Sample problems for the 1st quiz

Corrections to slides

- I have sorted out the slides before the lectures, and now use $\phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$ throughouts. So, if you obtain a previous version, you may find some mismatches between your slides and the slides I use in the lectures.

- Slide IDC1-11:

$$R_{xx}(\tau)\,\frac{\cos(2\pi f_c t)+\cos(2\pi f_c(2t+\tau))}{2} + R_{yy}(\tau)\,\frac{\cos(2\pi f_c t)-\cos(2\pi f_c(2t+\tau))}{2}$$
$$-R_{xy}(\tau)\,\frac{\sin(2\pi f_c(2t+\tau))-\sin(2\pi f_c t)}{2} - R_{yx}(\tau)\,\frac{\sin(2\pi f_c(2t+\tau))+\sin(2\pi f_c t)}{2}$$

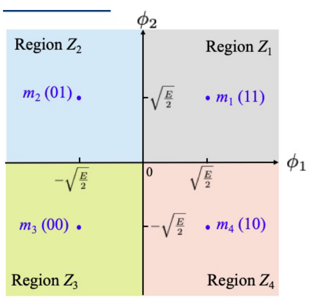
  should be replaced by

$$R_{xx}(\tau)\,\frac{\cos(2\pi f_c \tau)+\cos(2\pi f_c(2t+\tau))}{2} + R_{yy}(\tau)\,\frac{\cos(2\pi f_c \tau)-\cos(2\pi f_c(2t+\tau))}{2}$$
$$-R_{xy}(\tau)\,\frac{\sin(2\pi f_c(2t+\tau))-\sin(2\pi f_c \tau)}{2} - R_{yx}(\tau)\,\frac{\sin(2\pi f_c(2t+\tau))+\sin(2\pi f_c \tau)}{2}$$

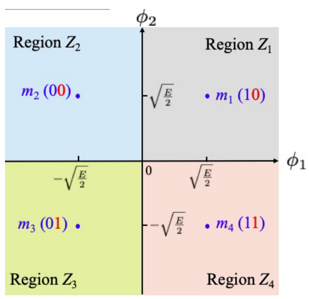
- IDC1-34: $R_{\tilde{s}\tilde{s}}(t+\tau,\tau)$ should be $R_{\tilde{s}\tilde{s}}(t+\tau,t)$.

- IDC1-38: $\begin{cases}\phi_1 = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\\ \phi_2 = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\end{cases}$ should be $\begin{cases}\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\\ \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\end{cases}$

- IDC1-40: In order to be consistent with the figures,



is now replaced by



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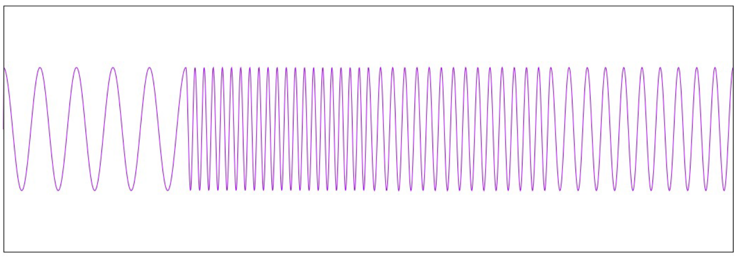
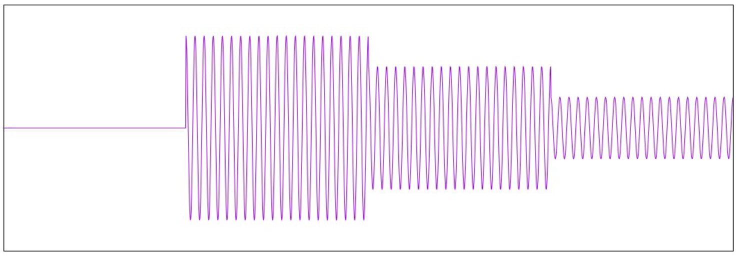
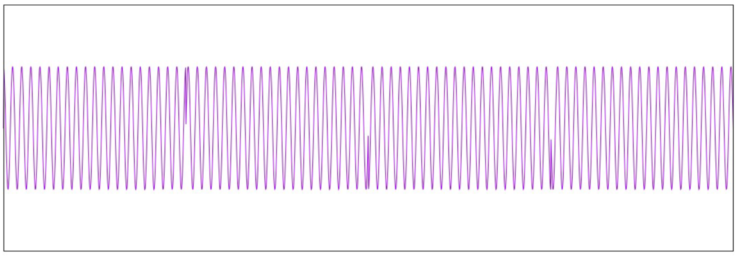
1. Answer the following questions.

