Four corrections to the slides:

- Slide IDC2-39:

$$\begin{aligned} P(\tilde{I}_{2k} \text{ error}) &= P(\tilde{I}_{2k} \text{ correct})P(\tilde{I}_{2k-1} \text{ error}) \\ &\quad + P(\tilde{I}_{2k} \text{ error})P(\tilde{I}_{2k-1} \text{ correct}) \end{aligned}$$

  should be

$$\begin{aligned} P(I_{2k} \text{ error}) &= P(\tilde{I}_{2k} \text{ correct})P(\tilde{I}_{2k-1} \text{ error}) \\ &\quad + P(\tilde{I}_{2k} \text{ error})P(\tilde{I}_{2k-1} \text{ correct}) \end{aligned}$$

- Slides IDC2-46 and IDC2-52:

$$\begin{aligned} s_{\text{GMSK}}(t) &= \sqrt{\frac{2E_b}{T_b}} \sum_{\ell=0}^{\infty} \big[a_{2\ell-1}(t) \cdot g(t-(2\ell-1)T_b) \cdot \cos(2\pi f_c t) \\ &\quad - a_{2\ell}(t) \cdot g(t-2\ell T_b) \cdot \sin(2\pi f_c t) \big] \end{aligned}$$

  is better "symbolized" as

$$\begin{aligned} s_{\text{GMSK}}(t) &= \sqrt{\frac{2E_b}{T_b}} \sum_{\ell=0}^{\infty} \big[a_{2\ell-1}(t) \star g(t-(2\ell-1)T_b) \cdot \cos(2\pi f_c t) \\ &\quad - a_{2\ell}(t) \star g(t-2\ell T_b) \cdot \sin(2\pi f_c t) \big], \end{aligned}$$

  where "$\star$" is the convolution operation. You may refer to the following paper about how GMSK is realized.

  [1] Mitsuru Ishizuka and Kenkichi Hirade, "Optimum Gaussian filter and deviated-frequency-locking scheme for coherent detection of MSK," *IEEE Trans. Comm.*, vol. COM-28, no. 6, pp. 850-857, June 1980.

- Slide IDC2-72: Remove "(coherent)".

- Slide IDC2-74: Replace "coherent matched filter" with "quadratic receiver using matched filter".

1. Prove the following statements.

   (a) Find all solutions of real-valued x and h for the identity $e^{jx\pi h} = xe^{j\pi h}$.

   (b) Suppose $I_n \in \{\pm 1\}$. Prove that $\{J_n \triangleq \prod_{k=0}^{n} I_k\}$ is uniform i.i.d. if $\{I_k\}$ is uniform i.i.d.

   (c) Suppose we have received

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} s_{1,k} \\ s_{2,k} \\ s_{3,k} \\ s_{4,k} \end{bmatrix}}_{\mathbf{s}_k} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}}_{\mathbf{w}},$$

where

$$\begin{bmatrix} s_{1,k} \\ s_{2,k} \\ s_{3,k} \\ s_{4,k} \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{E} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{k=1} \text{ or } \underbrace{\begin{bmatrix} 0 \\ \sqrt{E} \\ 0 \\ 0 \end{bmatrix}}_{k=2} \text{ or } \underbrace{\begin{bmatrix} 0 \\ 0 \\ \sqrt{E} \\ 0 \end{bmatrix}}_{k=3} \text{ or } \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{E} \end{bmatrix}}_{k=4} \text{ with equal probability,}$$

and w_1, w_2, w_3 and w_4 are zero-mean i.i.d. Gaussian with variance σ^2. Under this setting, the optimal decision rule is

$$\text{decision} = \arg \max_{1 \le k \le 4} f(\mathbf{x}|\mathbf{s}).$$

Prove that we can simplify the decision rule to

$$\text{decision} = \arg \max_{1 \le k \le 4} \prod_{i=1}^{4} e^{s_{i,k} x_i / \sigma^2} = \arg \max_{1 \le k \le 4} \exp\left\{ \frac{1}{\sigma^2} \sum_{k=1}^{4} s_{i,k} x_i \right\}.$$

Note: This sub-problem is a supplement to Slide IDC2-66.

Solution.

