|}{T}\right)^2}\right] \\
&= \arg\max_{\tau} e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k(\tau)\|^2} e^{-\frac{1}{N_0}\sum_{k=0}^{L_0-1}E\left(\frac{T-|\tau-\tau_0|}{T}\right)^2} E_\theta\left[e^{\frac{2}{N_0}\sum_{k=0}^{L_0-1}\boldsymbol{x}_k^T(\tau)\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)}\right]
\end{aligned}
$$

2. (a)



For the pseudo-noise sequence generator above, we have

$$s_{n+3} = c_2 s_{n+2} \oplus c_1 s_{n+1} \oplus c_0 s_n,$$

where each s_n and c_i are in $\{0,1\}$. Find the periods of the output sequences for the eight possible designs of $(c_0, c_1, c_2) \in \{0,1\}^3$.

Hint: The initial value of $s_n s_{n+1} s_{n+2}$ can be any values but 000. For example, you may set $s_0 s_1 s_2 = 001$ initially. Please ignore the initial transient state of the sequences. Find the first state $s_n s_{n+1} s_{n+2}$ that has appeared before and then the periods of the sequences can be identified afterwards.

(b) Check the autocorrelations of the periodic sequences generated from each of the eight designs.

Hint: The autocorrelation function is defined as

$$A(j) \triangleq \sum_{i=0}^{T-1}(-1)^{a_i}(-1)^{a_{(i+j) \bmod T}} \quad \text{for } 0 \le j \le T-1,$$

where T is the period of the sequence $a_0 a_1 a_2 \cdots$.

(c) Among the eight periodic sequences in (a), which satisfy the balance property, and which satisfy the run property?

Hint:

1. Balance property $\equiv$ the (absolute value of the) difference between the number of one's and the number of zeros is at most one.

2. Run property $\equiv N_{\ell+1} = \left\lceil \frac{N_\ell}{2} \right\rceil$ or $\left\lfloor \frac{N_\ell}{2} \right\rfloor$, where N_ℓ is the number of runs of length ℓ.

(d) There should be two m-sequences of length 7, which are denoted as $a_0a_1a_2a_3a_4a_5a_6$ and $b_0b_1b_2b_3b_4b_5b_6$. Check the cross-correlation between the two m-sequences, defined as

$$C(j) \triangleq \sum_{i=0}^{6} (-1)^{a_i}(-1)^{b_{(i+j) \bmod 7}} \quad \text{for } 0 \le j \le 6.$$

Solution.

(a) The periodic sequences are colored in red in the below table.

$c_0c_1c_2$	$s_0s_1s_2s_3s_4s_5s_6\cdots$	check when $s_{n+T}s_{n+1+T}s_{n+2+T} = s_ns_{n+1}s_{n+2}$	period T
000	$0010000\cdots$	state $s_4s_5s_6$ = state $s_3s_4s_5$ = 000	1
100	$001001\cdots$	state $s_3s_4s_5$ = state $s_0s_1s_2$ = 001	3
010	$001010\cdots$	state $s_3s_4s_5$ = state $s_1s_2s_3$ = 010	2
110	$0010111001\cdots$	state $s_7s_8s_9$ = state $s_0s_1s_2$ = 001	7
001	$001111\cdots$	state $s_3s_4s_5$ = state $s_2s_3s_4$ = 111	1
101	$0011101001\cdots$	state $s_7s_8s_9$ = state $s_0s_1s_2$ = 001	7
011	$0011011\cdots$	state $s_4s_5s_6$ = state $s_1s_2s_3$ = 011	3
111	00110011	state $s_4s_5s_6$ = state $s_0s_1s_2$ = 001	4

Note: Among the eight possible designs, two have period 7 (maximum-length), one has period 4, two have period 3, one has period 2, and two have period 1. Thus, the periods vary among different designs.

(b)

$c_0c_1c_2$	sequence of one period	j = 0	1	2	3	4	5	6
000	0	1						
100	001	3	-1	-1				
010	01	2	-2					
110	0010111	7	-1	-1	-1	-1	-1	-1
001	1	1						
101	0011101	7	-1	-1	-1	-1	-1	-1
011	011	3	-1	-1				
111	0011	4	0	-4	0			

Note: Except for the sequences of odd periods, the autocorrelation may be equal to a value other than T and -1.

(c) All eight sequences satisfy the balance property. Below is a tabularized summary of whether they satisfy the run property.

$c_0c_1c_2$	sequence	run property
000	0	Not applicable since only a single run
100	001	Satisfy as $N_1 = N_2 = 1$
010	01	Not satisfy because $N_1 = 2$ but $N_2 = 0$
110	0010111	Satisfy because $N_1 = 2$, $N_2 = 1$ and $N_3 = 1$
001	1	Not applicable since only a single run
101	0011101	Satisfy because $N_1 = 2$, $N_2 = 1$ and $N_3 = 1$
011	011	Satisfy because $N_1 = 1$ and $N_1 = 1$
111	0011	Not satisfy because $N_2 = 2$ but $N_1 = 0$

Note: Take sequence 011 as an example, which results in $N_1 = 1$, $N_2 = 1$, and $N_\ell = 0$ for $\ell \geq 3$. Then, $N_{\ell+1} = \lceil \frac{N_\ell}{2} \rceil$ or $\lfloor \frac{N_\ell}{2} \rfloor$ is always satisfied; hence, sequence 011 satisfies the run property.

