- Sample Problem 1 for Quiz 8: Here we provide the detail derivation of the power of the noise term for your reference, i.e.,

$$
\begin{aligned}
E\left[\left(\sum_{k=1}^{L} w_k z_k\right)^2\right] &= E\left[\left(\sum_{k=1}^{L} w_k z_k\right)\left(\sum_{k'=1}^{L} w_{k'} z_{k'}\right)\right] \\
&= E\left[\sum_{k=1}^{L}\sum_{k'=1}^{L} w_k w_{k'} z_k z_{k'}\right] \\
&= \sum_{k=1}^{L}\sum_{k'=1}^{L} w_k w_{k'} E\left[z_k z_{k'}\right]
\end{aligned}
$$

  Since

$$
E\left[z_k z_{k'}\right] = \begin{cases} E[z_k^2] = \sigma^2, & k = k' \\ E[z_k]E[z_{k'}] = 0, & k \neq k' \end{cases}
$$

  we continue

$$
E\left[\left(\sum_{k=1}^{L} w_k z_k\right)^2\right] = \sum_{k=1}^{L}\sum_{k'=1}^{L} w_k w_{k'} E\left[z_k z_{k'}\right] = \sum_{k=1}^{L} w_k^2 \cdot \sigma^2 = \sigma^2 \sum_{k=1}^{L} w_k^2.
$$

- There will be five problems in the second midterm. At least four of them will come from the sample problems for Quizzes 5, 6, 7, 8 and Midterm 2.

Additional Sample Problems for Midterm 2

1. Which of the following codes are prefix codes? Which of the following codeword lengths satisfy the Kraft-McMillan inequality? Let the three codewords respectively correspond to A, B and C. Decode 1110010010100 if the code is a prefix code.

   (a) $\{1, 0, 00\}$.

   (b) $\{1, 01, 00\}$.

   (c) $\{1, 10, 00\}$.

   Solution.

   (a) Codeword 0 is the prefix of codeword 00. Hence, it is not a prefix code. Its codeword lengths (i.e., 1, 1 and 2) do not satisfy the Kraft-McMillan inequality because $2^{-1} + 2^{-1} + 2^{-2} = \frac{5}{4} > 1$.

(b) No codewords are prefixes of other codewords (the code can form a tree with all codewords on leaves); hence, it is a prefix code. It must satisfy the Kraft-McMillan inequality. We can uniquely decode the sequence 1110010010100 as $(1, 1, 1, 00, 1, 00, 1, 01, 00) = AAACACABC$.

(c) Codeword 1 is the prefix of codeword 10. Hence, it is not a prefix code. Its codeword lengths (i.e., 1, 2 and 2) satisfy the Kraft-McMillan inequality because $2^{-1}+2^{-2}+2^{-2} = 1 \cancel{> 1}$.

2. (a) Give a binary prefix code, of which the longest codeword is 3 and which equates the Kraft-McMillan inequality.

(b) Determine the largest code size among all binary prefix codes that satisfy the requirement in (a).

Solution.

(a) Let n_j be the number of codewords of length j. Then, equality of the Kraft-McMillan inequality requires a saturated binary tree, i.e.,

$$n_1 \cdot 2^2 + n_2 \cdot 2 + n_3 = 2^3, \tag{1}$$

where $0 \leq n_1 < 2$, $0 \leq n_2 < 4$ and $2 \leq n_3 \leq 8$. Note that n_1 cannot be equal to 2 (similarly, n_2 cannot be equal to 4) because otherwise the tree cannot have codewords of length 3. The below table then lists all possible values of (n_1, n_2, n_3) that satisfies (1).

n_1	1	1	0	0	0	0
n_2	1	0	3	2	1	0
n_3	2	4	2	4	6	8
code size	4	5	5	6	7	8

Thus, a quick example is $\{000, 001, 010, 011, 100, 101, 110, 111\}$.

(b) The code size is equal to $n_1 + n_2 + n_3$. Thus the table in the solution of (a) indicates that the largest code size is $2^3 = 8$.

3. Prove that the entropy of a discrete random variable X satisfies the following inequalities. Also, give the necessary and sufficient condition under which equality holds.

(a) $H(X) \geq 0$

(b) $H(X) \leq \log_2(K)$, where K is the size of the support of X (i.e., X only takes on K possible values).

Solution. See Slides IDC 6-17 and 6-18.

