Further notes on the sample problems for Quiz 3

- Sample problem 2(b): The logic of the solution is that **all** three requirements must be fulfilled; hence, we simply use the previously obtained conditions to check what will be further required by the next requirement. So we wrote:

> (b) The FSK formulation in (1) ... no constraint on h and $f_c T_b$ is devised from memoryless requirement.
>
> For phase-continuity requirement, ... A sufficient and necessary condition for phase continuity is ...
>
> $$\underbrace{2f_cT_b \text{ is an integer}}_{\text{sub-condition 1}} \text{ or } \underbrace{h \text{ is an integer}}_{\text{sub-condition 2}}.$$
>
> With $\underbrace{\text{this condition in mind}}_{\text{i.e., two sub-conditions above}}$, coherent orthogonality requires
>
> ...
>
> $$\Leftrightarrow \begin{cases} \sin(2\pi h(n+1)) = \sin(2\pi hn), & \text{if } \underbrace{2f_cT_b \text{ integer}}_{\text{sub-condition 1}} \\ \sin(4\pi f_c(n+1)T_b) = \sin(4\pi f_c nT_b), & \text{if } \underbrace{h \text{ integer}}_{\text{sub-condition 2}} \end{cases} \text{ for every integer } n$$
>
> $\Leftrightarrow$ ...

- Correction to the solution for sample problem 3(b): "if $2h$ is an integer, then coherent orthogonality requires $\sin(4\pi f_c(n+1)T_b) = \sin(4\pi f_c nT_b)$, i.e., $2f_cT_b$ must be an integer ... both $h/2$ and $2f_cT_b$ to be an integer" should be corrected as "if $2h$ is an integer, then coherent orthogonality requires $\sin(4\pi f_c(n+1)T_b) = \sin(4\pi f_c nT_b)$, i.e., $4f_cT_b$ must be an integer ... both $h/2$ and $4f_cT_b$ to be an integer".

Sample Problems for Quiz 4

1. Find all solutions of real-valued x such that

$$\cos((1-x)\pi + \theta) = -(-1)^x \cos(\theta) \text{ for arbitrary } \theta.$$

Solution. Since the equation must hold for arbitrary θ, we take $\theta = 0$ for simplicity. Solving $\cos(\pi x) = (-1)^x$ yields that x must be an integer. As integer x fulfills the equation for arbitrary θ, the set of all solutions is the set of integers.

2. (a) A *general* binary continuous-phase signal such as binary continuous-phase modulation (CPM) can be formulated as

$$s(t) = \sum_{n=-\infty}^{\infty} g(t-nT_b)\cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q\,(t-nT_b)\right), \quad (1)$$

where $I_n \in \{\pm 1\}$,

$$g(t) = \begin{cases} 1, & 0 \le t < T_b; \\ 0, & \text{otherwsie,} \end{cases}$$

and $p(t)$ is a continue function satisfying

$$q(t) = \begin{cases} 0, & t \le 0; \\ \frac{1}{2}, & t \ge T_b. \end{cases} \tag{2}$$

Verify that the signal formulation in (1) is continuous in phase.

(b) Under $h = \frac{1}{2}$, can we specify $q(\cdot)$ (which is allowed to violate (2)) such that (1) can be used to formulate DPSK in Slide IDC2-81? Justify your answer by characterizing the relationship among I_k, b_k and d_k. Here, we assume $2E_b = T_b$ for simplicity.
Hint: Check $s(t)$ in Slide IDC2-83.

(c) Is the DPSK in (c) continuous in phase?