(a) We first solve x. Since $e^{jx\pi h} = xe^{j\pi h}$, we have $|e^{jx\pi h}| = |x|\,|e^{j\pi h}|$, which implies $|x| = 1$, i.e., $x = \pm 1$. We next solve h. When $x = 1$, the identity holds for every real h. When $x = -1$, we have

$$e^{-j\pi h} = -e^{j\pi h} \Rightarrow \begin{cases} \cos(\pi h) = -\cos(\pi h) \\ -\sin(\pi h) = -\sin(\pi h) \end{cases} \Rightarrow h = \pm \frac{(2\ell - 1)}{2} \text{ for integer } \ell.$$

Note: When $h = \pm\frac{(2\ell-1)}{2}$ for integer ℓ, the identity $e^{jx\pi h} = xe^{j\pi h}$ holds exactly for two values of x, i.e., $x = \pm 1$. We can thus apply this identity in the design of binary (digital) transmission as we have mentioned in class.

(b) First, we note that

$$\Pr(J_n = j_n | J_0 = j_0, J_1 = j_1, \ldots, J_{n-1} = j_{n-1}) = \Pr(I_n = j_n j_{n-1}) = \frac{1}{2}$$

remains equal to constant $1/2$, regardless of the values of j_0, j_1, $\ldots$, j_{n-1}. Hence, J_n is independent of J_0, J_1, $\ldots$, J_{n-1}. Secondly, uniformity of J_n can be proved by

$$\begin{aligned}
\Pr(J_n = j_n) &= \sum_{(j_0,\ldots,j_{n-1})\in\{\pm1\}^n} \Pr(J_0 = j_0, J_1 = j_1, \ldots, J_{n-1} = j_{n-1}) \\
&\qquad \times \Pr(J_n = j_n | J_0 = j_0, J_1 = j_1, \ldots, J_{n-1} = j_{n-1}) \\
&= \sum_{(j_0,\ldots,j_{n-1})\in\{\pm1\}^n} \Pr(J_0 = j_0, J_1 = j_1, \ldots, J_{n-1} = j_{n-1}) \times \frac{1}{2} \\
&= \frac{1}{2} \sum_{(j_0,\ldots,j_{n-1})\in\{\pm1\}^n} \Pr(J_0 = j_0, J_1 = j_1, \ldots, J_{n-1} = j_{n-1}) \\
&= \frac{1}{2}
\end{aligned}$$

for $j_n \in \{\pm1\}$.

Note: As long as $\{I_n\}$ is uniform i.i.d., $\{J_n\}$ is uniform i.i.d. Hence, $\sum_n I_n g(t - nT)$ and $\sum_n J_n g(t - nT)$ have exactly the same time-averaged PSD. This gives us the freedom of using $\{J_n\}$ in place of $\{I_n\}$ in the communications system when necessary. See Slide IDC2-41.

(c) Since w_1, w_2, w_3 and w_4 are independent, we have

$$\begin{aligned}
f(\mathbf{x}|\mathbf{s}_k) &= \prod_{i=1}^{4} f(x_i | s_{i,k}) \\
&= \prod_{i=1}^{4} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i - s_{i,k})^2}{2\sigma^2}\right\} \\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^4 \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{4}(x_i - s_{i,k})^2\right\} \\
&= \frac{1}{4\pi^2\sigma^4} \exp\left\{-\frac{1}{2\sigma^2}\left(\underbrace{\sum_{i=1}^{4} x_i^2}_{\text{irrelevant to } k} - 2\sum_{i=1}^{4} x_i s_{i,k} + \underbrace{\sum_{i=1}^{4} s_{i,k}^2}_{=E \text{ for every } k}\right)\right\} \\
&= \underbrace{\frac{1}{4\pi^2\sigma^4} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{4} x_i^2 + E\right)\right\}}_{\text{irrelevant to } k} \exp\left\{\frac{1}{\sigma^2}\sum_{i=1}^{4} x_i s_{i,k}\right\}
\end{aligned}$$

Hence, we can remove the multiplicative factor that is irrelevant to k and obtain

$$\text{decision} = \arg\max_{1\le k\le 4} f(\mathbf{x}|\mathbf{s}) = \arg\max_{1\le k\le 4} \exp\left\{\frac{1}{\sigma^2}\sum_{i=1}^{4} x_i s_{i,k}\right\}.$$

2. (This problem and the next problem will show you that the FSK formulation selected in the very beginning will have a fundamental impact on the final design. Each FSK formulation has its own advantage, and which one to use depends on the applications.)

