Three corrections and one supplement to slides

- IDC1-67: $\boldsymbol{s}_i = \begin{bmatrix} a_k\sqrt{E_0} \\ b_k\sqrt{E_0} \end{bmatrix}$ should be $\boldsymbol{s}_k = \begin{bmatrix} a_k\sqrt{E_0} \\ b_k\sqrt{E_0} \end{bmatrix}$.

- IDC2-15:

$$\begin{aligned} \bar{S}_B &= \bar{S}_{B,g_I} + \bar{S}_{B,g_Q}(f) \\ &= \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right)\right] + \frac{8E_b\cos^2(\pi T_b f)}{\pi^2(1-4_b^2 f^2)^2} \end{aligned}$$

  should be

$$\begin{aligned} \bar{S}_B(f) &= \bar{S}_{B,g_I}(f) + \bar{S}_{B,g_Q}(f) \\ &= \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right)\right] + \frac{8E_b\cos^2(\pi T_b f)}{\pi^2(1-4_b^2 f^2)^2} \end{aligned}$$

- IDC2-31:

$$\begin{cases} J_{2\ell}\sin\left(\frac{\pi}{2}\left[\left(\frac{t-2\ell T_b}{T_b}\right) + 2\ell\right]\right), & I_n = 1 \\ -J_{2\ell}\sin\left(\frac{\pi}{2}\left[-\left(\frac{t-2\ell T_b}{T_b}\right) + 2\ell + 2\right]\right), & I_n = -1 \end{cases}$$

  should be

$$\begin{cases} J_{2\ell}\sin\left(\frac{\pi}{2}\left[\left(\frac{t-2\ell T_b}{T_b}\right) + 2\ell\right]\right), & I_n = 1 \\ J_{2\ell}\sin\left(\frac{\pi}{2}\left[-\left(\frac{t-2\ell T_b}{T_b}\right) + 2\ell + 2\right]\right), & I_n = -1 \end{cases}$$

- Sample Problem 1(c) for Quiz 1 & Slide IDC1-11: Without prior knowledge on the wide-sense stationarity of $s(t)$ and $\tilde{s}(t)$, the general relation in autocorrelation functions are

$$\begin{aligned} &R_{ss}(t+\tau, t) \\ &= \frac{1}{2}[R_{xx}(t+\tau,t) + R_{yy}(t+\tau,t)]\cos(2\pi f_c \tau) \\ &\quad + \frac{1}{2}[R_{xx}(t+\tau,t) - R_{yy}(t+\tau,t)]\cos(2\pi f_c(2t+\tau)) \\ &\quad + \frac{1}{2}[R_{xy}(t+\tau,t) - R_{yx}(t+\tau,t)]\sin(2\pi f_c \tau) \\ &\quad - \frac{1}{2}[R_{xy}(t+\tau,t) + R_{yx}(t+\tau,t)]\sin(2\pi f_c(2t+\tau)) \end{aligned}$$

  As a result, all four cases below are possible.

  i) $s(t)$ WSS and $\tilde{s}(t)$ WSS;

  ii) $s(t)$ WSS and $\tilde{s}(t)$ non-WSS;

  iii) $s(t)$ non-WSS and $\tilde{s}(t)$ WSS, and

  iv) $s(t)$ non-WSS and $\tilde{s}(t)$ nonWSS, Examples for i) and iv) are either given or straightforward. Case iii) occurs when $R_{xx}(\tau) \neq 0$ for some τ but $y(t) = 0$, provided that $\tilde{s}(t) = x(t)$ is real and WSS. A trivial example for ii) can be constructed with $f_c = 0$, under which $s(t) = \mathrm{Re}\{\tilde{s}(t)\} = x(t)$ can be made WSS but $\tilde{s}(t) = x(t) + jy(t)$ is not. Alternatively, we can set $\tilde{s}(t) = e^{-j2\pi f_c t}$; then, $s(t) = \mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\} = 0$ is WSS but $\tilde{s}(t)$ is definitely non-WSS.

Sample Problems for the 2nd Quiz

1. Denote the Hilbert transform operation and the Fourier transform operation by $\mathcal{H}\{\cdot\}$ and $\mathcal{F}\{\cdot\}$, respectively. The inverse transforms of both are denoted by adding superscript "$^{-1}$". Note that the Hilbert transform is only defined for real-valued functions, while the Fourier transform is generally operated over complex domain.