Solution

(a) For phase-continuity, we observe that discontinuity can only occur at $t = \ell T_b$; thus, we require $\lim_{t \uparrow \ell T_b} s(t) = \lim_{t \downarrow \ell T_b} s(t)$. Derive

$$\begin{aligned}
&\lim_{t \uparrow \ell T_b} s(t) \\
&= \lim_{t \uparrow \ell T_b} \sum_{n=-\infty}^{\infty} g(t - nT_b) \cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q(t - nT_b)\right) \\
&= \lim_{t \uparrow \ell T_b} g(t - (\ell-1)T_b) \cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{\ell-2} I_k + 2\pi h I_{\ell-1} \cdot q(t - (\ell-1)T_b)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \pi h \sum_{k=-\infty}^{\ell-2} I_k + 2\pi h I_{\ell-1} \cdot q(\ell T_b - (\ell-1)T_b)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \pi h \sum_{k=-\infty}^{\ell-1} I_k\right),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t \downarrow \ell T_b} s(t) \\
&= \lim_{t \downarrow \ell T_b} \sum_{n=-\infty}^{\infty} g(t - nT_b) \cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q\left(t - nT_b\right)\right) \\
&= \lim_{t \downarrow \ell T_b} g(t - \ell T_b) \cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{\ell-1} I_k + 2\pi h I_\ell \cdot q(t - \ell T_b)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \pi h \sum_{k=-\infty}^{\ell-1} I_k\right).
\end{aligned}$$

Thus, $s(t)$ is continuous in phase.

Note: In the book titled *Digital Communicaitons* (5th edition) by Proakis and Salehi, GMSK is defined as $h = \frac{1}{2}$ and

$$\frac{\partial q(t)}{\partial t} = \frac{1}{\sqrt{\ln(2)}}\left[\Phi\left(-2\pi B\left(t - \frac{T}{2}\right)\right) - \Phi\left(-2\pi B\left(t + \frac{T}{2}\right)\right)\right].$$

(See Fig. 3.3-4 on p. 119.) In this definition of GMSK, it is the **phase** that is filtered through a Gaussian filter to smooth the transitions from one point to the next. This is indeed different from what was introduced in our textbook.

In order to clarify why there are two "different" definitions of GMSK, let me quote from the below article: `https://www.electronics-notes.com/articles/radio/modulation/what-is-gmsk-gaussian-minimum-shift-keying.php`

> There are two main ways in which GMSK modulation can be generated. The most obvious way (as introduced in Proakis and Salehi's book) is to filter the modulating signal using a Gaussian filter and then apply this to a *frequency* modulator where the modulation index is set to 0.5. This method is very simple and straightforward but it has the drawback that the modulation index must exactly equal 0.5. In practice this analogue method is not suitable because component tolerances drift and cannot be set exactly (See the figure therein).
>
> A second method is more widely used. Here what is known as a quadrature modulator is used (as introduced in our lectures). The term quadrature means that the phase of a signal is in quadrature or 90 degrees to another one. The quadrature modulator uses one signal that is said to be in-phase and another that is in quadrature to this. In view of the in-phase and quadrature elements this type of modulator is often said to be an I-Q modulator. Using this type of modulator the modulation index can be maintained at exactly 0.5 without the need for any settings or adjustments. This makes it much easier to use, and capable of providing the required level of performance without the need for adjustments. For demodulation the technique can be used in reverse (See the figure therein).

The textbook adopts the second method and passes a nonreturn-to-zero binary data stream through a baseband pulse-shaping-filter. You can also refer to [1], which gives (See Eq. (7) in [1]) that

$$e_0(t) = \mathrm{Re}\{h(t) \star E_i(t)e^{j2\pi f_c t}\}$$

with $E_i(t)$ being the baseband MSK and $h(t) = \frac{\sqrt{\pi}}{\alpha}\exp\left(-\frac{\pi^2}{\alpha^2}t^2\right)$ Gaussian filter (See Eq. (11) in [1]).

[1] Mitsuru Ishizuka and Kenkichi Hirade, "Optimum Gaussian filter and deviated-frequency-locking scheme for coherent detection of MSK," *IEEE Trans. Comm.*, vol. 28, no. 6, pp. 850-857, June 1980.