   (a) Which of the followings is ASK? Which is PSK? Which is FSK? Note that the digital data to be transmitted here are 00, 11, 10, 01.

   i. 

ii. 

iii. 

(b) Which of the following is the measuring unit for bandwidth efficiency?

i. bit/second/Hz

ii. bps

iii. Watt/Hz

(c) If the autocorrelation functions satisfy

$$R_{xx}(\tau) = f(\tau) \quad \text{and} \quad R_{yy}(\tau) = f(\tau) + 1$$

for some $f(\tau)$, can both

$$\tilde{s}(t) = x(t) + jy(t) \quad \text{and} \quad s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

wide-sense stationary? Justify your answer.

(d) Continue from (c). Given $f_c = 2$, plot the spectrum of $S(f)$ if $\tilde{S}(f)$ is equal to

$$\tilde{S}(f) = \begin{cases} 1 - f, & 0 \le f < 1; \\ 0, & \text{otherwise.} \end{cases}$$

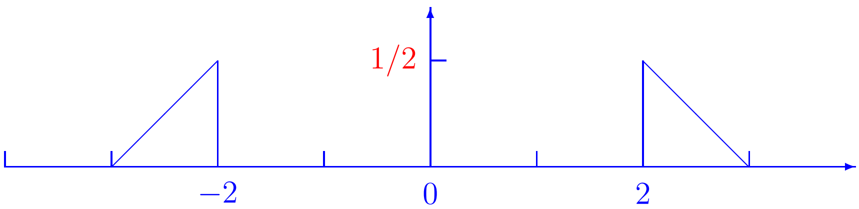
Solution.

(a) (i) is FSK, (ii) is ASK, and (iii) is PSK.

(b) (i)

(c) No because $R_{xx}(\tau)$ must be equal to $R_{yy}(\tau)$. So, in this particular case, only one of $\tilde{s}(t)$ and $s(t)$ can be wide-sense stationary.

(d)



2. Below are two key identities for Fourier analysis.

	$g(t) \rightleftharpoons G(f)$	wordy summary
1.	$g^*(t) \rightleftharpoons G^*(-f)$ $G^*(f) \rightleftharpoons g^*(-t)$	conjugate in one domain = conjugate & mirror in another document
2a.	$g(t)e^{j2\pi f_0 t} \rightleftharpoons G(f-f_0)$	constant phase-increase in t-domain =constant right shift in f-domain
2b.	$G(f)e^{-j2\pi f t_0} \rightleftharpoons g(t-t_0)$	constant phase-decrease in f-domain =constant right shift in t-domain

Use the two identifies to prove the following properties of Fourier transform. Note that "constant phase-increase" and "constant phase-decrease" are usually termed *linear phase* in documents.

(a) $g(t)$ is real iff $G(f)$ is a conjugate even function.

(b) $g(t)$ is pure imaginary iff $G(f)$ is a conjugate odd function.

(c) $G(f)$ is real iff $g(t)$ is a conjugate even function.

(d) $G(f)$ is pure imaginary iff $g(t)$ is a conjugate odd function.

(e) $g(t)$ is real and symmetric iff $G(f)$ is real and symmetric.

(f) For the math relation between passband and baseband,

$$S(f) = \mathcal{F}\{s(t)\} = \mathcal{F}\{\mathrm{Re}[\tilde{s}(t)e^{j2\pi f_c t}]\} = \frac{1}{2}[\tilde{S}(f-f_c) + \tilde{S}^*(-f-f_c)].$$

(g) For the matched filter, $\mathcal{F}^{-1}\{G^*(f)e^{-j2\pi fT}\} = g^*(T-t)$.

Solution.