   (a) Prove that if $\hat{p}(t) = \mathcal{H}\{p(t)\}$, then $\hat{p}(t) \star h(t) = \mathcal{H}\{p(t) \star h(t)\}$.
   Hint: $\hat{p}(t) = \mathcal{H}\{p(t)\}$ iff

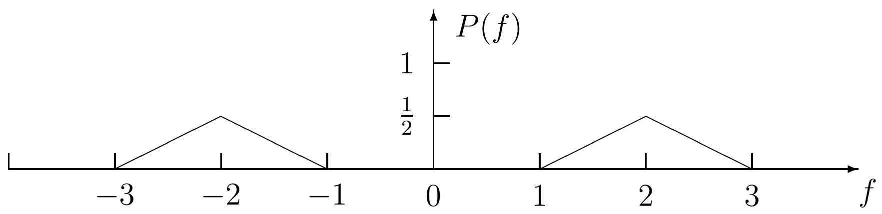
$$\big(j\mathcal{F}\{\hat{p}(t)\} = \big) \quad j\hat{P}(f) = \begin{cases} P(f), & f > 0 \\ -P(f), & f < 0 \end{cases} \quad \left(= \begin{cases} \mathcal{F}\{p(t)\}, & f > 0 \\ -\mathcal{F}\{p(t)\}, & f < 0 \end{cases} \right)$$

   (b) Prove $\hat{p}(t) = \mathcal{H}\{p(t)\}$ is orthogonal to $p(t)$ by using the identity:

$$\langle g_1(t), g_2(t)\rangle = \left(\int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df = \right) \langle G_1(f), G_2(f)\rangle,$$

   where $G_1(f) = \mathcal{F}\{g_1(t)\}$ and $G_2(f) = \mathcal{F}\{g_2(t)\}$.
   Hint: Use te hint in (a) and the fact that $p(t)$ is real.

   (c) Give $P(f) = \mathcal{F}\{p(t)\}$ as follows.



   Let

$$H(f) = \begin{cases} 1, & 2 < |f| < 3 \\ 0, & \text{otherwise} \end{cases}$$

   Plot $j\mathcal{F}\{\mathcal{H}\{p(t) \star h(t)\}\}$.

Solution.

   (a) From the hint, we know that $\underbrace{\hat{p}(t) \star h(t)}_{\hat{g}(t)} = \mathcal{H}\{\underbrace{p(t) \star h(t)}_{g(t)}\}$ iff

$$j\hat{G}(f) = \begin{cases} G(f), & f > 0 \\ -G(f), & f < 0 \end{cases}$$

The above equality holds because

$$j\hat{G}(f) = j\hat{P}(f)H(f) = \begin{cases} P(f)H(f), & f > 0 \\ -P(f)H(f), & f < 0 \end{cases} = \begin{cases} G(f), & f > 0 \\ -G(f), & f < 0 \end{cases}$$

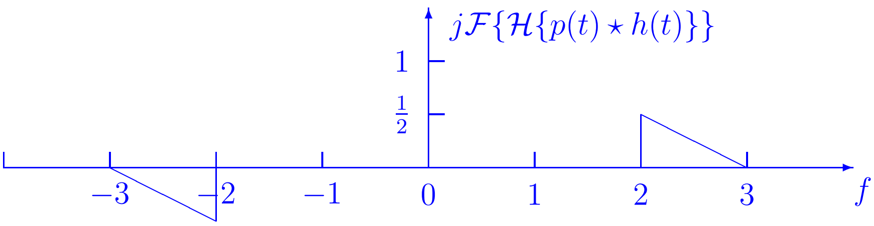
Note: A Hilbert transform pair remains a Hilbert transform pair after passing through the same linear filter.

(b)

$$\begin{aligned} j\langle \hat{p}(t), p(t)\rangle &= j\langle \hat{P}(f), P(f)\rangle \\ &= \int_{-\infty}^{\infty} j\hat{P}(f)P^*(f)df \\ &= \int_{-\infty}^{0} (-P(f))P^*(f)df + \int_{0}^{\infty} P(f)P^*(f)df \\ &= -\int_{-\infty}^{0} |P(f)|^2 df + \int_{0}^{\infty} |P(f)|^2 df \\ &= 0, \end{aligned}$$

where the last equality holds because the the Fourier transform $P(f)$ of real $p(t)$ must satisfy $P(f) = P^*(-f)$.

(c)



2. Find $\mathcal{F}\{\sin(\pi t/T) \cdot \mathbf{1}[0 \le t < T)\}$, where

$$\mathbf{1}[0 \le t < T) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

is the set indicator function.