(b) A PSK signal should have a constant frequency; thus, the derivative of the phase term must be a constant, i.e., for $nT_b \le t < (n+1)T_b$,

$$\frac{1}{2\pi}\frac{\partial\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q\left(t - nT_b\right)\right)}{\partial t} = f_c + hI_n q'(t) = \text{constant}.$$

Since $s(t)$ needs to be a function of I_n for $nT_b \le t < (n+1)T_b$, we cannot set $q(t) = 0$. So, we assign $q(t) = \frac{1}{2}$ for $0 \le t < T_b$ in order to obtain the desired π-difference for $I_n = \pm 1$, which gives

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{\infty} g(t - nT_b)\cos\left(2\pi f_c t + \frac{\pi}{2}\sum_{k=-\infty}^{n-1} I_k + \frac{\pi}{2} I_n\right) \\ &= \cos\left(2\pi f_c t + \frac{\pi}{2}\sum_{k=-\infty}^{n-1} I_k + \frac{\pi}{2} I_n\right) \text{ for } nT_b \le t < (n+1)T_b. \end{aligned}$$

In comparison with $s(t)$ in Slide IDC2-83, i.e.,

$$s(t) = \cos(2\pi f_c t + \theta + (1 - d_n)\pi) \text{ for } nT_b \le t < (n+1)T_b,$$

a feasible assignment under $I_n \in \{\pm 1\}$ and $d_n \in \{0, 1\}$ is

$$\begin{cases} \frac{\pi}{2}\sum_{k=-\infty}^{n-1} I_k - \frac{\pi}{2} = \theta \\ \frac{\pi}{2}I_n + \frac{\pi}{2} = (1-d_n)\pi \end{cases} \Rightarrow I_n = 2(1-d_n) - 1 = (-1)^{d_n}$$

Since $-(-1)^{d_n} = (-1)^{d_{n-1}}(-1)^{b_n}$, we obtain $-I_n = I_{n-1}(-1)^{b_n}$, which implies $(-1)^{b_n} = -I_n I_{n-1}$.

(c) With $q(t) = \frac{1}{2}$ for $0 \le t < T_b$ and $h = \frac{1}{2}$,

$$\begin{aligned}
&\lim_{t\uparrow \ell T_b} s(t) \\
&= \lim_{t\uparrow \ell T_b} \sum_{n=-\infty}^{\infty} g(t - nT_b)\cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q(t - nT_b)\right) \\
&= \lim_{t\uparrow \ell T_b} g(t - (\ell-1)T_b)\cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{\ell-2} I_k + 2\pi h I_{\ell-1} \cdot q(t - (\ell-1)T_b)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \pi h \sum_{k=-\infty}^{\ell-2} I_k + 2\pi h I_{\ell-1} \cdot q(\ell T_b - (\ell-1)T_b)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \pi h \sum_{k=-\infty}^{\ell-1} I_k\right),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\downarrow \ell T_b} s(t) \\
&= \lim_{t\downarrow \ell T_b} \sum_{n=-\infty}^{\infty} g(t - nT_b)\cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q\left(t - nT_b\right)\right) \\
&= \lim_{t\downarrow \ell T_b} g(t - \ell T_b)\cos\left(2\pi f_c t + \pi h \sum_{k=-\infty}^{\ell-1} I_k + 2\pi h I_\ell \cdot q(t - \ell T_b)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \pi h \sum_{k=-\infty}^{\ell-1} I_k + \pi h I_\ell\right).
\end{aligned}$$

Apparently, DPSK is not continuous in phase.

Note: It can be easily seen from Slide IDC2-81 that DPSK does not have phase-continuity. Thus, the above proof is somewhat superfluous (and is just for your reference). Since $\{a_k\}$ is uniform i.i.d. when $\{b_k\}$ is uniform i.i.d., the PSD of binary DPSK is the same as that of binary PSK, which decays at the speed of $1/f^2$ only, an anticipated result from its discontinuous phase.