(a) $g(t)$ real $\Longleftrightarrow g(t) = g^*(t) \overset{\text{by 1.}}{\Longleftrightarrow} G(f) = G^*(-f)$

(b) $g(t)$ imaginary $\Longleftrightarrow g(t) = -g^*(t) \overset{\text{by 1.}}{\Longleftrightarrow} G(f) = -G^*(-f)$

(c) $G(f)$ real $\Longleftrightarrow G(f) = G^*(f) \overset{\text{by 1.}}{\Longleftrightarrow} g(t) = g^*(-t)$

(d) $G(f)$ imaginary $\Longleftrightarrow G(f) = -G^*(f) \overset{\text{by 1.}}{\Longleftrightarrow} g(t) = -g^*(-t)$

(e) $g(t)$ is real and symmetric $\Longleftrightarrow g(t) = g^*(t) = g^*(-t) \overset{\text{by 1.}}{\Longleftrightarrow} G(f) = G^*(-f) = G^*(f)$

(f)

$$\begin{aligned}
S(f) &= \mathcal{F}\{\mathrm{Re}[\tilde{s}(t)e^{j2\pi f_c t}]\} \\
&= \mathcal{F}\left\{\frac{1}{2}\left[\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}^*(t)e^{-j2\pi f_c t}\right]\right\} \quad \text{(Remove non-linear Re}\{\cdot\}\text{ operation)} \\
&= \frac{1}{2}\mathcal{F}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} + \frac{1}{2}\mathcal{F}\left\{\tilde{s}^*(t)e^{-j2\pi f_c t}\right\} \\
&\qquad \text{(Exchange the order of linear "}\textstyle\int\text{" and "+" operations.)} \\
&= \frac{1}{2}\mathcal{F}\left\{\tilde{s}(t)\right\}\big|_{f\to f-f_c} + \frac{1}{2}\mathcal{F}\left\{\tilde{s}^*(t)\right\}\big|_{f\to f+f_c} \quad \text{(By 2.)} \\
&= \frac{1}{2}\,\tilde{S}(f)\Big|_{f\to f-f_c} + \frac{1}{2}\,\tilde{S}^*(-f)\Big|_{f\to f+f_c} \quad \text{(By 1.)} \\
&= \frac{1}{2}\tilde{S}(f-f_c) + \frac{1}{2}\tilde{S}^*(-(f+f_c)) \\
&= \frac{1}{2}\tilde{S}(f-f_c) + \frac{1}{2}\tilde{S}^*(-f-f_c).
\end{aligned}$$

(g)

$$\begin{aligned}
\mathcal{F}^{-1}\{G^*(f)e^{-j2\pi fT}\} &= \mathcal{F}^{-1}\{G^*(f)\}\big|_{t\to t-T} \quad \text{(By 2.)} \\
&= g^*(-t)|_{t\to t-T} \quad \text{(By 1.)} \\
&= g^*(-(t-T)) \\
&= g^*(T-t)
\end{aligned}$$

3. Define the inner product of two signals as

$$\langle \phi_1(t), \phi_2(t)\rangle \triangleq \int_0^T \phi_1(t)\phi_2^*(t)dt.$$

(a) Show that if f_cT is an integer, then

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \text{ and } \phi_2(t) = -\sqrt{\frac{2}{T}}\sin\left(2\pi f_c\left(t - \frac{T}{2}\right)\right)$$

for Offset QPSK are orthogonal.

(b) Show that if $2(f_1+f_2)T$ and $2(f_1-f_2)T$ are both integers, then

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t) \text{ and } \phi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)$$

for FSK are orthogonal.

(c) Show that the projections of a **zero-mean** white noise process $w(t)$ onto two orthonormal basis $\phi_1(t)$ and $\phi_2(t)$ are uncorrelated.
Hint: $R_w(\tau) = E[w(t+\tau)w^*(t)] = \frac{N_0}{2}\delta(\tau)$. In this sub-problem, $w(t)$, $\phi_1(t)$ and $\phi_2(t)$ are generally complex-valued functions.

Solution.