Hint: $\mathcal{F}\{\sin(2\pi f_0 t)\} = \frac{1}{2j}\left(\delta(f - f_0) - \delta(f + f_0)\right)$ and

$$\mathcal{F}\{\mathbf{1}[0 \le t < T)\} = T\text{sinc}(Tf)e^{-j\pi fT}$$

Solution.

(a)

$$
\begin{aligned}
&\mathcal{F}\{\sin(\pi t/T)\cdot \mathbf{1}[0\le t<T)\}\\
&= \frac{1}{2j}\left[\delta\left(f-\frac{1}{2T}\right)-\delta\left(f+\frac{1}{2T}\right)\right]\star T\mathrm{sinc}(Tf)e^{-j\pi Tf}\\
&= \frac{1}{2j}\left[T\mathrm{sinc}\left(T\left(f-\frac{1}{2T}\right)\right)e^{-j\pi T(f-1/2T)}-T\mathrm{sinc}\left(T\left(f+\frac{1}{2T}\right)\right)e^{-j\pi T(f+1/2T)}\right]\\
&= \frac{Te^{-j\pi Tf}}{2}\left(\mathrm{sinc}\left(Tf-\frac{1}{2}\right)+\mathrm{sinc}\left(Tf+\frac{1}{2}\right)\right)\\
&= \frac{Te^{-j\pi Tf}}{2}\left(\frac{\sin(\pi(Tf-1/2))}{\pi(Tf-1/2)}+\frac{\sin(\pi(Tf+1/2))}{\pi(Tf+1/2)}\right)\\
&= \frac{Te^{-j\pi Tf}}{2\pi}\left(\frac{-\cos(\pi Tf)}{(Tf-1/2)}+\frac{\cos(\pi Tf)}{(Tf+1/2)}\right)\\
&= \frac{T\cos(\pi Tf)e^{-j\pi Tf}}{2\pi}\left(\frac{-1}{(Tf-1/2)}+\frac{1}{(Tf+1/2)}\right)\\
&= \frac{T\cos(\pi Tf)e^{-j\pi Tf}}{2\pi}\frac{1}{(1/4-T^2f^2)}\\
&= \frac{2T\cos(\pi Tf)e^{-j\pi Tf}}{\pi(1-4T^2f^2)}
\end{aligned}
$$

3. Let

$$s(t)=\sum_{k=-\infty}^{\infty} I_k\cdot g(t-kT),$$

where $\{I_k\}_{k=-\infty}^{\infty}$ is a zero-mean, stationary, uncorrelated, complex-valued information sequence, i.e.,

$$E[I_kI_\ell^*]=\begin{cases}\sigma_I^2, & k=\ell;\\ 0, & k\neq\ell,\end{cases}$$

and $g(t)$ is a complex pulse shaping function. Note that we do not require $g(t)=0$ outside $[0,T)$.

(a) Show that the autocorrelation function of $s(t)$ is given by

$$R_{ss}(t+\tau,t)=\sigma_I^2\sum_{k=-\infty}^{\infty} g(t+\tau-kT)g^*(t-kT).$$

Hint:

$$R_{ss}(t+\tau,t) \;=\; E\left[\left(\sum_{k=-\infty}^{\infty} I_k\cdot g(t+\tau-kT)\right)\left(\sum_{\ell=-\infty}^{\infty} I_\ell\cdot g(t-\ell T)\right)^*\right]$$

(b) Show that the time-averaged PSD of $s(t)$ is equal to

$$\bar{S}_{ss}(f)=\frac{\sigma_I^2}{T}|G(f)|^2,$$

where

$$\bar{S}_{ss}(f)=\int_{-\infty}^{\infty}\bar{R}_{ss}(\tau)e^{-j2\pi f\tau}d\tau \quad\text{and}\quad \bar{R}_{ss}(\tau)=\frac{1}{T}\int_0^T R_{ss}(t+\tau,t)dt.$$

(c) Find the time-averaged PSD of the ASK signal

$$s(t) = \sum_{k=-\infty}^{\infty} I_k \cdot \cos(2\pi f_c t) \cdot \pi(t - kT),$$

where $\{I_k\}_{k=-\infty}^{\infty}$ is a zero-mean, stationary, uncorrelated, complex-valued information sequence, T is a multiple of $1/f_c$, and $\pi(t)$ is a pulse shaping function with Fourier transform $\Pi(f)$.

Hint: Express $s(t)$ as $\sum_{k=-\infty}^{\infty} I_k \cdot g(t - kT)$ for some properly chosen $g(t)$ and then use the formula in (b).

Solutions.