Consider a binary FSK signaling scheme defined by

$$s(t) = \sum_{k=-\infty}^{\infty} g(t - kT_b) \cos\left(2\pi f_c t + I_k \frac{\pi h}{T_b} t\right), \tag{1}$$

where $I_k \in \{\pm 1\}$ and $g(t) = \begin{cases} 1, & 0 \le t < T_b; \\ 0, & \text{otherwsie} \end{cases}$. Suppose $f_1 = 1.81$ MHz and $f_2 = 1.79$ MHz.

(a) Give an example of f_c, h and T_b that satisfy (1). Is the value of f_c unique? Is the value of h unique? Is the value of T_b unique?

(b) If we require

(i) memorylessness,

(ii) phase continuity and

(iii) coherent orthogonality between two waveforms corresponding to f_1 and f_2 over $[nT_b, (n+1)T_b)$ for all integers n,

what are the conditions for h and $f_c T_b$?

Hint: Discontinuity can only occur at $t = nT_b$; thus, we require

$$\lim_{t \uparrow nT_b} s(t) = \lim_{t \downarrow nT_b} s(t)$$

for arbitrary I_{n-1} and I_n and for arbitrary integer n.

(c) Check whether Sunde's FSK described in Slide IDC2-4 fulfills the condition in (b).

(d) In modern communications, only $f_c \gg 1/T_b$ is guaranteed, while $2f_c T_b$ may not be an integer. Also, coherent orthogonality is actually not as important as phase-continuity. When only phase-continuity is required and $2f_c T_b$ is not an integer, what is the largest transmission rate R_b (bits per second) attainable under the FSK formulation in (1)?

Solution.

(a)

$$\cos\left(2\pi f_c t + I_k \frac{\pi h}{T_b} t\right) = \cos\left(2\pi \left[f_c + I_k \frac{h}{2T_b}\right] t\right),$$

implies that $f_1 = f_c + \frac{h}{2T_b}$ and $f_2 = f_c - \frac{h}{2T_b}$. Thus, summing and subtracting the two above equations yield $f_1 + f_2 = 2f_c$ and $f_1 - f_2 = h/T_b$. As a result, $f_c = 1.8$ MHz is unique; but, we have $h/T_b = 20$ KHz, and hence, h and T_b can be adjusted (e.g., $h = 1$ and $T_b = 50$ μs).

Note: h/T_b is a constant. Hence, a smaller h gives a smaller T_b, which in turns gives a higher transmission rate $R_b = 1/T_b$.

(b) The FSK formulation in (1) guarantees a memoryless BFSK; hence, no constraint on h and $f_c T_b$ is devised from memoryless requirement.

For phase-continuity requirement, we observe that discontinuity can only occur at $t = nT_b$; thus, we require

$$\lim_{t \uparrow nT_b} s(t) = \lim_{t \downarrow nT_b} s(t).$$

Derive

$$\begin{aligned}
\lim_{t \uparrow nT_b} s(t) &= \lim_{t \uparrow nT_b} \sum_{k=-\infty}^{\infty} g(t - kT_b) \cos\left(2\pi f_c t + I_k \frac{\pi h}{T_b} t\right) \\
&= \cos(2\pi f_c n T_b + I_{n-1} \pi h n) \quad \text{(i.e., } k = n-1 \text{ when } (n-1)T_b \le t < nT_b)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t \downarrow nT_b} s(t) = s(nT_b) &= \sum_{k=-\infty}^{\infty} g(nT_b - kT_b) \cos\left(2\pi f_c n T_b + I_k \frac{\pi h}{T_b} n T_b\right) \\
&= \cos(2\pi f_c n T_b + I_n \pi h n) \quad \text{(i.e., } k = n).
\end{aligned}$$