(a)

$$
\begin{aligned}
&\langle \phi_1(t), \phi_2(t)\rangle \\
&= -\int_0^T \underbrace{\sqrt{\frac{2}{T}}\cos(2\pi f_c t)}_{\phi_1(t)} \underbrace{\sqrt{\frac{2}{T}}\sin\left(2\pi f_c\left(t-\frac{T}{2}\right)\right)}_{\phi_2(t)} dt \\
&= -\frac{2}{T}\int_0^T \cos(2\pi f_c t)\sin(2\pi f_c t - \pi f_c T)\, dt \\
&= \begin{cases} \frac{2}{T}\int_0^T \cos(2\pi f_c t)\sin(2\pi f_c t)\, dt, & f_c T \text{ odd} \\ -\frac{2}{T}\int_0^T \cos(2\pi f_c t)\sin(2\pi f_c t)\, dt, & f_c T \text{ even} \end{cases} \\
&= \pm\frac{1}{T}\int_0^T \sin(4\pi f_c t)\, dt \\
&= \pm\frac{1-\cos(4\pi f_c T)}{4\pi f_c T} \\
&= 0.
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\langle \phi_1(t), \phi_2(t)\rangle \\
&= \frac{2}{T}\int_0^T \cos(2\pi f_1 t)\cos(2\pi f_2 t)\, dt \\
&= \frac{2}{T}\int_0^T \frac{\cos(2\pi(f_1+f_2)t) + \cos(2\pi(f_1-f_2)t)}{2} dt \\
&= \frac{\sin(2\pi(f_1+f_2)T)}{2\pi(f_1+f_2)T} + \frac{\sin(2\pi(f_1-f_2)T)}{2\pi(f_1-f_2)T} \\
&= 0
\end{aligned}
$$

Note: The smallest positive integer (for $2(f_1-f_2)T$ to be equal to) is 1; as a result, the minimum shift that makes $\phi_1(t)$ and $\phi_2(t)$ orthogonal is $f_1 - f_2 = \frac{1}{2T}$, which implies $f_1 = \frac{k+1}{4T}$ and $f_2 = \frac{k-1}{4T}$ for integer k.

However, if both f_1 and f_2 are required to be multiples of $\frac{1}{T}$, then the smallest positive integer that is equal to $2(f_1-f_2)T$ becomes 2; as a result, the minimum shift that makes $\phi_1(t)$ and $\phi_2(t)$ orthogonal becomes $f_1 - f_2 = \frac{1}{T}$, which implies $f_1 = \frac{k+2}{4T}$ and $f_2 = \frac{k-2}{4T}$ for $k = 2, 6, 10, \ldots$ (cf. Slide IDC1-22).

(c) Let $w_1 = \langle w(t), \phi_1(t)\rangle$ and $w_2 = \langle w(t), \phi_2(t)\rangle$. By definition, w_1 and w_2 are uncorrelated if $E[w_1 w_2^*] = E[w_1]E[w_2^*] = 0$, where the last equality holds because $w(t)$ is a zero-mean process. Hence, we derive

$$\begin{aligned}
E[w_1 w_2] &= E\left[\langle w(t), \phi_1(t)\rangle \cdot (\langle w(t), \phi_2(t)\rangle)^*\right] \\
&= E\left[\left(\int_{-\infty}^{\infty} w(t)\phi_1^*(t)dt\right) \cdot \left(\int_{-\infty}^{\infty} w(s)\phi_2^*(s)ds\right)^*\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[w(t)w^*(s)\right]\phi_1^*(t)\phi_2(s)dtds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(t-s)\phi_1^*(t)\phi_2(s)dtds \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} \phi_1^*(t)\phi_2(t)dt \\
&= \frac{N_0}{2}\langle \phi_2(t), \phi_1(t)\rangle \\
&= 0
\end{aligned}$$

Note: If $w(t)$ is also a Gaussian process, then the uncorrelatedness of w_1 and w_2 implies their independence.

4. Give

$$\tilde{s}(t) = \sum_{k=1}^{N} \tilde{s}_k \cdot \tilde{\phi}_k(t),$$

where $\{\tilde{\phi}_k(t)\}_{k=1}^{N}$ are complex-valued orthonormal basis. Show that

$$\|\tilde{s}(t)\|^2 = \langle \tilde{s}(t), \tilde{s}(t)\rangle$$

is equal to $\sum_{k=1}^{N} |\tilde{s}_k|^2$.