4. (Not a part of the exam but only for your reference) Prove that the function defined over $p \in [0,1]$ and satisfying three axioms listed below must be of the form:

$$I(p) = -c \cdot \log_b(p),$$

where c is a positive constant and the base b of the logarithm is any number larger than one.

i) $I(p)$ is monotonically decreasing in p;

ii) $I(p_1 \times p_2) = I(p_1) + I(p_2)$;

iii) $I(p)$ is a continuous function of p for $0 \leq p \leq 1$;

Proof.

Step 1: Claim. For integer value of $n = 1, 2, 3, \ldots$,

$$I\left(\tfrac{1}{n}\right) = -c \cdot \log_b\left(\tfrac{1}{n}\right),$$

where $c > 0$ is a constant.

Proof: First note that for $n = 1$, Axiom *ii)* directly verifies the claim since it yields that $I(1) = I(1) + I(1)$. Thus $I(1) = 0 = -c \log_b(1)$.

Now let n be a fixed positive integer greater than 1. Axioms *i)* and *ii)* respectively imply

$$n < m \quad \Longrightarrow \quad I\left(\tfrac{1}{n}\right) < I\left(\tfrac{1}{m}\right) \tag{2}$$

and

$$I\left(\tfrac{1}{mn}\right) = I\left(\tfrac{1}{m}\right) + I\left(\tfrac{1}{n}\right) \tag{3}$$

where $n, m = 1, 2, 3, \cdots$. Now using (3), we can show by induction on k that

$$I\left(\tfrac{1}{n^k}\right) = k \cdot I\left(\tfrac{1}{n}\right) \tag{4}$$

for all non-negative integers k.

Now for any positive integer r, there exists a non-negative integer k such that

$$n^k \leq 2^r < n^{k+1}.$$

By (2), we obtain

$$I\left(\tfrac{1}{n^k}\right) \leq I\left(\tfrac{1}{2^r}\right) < I\left(\tfrac{1}{n^{k+1}}\right),$$

which together with (4), yields

$$k \cdot I\left(\tfrac{1}{n}\right) \leq r \cdot I\left(\tfrac{1}{2}\right) < (k+1) \cdot I\left(\tfrac{1}{n}\right).$$

Hence, since $I(\frac{1}{n}) > I(1) = 0$,

$$\frac{k}{r} \leq \frac{I(\frac{1}{2})}{I(\frac{1}{n})} \leq \frac{k+1}{r}.$$

On the other hand, by the monotonicity of the logarithm, we obtain

$$\log_b n^k \leq \log_b 2^r \leq \log_b n^{k+1} \quad \Longleftrightarrow \quad \frac{k}{r} \leq \frac{\log_b(2)}{\log_b(n)} \leq \frac{k+1}{r}.$$

Therefore,

$$\left| \frac{\log_b(2)}{\log_b(n)} - \frac{I(\frac{1}{2})}{I(\frac{1}{n})} \right| < \tfrac{1}{r}.$$

Since n is fixed, and r can be made arbitrarily large, we can let $r \to \infty$ to get:

$$I\left(\tfrac{1}{n}\right) = c \cdot \log_b(n).$$

where $c = I(1/2)/\log_b(2) > 0$. This completes the proof of the claim.

Step 2: Claim. $I(p) = -c \cdot \log_b(p)$ for positive rational number p, where $c > 0$ is a constant.

Proof: A positive rational number p can be represented by a ratio of two integers, i.e., $p = r/s$, where r and s are both positive integers. Then Axiom *ii)* yields that

$$I\left(\tfrac{1}{s}\right) = I\left(\tfrac{r}{s}\tfrac{1}{r}\right) = I\left(\tfrac{r}{s}\right) + I\left(\tfrac{1}{r}\right),$$

which, from Step 1, implies that

$$I(p) = I\left(\tfrac{r}{s}\right) = I\left(\tfrac{1}{s}\right) - I\left(\tfrac{1}{r}\right) = c \cdot \log_b s - c \cdot \log_b r = -c \cdot \log_b p.$$

Step 3: For any $p \in [0, 1]$, it follows by continuity and the density of the rationals in the reals that

$$I(p) = \lim_{a \uparrow p,\ a \text{ rational}} I(a) = \lim_{b \downarrow p,\ b \text{ rational}} I(b) = -c \cdot \log_b(p).$$