(a)

$$\begin{aligned}
R_{ss}(t+\tau, t) &= E\left[\left(\sum_{k=-\infty}^{\infty} I_k \cdot g(t+\tau-kT)\right)\left(\sum_{\ell=-\infty}^{\infty} I_\ell \cdot g(t-\ell T)\right)^*\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} E\left[I_k I_\ell^*\right] \cdot g(t+\tau-kT) g^*(t-\ell T) \\
&= \sigma_I^2 \sum_{k=-\infty}^{\infty} g(t+\tau-kT) g^*(t-\ell T)
\end{aligned}$$

(b)

$$\begin{aligned}
\bar{S}_{ss}(f) &= \int_{-\infty}^{\infty} \bar{R}_{ss}(\tau) e^{-\jmath 2\pi f \tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T} \int_0^T R_{ss}(t+\tau, t) dt \right) e^{-\jmath 2\pi f \tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T} \int_0^T \left[\sigma_I^2 \sum_{k=-\infty}^{\infty} g(t+\tau-kT) g^*(t-kT) \right] dt \right) e^{-\jmath 2\pi f \tau} d\tau \\
&= \frac{\sigma_I^2}{T} \sum_{k=-\infty}^{\infty} \int_0^T \left(\int_{-\infty}^{\infty} g(t+\tau-kT) e^{-\jmath 2\pi f \tau} d\tau \right) g^*(t-kT) dt \\
& \quad \text{(Let } s = t+\tau-kT.\text{)} \\
&= \frac{\sigma_I^2}{T} \sum_{k=-\infty}^{\infty} \int_0^T \left(\int_{-\infty}^{\infty} g(s) e^{-\jmath 2\pi f (s-t+kT)} ds \right) g^*(t-kT) dt \\
&= \frac{\sigma_I^2}{T} \sum_{k=-\infty}^{\infty} \int_0^T \left(\int_{-\infty}^{\infty} g(s) e^{-\jmath 2\pi f s} ds \right) g^*(t-kT) e^{\jmath 2\pi f (t-kT)} dt \\
&= \frac{\sigma_I^2}{T} G(f) \sum_{k=-\infty}^{\infty} \int_0^T g^*(t-kT) e^{\jmath 2\pi f (t-kT)} dt \\
& \quad \text{(Let } u = t-kT.\text{)} \\
&= \frac{\sigma_I^2}{T} G(f) \sum_{k=-\infty}^{\infty} \int_{-kT}^{(1-k)T} g^*(u) e^{\jmath 2\pi f u} dt \\
&= \frac{\sigma_I^2}{T} G(f) \left(\int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f u} dt \right)^* \\
&= \frac{\sigma_I^2}{T} G(f) G^*(f) \\
&= \frac{\sigma_I^2}{T} |G(f)|^2
\end{aligned}$$

Note: For general zero-mean stationary uncorrelated complex $\{I_k\}$ and complex $g(t)$, we have $S_{ss}(f) = \frac{\sigma_I^2}{T}|G(f)|^2$.

(c) Since T is a multiple of $1/f_c$, we obtain

$$\begin{aligned}
s(t) &= \sum_{k=-\infty}^{\infty} I_k \cdot \cos(2\pi f_c t) \cdot \pi(t-kT) \\
&= \sum_{k=-\infty}^{\infty} I_k \cdot \underbrace{\cos(2\pi f_c (t-kT)) \cdot \pi(t-kT)}_{g(t-kT)} \\
&= \sum_{k=-\infty}^{\infty} I_k \cdot g(t-kT)
\end{aligned}$$

where $g(t) = \cos(2\pi f_c t) \cdot \pi(t)$. It can be derived that

$$\begin{aligned} G(f) &= \mathcal{F}\{\cos(2\pi f_c t) \cdot \pi(t)\} \\ &= \frac{1}{2}\left(\delta(f - f_c) + \delta(f + f_c)\right) \star \Pi(f) \\ &= \frac{1}{2}\Pi(f - f_c) + \frac{1}{2}\Pi(f + f_c). \end{aligned}$$

Therefore,

$$\bar{S}_{ss}(f) = \frac{\sigma_I^2}{4T}\left|\Pi(f - f_c) + \Pi(f + f_c)\right|^2.$$

Note: Setting $f_c T$ to be an integer makes easy the derivation of the (time-averaged) PSD of $s(t)$.