Hint: In parellel to Slide IDC1-20,

$$\langle \tilde{s}_k \cdot \tilde{\phi}_k(t), \tilde{s}_{k'} \cdot \tilde{\phi}_{k'}(t)\rangle = \tilde{s}_k \tilde{s}_{k'}^* \cdot \langle \tilde{\phi}_k(t), \tilde{\phi}_{k'}(t)\rangle$$

when the inner product operation is extended to the complex domain.

Solution.

$$\begin{aligned}
\langle \tilde{s}(t), \tilde{s}(t)\rangle &= \left\langle \sum_{k=1}^{N} \tilde{s}_k \cdot \tilde{\phi}_k(t), \sum_{k'=1}^{N} \tilde{s}_{k'} \cdot \tilde{\phi}_{k'}(t)\right\rangle \\
&= \sum_{k=1}^{N}\sum_{k'=1}^{N} \left\langle \tilde{s}_k \cdot \tilde{\phi}_k(t), \tilde{s}_{k'} \cdot \tilde{\phi}_{k'}(t)\right\rangle \\
&= \sum_{k=1}^{N}\sum_{k'=1}^{N} \tilde{s}_k (\tilde{s}_{k'})^* \cdot \left\langle \tilde{\phi}_k(t), \tilde{\phi}_{k'}(t)\right\rangle \\
&= \sum_{k=1}^{N} \tilde{s}_k (\tilde{s}_k)^* \\
&= \sum_{k=1}^{N} |\tilde{s}_k|^2.
\end{aligned}$$

5. Suppose $s(t) = \mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\}$. Then, we have

$$S(f) = \frac{1}{2}\left[\tilde{S}(f - f_c) + \tilde{S}^*(-f - f_c)\right],$$

where $S(f)$ and $\tilde{S}(f)$ are respectively the Fourier transforms of $s(t)$ and $\tilde{s}(t)$.

(a) Is $\tilde{S}(f)$ always real-valued? Is $S(f)$ always real-valued? Justify your answer.

(b) Further assume that $\tilde{s}(t) = x(t)$ with $x(t)$ being a real-valued wide-sense stationary (WSS) random process with $R_{xx}(\tau) \neq 0$ for some τ. Is $s(t)$ a WSS random process? Justify your answer.

(c) Continue from (b). Determine the time-averaged autocorrelation function $\bar{R}_{ss}(\tau)$ of $s(t)$ in (b), where

$$\bar{R}_{ss}(\tau) := \lim_{W\to\infty} \frac{1}{2W}\int_{-W}^{W} R_{ss}(t+\tau, t)dt,$$

and $R_{ss}(t+\tau, t) := E[s(t+\tau)s(t)]$. Show that

$$\bar{R}_{ss}(\tau) = \frac{1}{2}\mathrm{Re}\left\{R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c \tau}\right\}.$$

Hint:

$$\lim_{W\to\infty} \frac{1}{2W}\int_{-W}^{W} \cos(2\pi f_c(2t+\tau))dt = 0.$$

Solution.

(a) Both $\tilde{S}(f)$ and $S(f)$ are not necessarily real-valued. $\tilde{S}(f)$ is real-valued if, and only if, $\tilde{s}^*(-t) = \tilde{s}(t)$, and $S(f)$ is real-valued if, and only if, $s^*(-t)\ \big(= s(-t)\big)\ = s(t)$.

(b)

$$\begin{aligned}
R_{ss}(t+\tau, t) &= E[s(t+\tau)s(t)] \\
&= E\left[\mathrm{Re}\left\{x(t+\tau)e^{j2\pi f_c(t+\tau)}\right\} \cdot \mathrm{Re}\left\{x(t)e^{j2\pi f_c t}\right\}\right] \\
&= E[x(t+\tau)x(t)]\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&\qquad \text{(Because } x(t+\tau) \text{ and } x(t) \text{ are real.)} \\
&= R_{xx}(\tau)\cdot\frac{1}{2}\left[\cos(2\pi f_c(2t+\tau)) + \cos(2\pi f_c \tau)\right]
\end{aligned}$$

indicates that $R_{ss}(t+\tau, t)$ is in general a function of both t and τ; hence, $s(t)$ is not a WSS random process.