As a result, phase-continuity requires

$$\begin{aligned}
& (\forall\ I_{n-1}, I_n \in \{\pm 1\})\ \cos(2\pi f_c n T_b + I_{n-1}\pi h n) = \cos(2\pi f_c n T_b + I_n \pi h n) \\
\Leftrightarrow\ & (\forall\ I_{n-1}, I_n \in \{\pm 1\})\ \cos(2\pi f_c n T_b)\cos(I_{n-1}\pi h n) - \sin(2\pi f_c n T_b)\sin(I_{n-1}\pi h n) \\
& \qquad = \cos(2\pi f_c n T_b)\cos(I_n \pi h n) - \sin(2\pi f_c n T_b)\sin(I_n \pi h n)
\end{aligned}$$

for every integer n, which is equivalent to

$$\sin(2\pi f_c n T_b)\sin(\pi h n) = 0$$

for every integer n. A sufficient and necessary condition for phase continuity is

$$2\pi f_c n T_b \bmod \pi = 0 \text{ or } \pi h n \bmod \pi = 0 \text{ for every integer } n,$$

i.e.,

$$2 f_c T_b \text{ is an integer or } h \text{ is an integer.}$$

With this condition in mind, coherent orthogonality requires

$$\begin{aligned}
& \int_{nT_b}^{(n+1)T_b} \cos\left(2\pi f_c t + \frac{\pi h}{T_b} t\right) \cos\left(2\pi f_c t - \frac{\pi h}{T_b} t\right) dt = 0 \\
\Leftrightarrow\ & \int_{nT_b}^{(n+1)T_b} \cos(4\pi f_c t)\, dt + \int_{nT_b}^{(n+1)T_b} \cos\left(2\frac{\pi h}{T_b} t\right) dt = 0 \\
\Leftrightarrow\ & \frac{1}{4\pi f_c} \sin(4\pi f_c t)\Big|_{nT_b}^{(n+1)T_b} + \frac{T_b}{2\pi h} \sin\left(2\frac{\pi h}{T_b} t\right)\Big|_{nT_b}^{(n+1)T_b} = 0 \\
\Leftrightarrow\ & \frac{1}{4\pi f_c}\left(\sin(4\pi f_c (n+1) T_b) - \sin(4\pi f_c n T_b)\right) \\
& + \frac{T_b}{4\pi h}\left(\sin(2\pi h(n+1)) - \sin(2\pi h n)\right) = 0 \text{ for every integer } n \\
\Leftrightarrow\ & \begin{cases} \sin(2\pi h(n+1)) = \sin(2\pi h n), & \text{if } 2 f_c T_b \text{ integer} \\ \sin(4\pi f_c (n+1) T_b) = \sin(4\pi f_c n T_b), & \text{if } h \text{ integer} \end{cases} \text{ for every integer } n \\
\Leftrightarrow\ & \begin{cases} 2h \text{ integer}, & \text{if } 2 f_c T_b \text{ integer} \\ 2 f_c T_b \text{ integer}, & \text{if } h \text{ integer} \end{cases}
\end{aligned}$$

To sum up, the three requirements dictate that both $2h$ and $2T_b f_c$ are integers.

(c) In Sunde's FSK, $f_1 = k/T_b$ and $f_2 = \ell/T_b$ are both multiples of $1/T_b$. We confirm that $2f_cT_b = (f_1 + f_2)T_b = k + \ell$ is an integer, and $2h = 2(f_1 - f_2)T_b = 2(k - \ell)$ is an integer. Hence, the three requirements are all satisfied.

(d) As mentioned in the solution of (a), we shall make h as small as possible (in order to approach the largest transmission rate). In the solution of (b), we obtain that phase-continuity dictates

$$2f_cT_b \text{ is an integer or } h \text{ is an integer.}$$

As $2f_cT_b$ is not an integer, the smallest h attainable is $h = 1$. As such, $h/T_b = hR_b =$ 20 KHz implies $R_{b,\max} = 20/h = 20$ Kbps.

Note: When $f_c \gg \frac{1}{T_b}$, we have

$$\int_{nT_b}^{(n+1)T_b} \cos(4\pi f_c t)\, dt \approx 0.$$

Thus, setting $h = 1$ still results in coherent near-orthogonality. As a result, at $R_b = 20$ Kbps, the FSK formulation in (1) can fulfill memorylessness, phase-continuity and coherent near-orthogonality.