3. For digital communications, we can ignore completely the waveforms and work on the system design over the projections (i.e., over the signal constellation). Suppose a N-dimensional constellation is constructed. A system designer chooses two constellation

points $\mathbf{s}_1$ and $\mathbf{s}_2$ for binary transmission over the AWGN channel. The additive noise vector $\mathbf{n}$ has the pdf

$$\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\{-\|\mathbf{n}\|^2/(2\sigma^2)\}.$$

(a) Find the optimal decision rule for the received vector $\mathbf{x} = \mathbf{s} + \mathbf{n}$, where $\mathbf{s}$ is either $\mathbf{s}_1$ or $\mathbf{s}_2$ with equal probability.
Hint: MAP decision rule.

(b) Let $\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} \sqrt{E_b}\alpha \\ \sqrt{E_b(1-\alpha^2)} \end{bmatrix}$ as in Sample Problem 4 for Quiz 3. What is the optimal decision rule.

(c) Derive the error probability of the optimal decision rule in (a).

Solution.

(a)

$$\begin{aligned} \hat{\mathbf{s}}_{\text{MAP}} &= \max_{m=1,2} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\{-\|\mathbf{x} - \mathbf{s}_m\|^2/(2\sigma^2)\} \\ &= \min_{m=1,2} \|\mathbf{x} - \mathbf{s}_m\|^2 \end{aligned}$$

Hence, $\hat{\mathbf{s}}_{\text{MAP}} = \mathbf{s}_1$ if $\|\mathbf{x} - \mathbf{s}_1\|^2 \leq \|\mathbf{x} - \mathbf{s}_2\|^2$, and $\hat{\mathbf{s}}_{\text{MAP}} = \mathbf{s}_2$, otherwise. Derive

$$\begin{aligned} &\|\mathbf{x} - \mathbf{s}_1\|^2 \leq \|\mathbf{x} - \mathbf{s}_2\|^2 \\ \Leftrightarrow \quad &\|\mathbf{x}\|^2 + \|\mathbf{s}_1\|^2 - 2\langle \mathbf{x}, \mathbf{s}_1 \rangle \leq \|\mathbf{x}\|^2 + \|\mathbf{s}_2\|^2 - 2\langle \mathbf{x}, \mathbf{s}_2 \rangle \\ \Leftrightarrow \quad &\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2 \leq 2\langle \mathbf{x}, \mathbf{s}_1 \rangle - 2\langle \mathbf{x}, \mathbf{s}_2 \rangle = 2\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \end{aligned}$$

Consequently, the optimal decision rule is

$$\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2}$$

Note: The boundary of the two decision regions is characterized by

$$\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle = \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2}.$$

Note that the hyperplain

$$\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle = \text{constant}$$

is perpendicular to the line connecting the two points. Hence, the boundary is exactly the *perpendicular bisector* of the two points.

(b) Since $\|\mathbf{s}_1\|^2 = \|\mathbf{s}_2\|^2$, and $\mathbf{s}_1 - \mathbf{s}_2 = \begin{bmatrix} \sqrt{E_b}(1-\alpha) \\ -\sqrt{E_b(1-\alpha^2)} \end{bmatrix}$

the optimal decision rule is

$$\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle = x_1\sqrt{E_b}(1-\alpha) - x_2\sqrt{E_b(1-\alpha^2)} \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} 0$$

$$\left(\Leftrightarrow \quad \langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \times \frac{\sqrt{1-\alpha^2}}{(1-\alpha)\sqrt{E_b}} = x_1\sqrt{1-\alpha^2} - x_2(1+\alpha) \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} 0 \right)$$