Note: Since $\tilde{s}(t) = x(t) + jy(t) = x(t)$ is WSS and $R_{yy}(\tau) = 0 \neq R_{xx}(\tau)$ (as $y(t) = 0$), $s(t)$ cannot be WSS! Recall again that $R_{xx}(\tau) \neq R_{yy}(\tau)$ implies at least one of $s(t)$ and $\tilde{s}(t)$ is not WSS.

(c) It follows from the solution to (b) that

$$\begin{aligned}\bar{R}_{ss}(\tau) &= \lim_{W\to\infty}\frac{1}{2W}\int_{-W}^{W} R_{ss}(t+\tau,t)dt \\ &= R_{xx}(\tau)\cdot\frac{1}{2}\left[\lim_{W\to\infty}\frac{1}{2W}\int_{-W}^{W}\cos(2\pi f_c(2t+\tau))dt\right. \\ &\quad \left. + \lim_{W\to\infty}\frac{1}{2W}\int_{-W}^{W}\cos(2\pi f_c\tau)dt\right] \\ &= \frac{1}{2}R_{xx}(\tau)\cos(2\pi f_c\tau) \\ &= \frac{1}{2}\mathrm{Re}\left\{R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}\right\},\end{aligned}$$

where the last step follows from $R_{\tilde{s}\tilde{s}}(\tau) = R_{xx}(\tau)$.

Note: Even in a general non-WSS situation, we still have $\bar{R}_{ss}(\tau) = \frac{1}{2}\mathrm{Re}\left\{R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}\right\}$ from the time-averaged perspective.

6. Suppose

$$s_i(t) = \begin{cases}\sqrt{\frac{2E}{T}}\cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right], & 0 \le t < T \\ 0, & \text{elsewhere}\end{cases}$$

where $i = 1, 2, 3, 4$, $f_c T$ is an integer, E is the transmitted energy per QPSK symbol, and T is the symbol duration.

(a) Give the orthonormal basis

$$\begin{cases}\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t); \\ \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t).\end{cases}$$

Determine $\boldsymbol{s}_i = \begin{bmatrix}\langle s_i(t), \phi_1(t)\rangle \\ \langle s_i(t), \phi_2(t)\rangle\end{bmatrix}$.

(b) Suppose $x(t) = s(t) + w(t)$, where $w(t)$ is a zero-mean white Gaussian noise with variance $\frac{N_0}{2}$. After performing projection onto $\phi_1(t)$ and $\phi_2(t)$, we obtain

$$\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}s_1 \\ s_2\end{bmatrix} + \begin{bmatrix}w_1 \\ w_2\end{bmatrix}.$$

Under the assumptions that each of $\{s_i(t)\}_{i=1}^4$ was used with equal probability, we obtain that the maximum likelihood (i.e., optimal) decision is given by

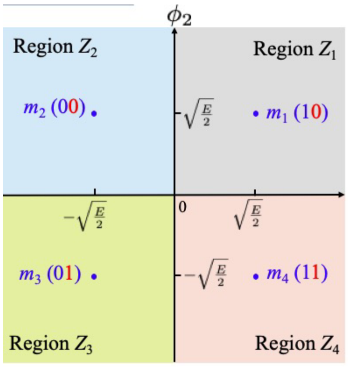
$$x_1 \underset{+\sqrt{E/2}}{\overset{-\sqrt{E/2}}{\lessgtr}} 0 \quad \text{and} \quad x_2 \underset{+\sqrt{E/2}}{\overset{-\sqrt{E/2}}{\lessgtr}} 0.$$

We thus derive from the decision rule that

$$P(s_1 \text{ error}) = P(s_2 \text{ error}) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right),$$

where $\Phi(\cdot)$ is the cdf of the standard normal random variable, and $E_b = \frac{1}{2}E$ is the equivalent transmitted energy per message bit.