3. Consider a binary FSK signaling scheme defined by

$$s(t) = \sum_{n=-\infty}^{\infty} g(t - nT_b) \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t - nT_b}{T_b}\right)\right), \tag{2}$$

where $I_n \in \{\pm 1\}$ and $g(t) = \begin{cases} 1, & 0 \le t < T_b; \\ 0, & \text{otherwsie} \end{cases}$. Suppose $f_1 = 1.81$ MHz and $f_2 = 1.79$ MHz.

(a) Give an example of f_c, h and T_b that satisfy (2). Is the value of f_c unique? Is the value of h unique? Is the value of T_b unique?

(b) If we require

(i) memorylessness,

(ii) phase continuity and

(iii) coherent orthogonality between two waveforms corresponding to f_1 and f_2 over $[nT_b, (n+1)T_b)$ for all integers n,

what are the conditions for h and f_cT_b?

Hint: Discontinuity can only occur at $t = nT_b$; thus, we require

$$\lim_{t \uparrow nT_b} s(t) = \lim_{t \downarrow nT_b} s(t)$$

for arbitrary I_{n-1} and I_n and for arbitrary integer n.

(c) In modern communications, only $f_c \gg 1/T_b$ is guaranteed, while $2f_cT_b$ may not be an integer. Also, coherent orthogonality is actually not as important as phase-continuity. When only phase-continuity is required and $2f_cT_b$ is not an integer, what is the largest transmission rate R_b (bits per second) attainable under the FSK formulation in (2)?

Solution.

(a)

$$\begin{aligned}
&\cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t-nT_b}{T_b}\right)\right) \\
&\quad = \cos\left(2\pi\left[f_c + I_n \frac{h}{2T_b}\right] t + \sum_{k=-\infty}^{n-1} I_k \pi h - n I_n \pi h\right),
\end{aligned}$$

implies that $f_1 = f_c + \frac{h}{2T_b}$ and $f_2 = f_c - \frac{h}{2T_b}$. Thus, summing and subtracting the two above equations yield $f_1 + f_2 = 2f_c$ and $f_1 - f_2 = h/T_b$. As a result, $f_c = 1.8$ MHz is unique; but, we have $h/T_b = 20$ KHz, and hence, h and T_b can be adjusted (e.g., $h = 1$ and $T_b = 50$ μs).

(b) Memorylessness requires $s(t)$ to be independent of $\{I_k\}_{k=-\infty}^{n-1}$ for $nT_b \le t < (n+1)T_b$. In this period of t, we have

$$\begin{aligned}
s(t) &= \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t-nT_b}{T_b}\right)\right) \\
&= \cos\left(2\pi\left[f_c + I_n \frac{h}{2T_b}\right] t + \sum_{k=-\infty}^{n-1} I_k \pi h - I_n \pi h n\right).
\end{aligned}$$

Accordingly, memorylessness implies $\left(\sum_{k=-\infty}^{n-1} I_k \pi h\right) \bmod 2\pi =$ constant for arbitrary $\sum_{k=-\infty}^{n-1} I_k$, which gives that h must be an even integer.

For phase-continuity, we observe that discontinuity can only occur at $t = \ell T_b$; thus, we require $\lim_{t\uparrow \ell T_b} s(t) = \lim_{t\downarrow \ell T_b} s(t)$. Derive

$$\begin{aligned}
&\lim_{t\uparrow \ell T_b} s(t) \\
&= \lim_{t\uparrow \ell T_b} \sum_{n=-\infty}^{\infty} g(t-nT_b) \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t-nT_b}{T_b}\right)\right) \\
&= \cos\left(2\pi f_c \ell T_b + \sum_{k=-\infty}^{\ell-2} I_k \pi h + I_{\ell-1} \pi h \left(\frac{\ell T_b - (\ell-1)T_b}{T_b}\right)\right) \qquad \text{(i.e., } n = \ell - 1\text{)} \\
&= \cos\left(2\pi f_c \ell T_b + \sum_{k=-\infty}^{\ell-1} I_k \pi h\right),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\downarrow \ell T_b} s(t) = s(\ell T_b) \\
&\quad = \sum_{n=-\infty}^{\infty} g(t-nT_b) \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t-nT_b}{T_b}\right)\right)\Bigg|_{t=\ell T_b} \\
&\quad = \cos\left(2\pi f_c \ell T_b + \sum_{k=-\infty}^{\ell-1} I_k \pi h\right) \qquad \text{(i.e., } n = \ell\text{)}.
\end{aligned}$$