(c) When $\mathbf{s} = \mathbf{s}_1$, we have $\mathbf{x} = \mathbf{s}_1 + \mathbf{n}$ and an erroneous decision is made if

$$\begin{aligned}\langle \mathbf{s}_1 + \mathbf{n}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \le \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2} \quad \equiv \quad N \triangleq \langle \mathbf{n}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \; &\le \; \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2} - \langle \mathbf{s}_1, \mathbf{s}_1 - \mathbf{s}_2 \rangle \\ &= \; -\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\|^2\end{aligned}$$

Since N is zero-mean Gaussian distributed with variance

$$E[(\langle \mathbf{n}, \mathbf{s}_1 - \mathbf{s}_2 \rangle)^2] = E[(\mathbf{s}_1 - \mathbf{s}_2)^T \mathbf{n}\mathbf{n}^T (\mathbf{s}_1 - \mathbf{s}_2)] = \sigma^2 \|\mathbf{s}_1 - \mathbf{s}_2\|^2,$$

we derive

$$\Pr\left[N \le -\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\|^2\right] = \Phi\left(\frac{-\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\|^2 - 0}{\sigma\|\mathbf{s}_1 - \mathbf{s}_2\|}\right) = \Phi\left(-\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|}{2\sigma}\right).$$

On the other hand, when $\mathbf{s} = \mathbf{s}_2$, we have $\mathbf{x} = \mathbf{s}_2 + \mathbf{n}$ and an erroneous decision is made if

$$\begin{aligned}\langle \mathbf{s}_2 + \mathbf{n}, \mathbf{s}_1 - \mathbf{s}_2 \rangle > \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2} \quad \equiv \quad N \triangleq \langle \mathbf{n}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \; &> \; \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2} - \langle \mathbf{s}_2, \mathbf{s}_1 - \mathbf{s}_2 \rangle \\ &= \; \frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\|^2\end{aligned}$$

We thus have

$$\Pr\left[N > \frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\|^2\right] = \Pr\left[N < -\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\|^2\right] = \Phi\left(-\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|}{2\sigma}\right).$$

As a result, the overall decision error rate is

$$\underbrace{\frac{1}{2}}_{\Pr(\mathbf{s}=\mathbf{s}_1)} \Phi\left(-\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|}{2\sigma}\right) + \underbrace{\frac{1}{2}}_{\Pr(\mathbf{s}=\mathbf{s}_2)} \Phi\left(-\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|}{2\sigma}\right) = \Phi\left(-\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|}{2\sigma}\right).$$

Note: The optimal error is only a function of the distance of the two constellation points, and is irrelevant to their locations.

4. (a) Continue from Problem 3. Suppose

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} s_{1,1} \\ s_{1,2} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ 0 \\ s_{2,3} \\ s_{2,4} \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix}$$

are four dimensional vectors, and the receiver can only observe $\ell_1^2 \triangleq x_1^2 + x_2^2$ and $\ell_2^2 \triangleq x_3^2 + x_4^2$. The receiver thus adopts the decision rule as:

$$\ell_1^2 \quad \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} \quad \ell_2^2$$

What is the probability of erroneous decision?

Hint: The pdf of $\ell \triangleq \sqrt{n_3^2 + n_4^2}$ is $\frac{\ell}{\sigma^2} e^{-\frac{\ell^2}{2\sigma^2}}$ for $\ell \ge 0$.

(b) Subject to $\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2 = E$, where E is constant, find the optimal power allocation $\|\mathbf{s}_1\|^2$ such that the error rate in (a) is minimized.

Solution.