Now suppose Gray labelling is adopted as



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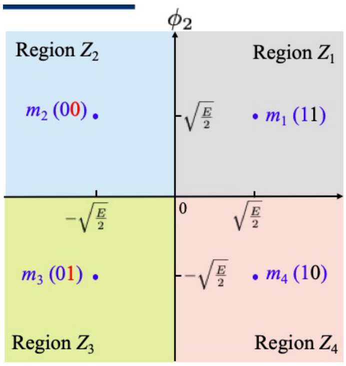
What is the error rate of the second message bit?

(c) Continue from (b). If we change to natural labelling as



,

what is the error rate of the second message bit?

Hint: [s_1 error] and [s_2 error] are independent events since w_1 and w_2 are independent noises.

(a) From

$$\sqrt{\frac{2E}{T}}\cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right]$$
$$= \sqrt{E}\cos((2i-1)\tfrac{\pi}{4})\cdot\underbrace{\sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)}_{\phi_1(t)} + \sqrt{E}\sin((2i-1)\tfrac{\pi}{4})\cdot\underbrace{\left(-\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)\right)}_{\phi_2(t)},$$

we know $\boldsymbol{s}_i = \begin{bmatrix}\sqrt{E}\cos((2i-1)\frac{\pi}{4})\\ \sqrt{E}\sin((2i-1)\frac{\pi}{4})\end{bmatrix}$.

(b) Let the first bit and the second bit be denoted as b_1 and b_2, respectively. Denote the estimate of the two bits by $\hat{b}_1$ and $\hat{b}_2$. Then,

$$\begin{aligned}
&P(b_2 \text{ error}) \\
&= P(b_2=0)P(\hat{b}_2=1|b_2=0)+P(b_2=1)P(\hat{b}_2=0|b_2=1) \\
&= P(\hat{b}_2=1 \wedge b_2=0)+P(\hat{b}_2=0 \wedge b_2=1) \\
&= P(\hat{s}_2=-\sqrt{E/2} \wedge s_2=\sqrt{E/2})+P(\hat{s}_2=\sqrt{E/2} \wedge s_2=-\sqrt{E/2}) \\
&= P(s_2 \text{ error}) \quad (\text{I.e., } P(\hat{s}_2 \neq s_2)) \\
&= \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
&P(b_2 \text{ error}) \\
&= P(b_2=0)P(\hat{b}_2=1|b_2=0)+P(b_2=1)P(\hat{b}_2=0|b_2=1) \\
&= P(\hat{b}_2=1 \wedge b_2=0)+P(\hat{b}_2=0 \wedge b_2=1) \\
&= P(\hat{s}_1\hat{s}_2=E/2 \wedge s_1s_2=-E/2)+P(\hat{s}_1\hat{s}_2=-E/2 \wedge s_1s_2=E/2) \\
&= P(s_1 \text{ correct} \wedge s_2 \text{ error} \wedge s_1s_2=-E/2)+P(s_1 \text{ error} \wedge s_2 \text{ correct} \wedge s_1s_2=-E/2) \\
&+ P(s_1 \text{ correct} \wedge s_2 \text{ error} \wedge s_1s_2=E/2)+P(s_1 \text{ error} \wedge s_2 \text{ correct} \wedge s_1s_2=E/2) \\
&= P(s_1 \text{ correct} \wedge s_2 \text{ error})+P(s_1 \text{ error} \wedge s_2 \text{ correct}) \\
&= P(s_1 \text{ correct})P(s_2 \text{ error})+P(s_1 \text{ error})P(s_2 \text{ correct}) \quad (\text{They are independent events.}) \\
&= 2\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\left[1-\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\right]
\end{aligned}$$

Note: Although the BER of the second message bit for natural labelling (in (c)) is always worse than that for Gray labelling (in (b)), the error rate of the first message bit is the same for (b) and (c). Also, the symbol error rates of (b) and (c) are identical (since symbol error rate has nothing to do with bit labelling).

7. The time-averaged power spectrum density of the M-ary PSK signaling scheme is given by $2E \cdot \text{sinc}^2(fT)$, where E is the symbol energy and T is the symbol period.

   (a) Find the null-to-null bandwidth.

   (b) Find the bandwidth efficiency based on null-to-null bandwidth.

Solution.

(a) $\frac{2}{T}$.

(b) $\frac{1}{2}\log_2(M)$ (cf. Slide IDC1-65)