Thus, no condition is imposed upon h and f_cT_b from phase-continuity requirement. (Note: This is an anticipated result because the formulation in (2) is a CPFSK signaling scheme.)

From Slide IDC2-24, we obtain

$$\begin{aligned}
&\int_{nT_b}^{(n+1)T_b} \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + \pi h\left(\frac{t-nT_b}{T_b}\right)\right) \\
&\qquad \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h - \pi h\left(\frac{t-nT_b}{T_b}\right)\right) dt = 0 \\
&\Leftrightarrow \int_{nT_b}^{(n+1)T_b} \cos\left(4\pi f_c t + 2\sum_{k=-\infty}^{n-1} I_k \pi h\right) dt + \int_{nT_b}^{(n+1)T_b} \cos\left(2\pi h\left(\frac{t-nT_b}{T_b}\right)\right) dt = 0 \\
&\Leftrightarrow \frac{1}{4\pi f_c}\left(\sin\left(4\pi f_c(n+1)T_b + 2\sum_{k=-\infty}^{n-1} I_k \pi h\right) - \sin\left(4\pi f_c nT_b + 2\sum_{k=-\infty}^{n-1} I_k \pi h\right)\right) \\
&\qquad + \frac{T_b}{2\pi h}\sin(2\pi h) = 0
\end{aligned}$$

Thus, if $2h$ is an integer, then coherent orthogonality requires $\sin(4\pi f_c(n+1)T_b) = \sin(4\pi f_c nT_b)$, i.e., $2f_cT_b$ must be an integer.

To sum up, memorylessness, phase-continuity and coherent orthogonality can be secured by setting both $h/2$ and $2f_cT_b$ to be an integer.

(c) Since phase-continuity is guaranteed straightforwardly by the formulation in (2), we can make h arbitrarily small. Hence, $R_b = 20/h$ Kbps can approach infinity (at the price of coherent non-orthogonality).

Note: When $f_c \gg \frac{1}{T_b}$, we have

$$\int_{nT_b}^{(n+1)T_b} \cos\left(4\pi f_c t + 2\sum_{k=-\infty}^{n-1} I_k \pi h\right) dt \approx 0.$$

Thus, taking $h = 1/2$ doubles the transmission rate but still maintains coherent orthogonality as $\langle \phi_1(t), \phi_2(t)\rangle = \frac{\sin(2\pi h)}{2\pi h} = \text{sinc}(2h) = 0$. We can further increase the transmission rate by making h smaller than $1/2$ at the price of a higher error performance due to coherent non-orthogonality.

For example, in the bluetooth standard, binary GFSK is adopted. The *frequency deviation* range $f_d \triangleq (f_1 - f_2)/2 = f_1 - f_c = f_c - f_2$ of the bluetooth standard is between 140 KHz and 175 KHz. The symbol rate of 1 mega symbols per second (Ms/s) corresponds to a data rate of 1 Mb/s, which results in the *modulation index* h between $0.28 = \frac{2f_{d,\min}}{1/T_b} = \frac{2\times 140 \text{ KHz}}{1Mbps}$ and $0.35 = \frac{2f_{d,\max}}{1/T_b} = \frac{2\times 175 \text{ KHz}}{1Mbps}$.

Note that at the bottom line of page 387, the textbook names the quantity h as the *deviation ratio*. But usually, it is referred to as the *modulation index* (See Section 3.3-1 of the book *Digital Communications* by Proakis and Salehi, or see the bluetooth specification). In terminologies, f_d is generally referred to as the *peak frequency deviation* (or simply *frequency deviation*), and $f_1 - f_2 = 2f_d$ is called the *frequency separation*. Also, in the textbook, the Gaussian filter in GMSK is truncated at $t = \pm 2.5T_b$; but in *Digital Communcations* (bottom line of p. 118), the suggested truncation is perfomed at $t = \pm 1.5T_b$. Thus, the truncation window is an engineering choice.