(a) When $\mathbf{s} = \mathbf{s}_1$, an erroneous decision is made if $\ell_1^2 \le \ell_2^2$. Thus, the error rate given $\mathbf{s} = \mathbf{s}_1$ is

$$\begin{aligned}
\Pr[\ell_1^2 \le \ell_2^2 | \mathbf{s} = \mathbf{s_1}] &= \Pr\left(x_1^2 + x_2^2 \le x_3^2 + x_4^2\right) \quad \text{with } \begin{cases} x_1 \sim \mathcal{N}(s_{1,1}, \sigma^2) \\ x_2 \sim \mathcal{N}(s_{1,2}, \sigma^2) \\ x_3 \sim \mathcal{N}(0, \sigma^2) \\ x_4 \sim \mathcal{N}(0, \sigma^2) \end{cases} \\
&= \Pr\left(x_1^2 + x_2^2 \le \ell_2^2\right) \quad \text{with } \begin{cases} x_1 \sim \mathcal{N}(s_{1,1}, \sigma^2) \\ x_2 \sim \mathcal{N}(s_{1,2}, \sigma^2) \\ \ell_2 \text{ Rayleigh with } E[\ell_2^2] = 2\sigma^2 \end{cases} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1 - s_{1,1})^2}{2\sigma^2}} e^{-\frac{(x_2 - s_{1,2})^2}{2\sigma^2}} \left(\int_{\sqrt{x_1^2 + x_2^2}}^{\infty} \frac{\ell_2}{\sigma^2} e^{-\frac{\ell_2^2}{2\sigma^2}} d\ell_2 \right) dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1 - s_{1,1})^2}{2\sigma^2}} e^{-\frac{(x_2 - s_{1,2})^2}{2\sigma^2}} \left(e^{-\frac{x_1^2 + x_2^2}{2\sigma^2}} \right) dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{2x_1^2 - 2s_{1,1}x_1 + s_{1,1}^2 + 2x_2^2 - 2s_{1,2}x_2 + s_{1,2}^2}{2\sigma^2}} dx_1 dx_2 \\
&= \frac{1}{2} e^{-\frac{(s_{1,1}^2 + s_{1,2}^2)}{4\sigma^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\frac{(x_1 - \frac{1}{2}s_{1,1})^2}{\sigma^2}} e^{-\frac{(x_2 - \frac{1}{2}s_{1,2})^2}{\sigma^2}} dx_1 dx_2 \\
&= \frac{1}{2} e^{-\frac{(s_{1,1}^2 + s_{1,2}^2)}{4\sigma^2}} = \frac{1}{2} e^{-\frac{\|\mathbf{s}_1\|^2}{4\sigma^2}}.
\end{aligned}$$

Similarly, when $\mathbf{s} = \mathbf{s}_2$, the error rate given $\mathbf{s} = \mathbf{s}_2$ is

$$\frac{1}{2} e^{-\frac{(s_{2,3}^2 + s_{2,4}^2)}{4\sigma^2}} = \frac{1}{2} e^{-\frac{\|\mathbf{s}_2\|^2}{4\sigma^2}}.$$

As a result, the overall error rate is

$$\frac{1}{4} e^{-\frac{\|\mathbf{s}_1\|^2}{4\sigma^2}} + \frac{1}{4} e^{-\frac{\|\mathbf{s}_2\|^2}{4\sigma^2}}.$$

(b)

$$\begin{aligned}
&\arg \min_{0 \le \|\mathbf{s}_1\|^2 \le E} \left(\frac{1}{4} e^{-\frac{\|\mathbf{s}_1\|^2}{4\sigma^2}} + \frac{1}{4} e^{-\frac{E - \|\mathbf{s}_1\|^2}{4\sigma^2}} \right) \\
&= 4\sigma^2 \cdot \arg \min_{0 \le P_1 \le E_1} \left(e^{-P_1} + e^{-(E_1 - P_1)} \right) \quad (P_1 = \frac{\|\mathbf{s}_1\|^2}{4\sigma^2} \text{ and } E_1 = \frac{E}{4\sigma^2})
\end{aligned}$$

Since $e^{-P_1} + e^{-(E_1 - P_1)}$ is convex, equating its derivative to zero yields $P_1^* = \frac{1}{2}E_1$, which implies the optimal power allocation that minimizes the error rate is $\|\mathbf{s}_1\|^2 = \frac{E}{2}$.