4. (a) Suppose the channel follows $x(t) = s(t) + w(t)$, where $s(t)$ is equal to either $m_1(t)$ or $m_2(t)$ with equal probability, and $w(t)$ is an additive noise. After the reception of $x(t)$, the detector performs "projection" onto $\phi(t)$-axis, i.e.,

$$\underbrace{\langle x(t), \phi(t)\rangle}_{x} = \underbrace{\langle s(t), \phi(t)\rangle}_{s} + \underbrace{\langle w(t), \phi(t)\rangle}_{w},$$

where s is equal to either $m_1 = \langle m_1(t), \phi(t)\rangle$ or $m_2 = \langle m_2(t), \phi(t)\rangle$ with equal probability, and w is the noise with pdf $f(w)$. Let $m_1 < m_2$. Assume $f(w)$ is symmetric with respect to $w = w_0$ and is strictly decreasing for $w > w_0$. Find the best decision rule (in the sense of minimizing the detection error probability) about the transmitted message (i.e., m_1 or m_2) based upon x.

(b) Suppose $m_1 = -\frac{d}{2}$, $m_2 = \frac{d}{2}$ and $w_0 = 0$. Let w be Gaussian distributed with mean zero and variance σ^2. Show that the error rate in (a) is

$$\Phi\left(-\frac{d}{2\sigma}\right)$$

where $\Phi(\cdot)$ is the cdf of the standard normal.

Hint: $\int_{-\infty}^{r} \mathcal{N}(\mu, \sigma^2) = \Phi\left(\frac{r - \mu}{\sigma}\right)$

Solution.

(a)

$$\begin{aligned} \hat{m} &= \arg\max\{\Pr(x|m_1), \Pr(x|m_2)\} \\ &= \arg\max\{f(x - m_1), f(x - m_2)\} \end{aligned}$$

implies

$$x \underset{m_2}{\overset{m_1}{\lessgtr}} w_0 + \frac{(m_1 + m_2)}{2}.$$

Note: The optimal decision rule for $m_1 = -m_2$ and $w_0 = 0$ is

$$x \underset{m_2}{\overset{m_1}{\lessgtr}} 0.$$

Hence, the above simple rule remains optimal whenever the pdf of w peaks at zero and is strictly decreasing for positive argument. If w does not have zero mean or $m_1 \neq -m_2$, then the threshold needs to be adjusted by the amount of the $\underbrace{\text{mid-point of } s}_{(m_1+m_2)/2}$ and the $\underbrace{\text{median value}}_{w_0}$ of w.

(b)

$$\begin{aligned}
\text{Error rate} &= \Pr\left(s = -\frac{d}{2}\right)\Pr\left(x > 0 \middle| s = -\frac{d}{2}\right) + \Pr\left(s = \frac{d}{2}\right)\Pr\left(x < 0 \middle| s = \frac{d}{2}\right) \\
&= \frac{1}{2}\Pr\left(s + w > 0 \middle| s = -\frac{d}{2}\right) + \frac{1}{2}\Pr\left(s + w < 0 \middle| s = \frac{d}{2}\right) \\
&= \frac{1}{2}\Pr\left(w > \frac{d}{2}\right) + \frac{1}{2}\Pr\left(w < -\frac{d}{2}\right) \\
&= \frac{1}{2}\Pr\left(w < -\frac{d}{2}\right) + \frac{1}{2}\Pr\left(w < -\frac{d}{2}\right) \\
&\quad \text{(By symmetry of the pdf of zero-mean Gaussian)} \\
&= \Pr\left(w < -\frac{d}{2}\right) \\
&= \Phi\left(\frac{-\frac{d}{2} - 0}{\sigma}\right) \\
&= \Phi\left(-\frac{d}{2\sigma}\right)
\end{aligned}$$

Note: For IDC2-6, we have $d = \sqrt{E_b} - (-\sqrt{E_b}) = 2\sqrt{E_b}$ and $\sigma^2 = N_0$. Thus, the error rate is

$$\Phi\left(-\frac{d}{2\sigma}\right) = \Phi\left(-\frac{2\sqrt{E_b}}{2\sqrt{N_0}}\right) = \Phi\left(-\sqrt{\frac{E_b}{N_0}}\right).$$

For IDC1-30, we have $d = 2\sqrt{E_b}$ but $\sigma^2 = \frac{N_0}{2}$. Thus, the error rate

$$\Phi\left(-\frac{d}{2\sigma}\right) = \Phi\left(-\frac{2\sqrt{E_b}}{2\sqrt{N_0/2}}\right) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right).$$

For IDC1-62, we are given that $\sigma^2 = \frac{N_0}{2}$ for varying distances among different pairs of constellation points; thus,

$$\Phi\left(-\frac{d}{2\sigma}\right) = \Phi\left(-\frac{d}{2\sqrt{N_0/2}}\right) = \Phi\left(-\frac{d}{\sqrt{2N_0}}\right)$$

is the basic form for each component.