4. Suppose we transmit either $\sqrt{E_b}\phi_1(t)$ or $\sqrt{E_b}\phi_2(t)$ with equal probability; however, $\phi_1(t)$ and $\phi_2(t)$ are no longer orthogonal (perhaps due to setting modulation index $h = 0.28$ to 0.35). Let $\langle \phi_1(t), \phi_1(t)\rangle = \langle \phi_2(t), \phi_2(t)\rangle = 1$, and $w(t)$ is the additive white Gaussian noise process with two-sided PSD $N_0/2$. Express the optimal error performance as a function of $\langle \phi_1(t), \phi_2(t)\rangle$, provided $|\langle \phi_1(t), \phi_2(t)\rangle| < 1$.

   Note: Here, all functions (i.e., $\phi_1(t)$, $\phi_2(t)$ and $w(t)$) are real-valued and the receiver knows $\phi_1(t)$ and $\phi_2(t)$ and hence has the knowledge of $\langle \phi_1(t), \phi_2(t)\rangle$.

   Solution. For convenience, denote $\alpha = \langle \phi_1(t), \phi_2(t)\rangle$.

   Based on the Gram-Schmidt procedure, we let $\psi_1(t) = \phi_1(t)$ and

$$\psi_2(t) = \frac{\phi_2(t) - \langle \phi_2(t), \psi_1(t)\rangle\psi_1(t)}{\sqrt{1 - (\langle \phi_1(t), \phi_2(t)\rangle)^2}} = \frac{\phi_2(t) - \alpha\phi_1(t)}{\sqrt{1-\alpha^2}}.$$

   Then, the transmitter transmits either $\sqrt{E_b}\psi_1(t)$ or

$$\sqrt{E_b}\alpha\psi_1(t) + \sqrt{E_b(1-\alpha^2)}\psi_2(t).$$

   The optimal receiver will perform

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \langle x(t), \psi_1(t)\rangle \\ \langle x(t), \psi_2(t)\rangle \end{bmatrix} = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} \sqrt{E_b}\alpha \\ \sqrt{E_b(1-\alpha^2)} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

   where w_1 and w_2 are zero-mean independent Gaussian distributed with variance $\sigma^2 = \frac{N_0}{2}$. The optimal decision rule is

$$y = \sqrt{1-\alpha^2}x_1 - (1+\alpha)x_2 \quad \underset{\sqrt{E_b}\phi_1(t) \text{ is transmitted}}{\overset{\sqrt{E_b}\phi_2(t) \text{ is trasmitted}}{\lessgtr}} \quad 0$$

   Since

$$y = \left(\sqrt{(1-\alpha^2)E_b} \text{ or } -\sqrt{(1-\alpha^2)E_b}\right) + w$$

   with w zero-mean Gaussian of variance $(1+\alpha)N_0$, the error probability is equal to[1]

$$\Phi\left(-\sqrt{\frac{(1-\alpha^2)E_b}{(1+\alpha)N_0}}\right) = \Phi\left(-\sqrt{(1-\alpha)\frac{E_b}{N_0}}\right).$$

   Note: For CPFSK, $\langle \phi_1(t), \phi_2(t)\rangle = \text{sinc}(2h)$. Hence, the error rate, as a function of the modulation index h, is

$$\Phi\left(-\sqrt{(1-\text{sinc}(2h))\frac{E_b}{N_0}}\right).$$

   The performance loss, in comparison with orthogonal MSK, is $-10\log_{10}(1 - \text{sinc}(2h))$ dB. For the bluetooth standard that adopts $h = 0.28$ to 0.35, the performance loss is around 2.53 dB to 4.34 dB (but we gain a better transmission rate).

[1] Recall that given $y = \pm\sqrt{E_b} + w$ and w being zero-mean Gaussian distributed with variance σ^2, the error rate is $\Phi(-\sqrt{E_b/\sigma^2})$.