By this rule, we may regard sequence 0 as $N_1 = 1$ and $N_\ell = 0$ for $\ell \geq 2$, and claim that it satisfies the run property, although this is kind of tricky. I would personally favor the answer of non-applicability because there is only one single bit in the sequence.

(d) With $a_0a_1a_2a_3a_4a_5a_6 = 0010111$ and $b_0b_1b_2b_3b_4b_5b_6 = 0011101$, we have

j	$C(j)$	
0	3	
1	-1	
2	$\cancel{3}$	-1
3	$\cancel{3}$	-5
4	$\cancel{-5}$	3
5	$\cancel{-1}$	3
6	-1	

Note: Unlike the autocorrelation that always equals -1 for non-zero delay (i.e., $j \neq 0$), the absolute value of the cross-correlation may be as large as 5.

3. $c(t) = \sum_{i=-\infty}^{\infty} c_i \cdot g(t - iT_c)$, where $c_i \in \{\pm 1\}$ is a deterministic given value, and assume $g(t) = 0$ for t outside $[0, T_c)$

(a) Give a $g(t)$ that guarantees $c^2(t) = 1$.

(b) Based on the $g(t)$ in (a), if $\{c_i\}_{i=-\infty}^{\infty}$ satisfies

$$R(\ell) \triangleq \sum_{i=0}^{n-1} c_i c_{i-\ell} = -1$$

for $1 \leq \ell \leq n-1$, show that for every integer ℓ,

$$R_c(\ell T_c) = \frac{1}{nT_c} \int_0^{nT_c} c(t)c(t - \ell T_c)dt = \begin{cases} 1, & \ell = 0; \\ -\frac{1}{n}, & 1 \leq \ell \leq n-1. \end{cases}$$

Solution.

(a)

$$\begin{aligned} c^2(t) &= \left(\sum_{i=-\infty}^{\infty} c_i \cdot g(t - iT_c) \right)^2 \\ &= \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_i c_k \cdot g(t - iT_c) g(t - kT_c) \\ &= \sum_{i=-\infty}^{\infty} c_i^2 \cdot g^2(t - iT_c) \quad \text{(Because } g(t - iT_c)g(t - kT_c) = 0 \text{ for } i \neq k) \\ &= \sum_{i=-\infty}^{\infty} g^2(t - iT_c) \quad \text{(Because } c_i^2 = 1) \end{aligned}$$

Thus, $g^2(t) = 1$ for $0 \le t < T_c$. A setting that satisfies $g^2(t) = 1$ for $0 \le t < T_c$ is

$$g(t) = \begin{cases} 1, & 0 \le t < T_c; \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, we can set

$$g(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}T_c; \\ -1, & \frac{1}{2}T_c \le t < T_c; \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$\begin{aligned}
R_c(\ell T_c) &= \frac{1}{nT_c}\int_0^{nT_c} c(t)c(t-\ell T_c)dt \\
&= \frac{1}{nT_c}\int_0^{nT_c} c(t)c(t-\ell T_c)dt \\
&= \frac{1}{nT_c}\int_0^{nT_c} \left(\sum_{i=-\infty}^{\infty} c_i \cdot g(t-iT_c)\right)\left(\sum_{k=-\infty}^{\infty} c_k \cdot g(t-\ell T_c - kT_c)\right) dt \\
&= \frac{1}{nT_c}\int_0^{nT_c} \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_i c_k g(t-iT_c)g(t-\ell T_c-kT_c)dt \\
&= \frac{1}{nT_c}\int_0^{nT_c} \sum_{i=-\infty}^{\infty} c_i c_{i-\ell} g^2(t-iT_c)dt \\
&\quad \text{(Because } g(t-iT_c)g(t-\ell T_c-kT_c) = 0 \text{ for } i \ne \ell + k.\text{)} \\
&= \frac{1}{nT_c}\int_0^{nT_c} \sum_{i=0}^{n-1} c_i c_{i-\ell} g^2(t-iT_c)dt \\
&\quad \text{(Because given } 0 \le t < nT_c,\ g^2(t-iT_c) = 0 \text{ for } i < 0 \text{ or } i \ge n.\text{)} \\
&= \frac{1}{nT_c}\sum_{i=0}^{n-1} c_i c_{i-\ell}\int_0^{nT_c} g^2(t-iT_c)dt \\
&= \frac{1}{n}\sum_{i=0}^{n-1} c_i c_{i-\ell} \quad \text{(See the note below.)} \\
&= \begin{cases} 1, & \ell = 0; \\ -\frac{1}{n}, & 1 \le \ell \le n-1. \end{cases}
\end{aligned}$$

Note: This indicates that the value of $\sum_{i=0}^{n-1} c_i c_{i-\ell}$ decides the value of $R_c(\ell T_c)$ for every integer ℓ.