Corrections to slides and a reminder for midterm 1

- Slide IDC3-35:

$$\begin{aligned} &\max_{\{(P_1,P_2,\cdots,P_n):\sum_{n=1}^{n} P_n \le P\}} \Big(\cdot \Big) \\ &= \max_{\{(P_1,P_2,\cdots,P_n):\sum_{n=1}^{n} P_n = P\}} \Big(\cdot \Big) \\ &= \max_{\{(P_1,P_2,\cdots,P_n):\sum_{n=1}^{n} P_n = P\}} \Big[\cdot \Big] \end{aligned}$$

  shall be replaced by

$$\begin{aligned} &\max_{\{(P_1,P_2,\cdots,P_N):\sum_{n=1}^{N} P_n \le P\}} \Big(\cdot \Big) \\ &= \max_{\{(P_1,P_2,\cdots,P_N):\sum_{n=1}^{N} P_n = P\}} \Big(\cdot \Big) \\ &= \max_{\{(P_1,P_2,\cdots,P_N):\sum_{n=1}^{N} P_n = P\}} \Big[\cdot \Big] \end{aligned}$$

- Slide IDC3-36:

$$\frac{\log_2(e)}{2N} \frac{g_j^2/(\Gamma\sigma_j^2)}{1+g_j^2/(\Gamma\sigma_j^2)} - \lambda \begin{cases} = 0, & \text{if } P_j^* > 0 \\ \ge 0, & \text{if } P_j^* = 0 \end{cases}$$

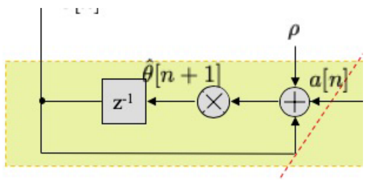
  shall be replaced by

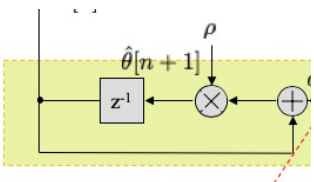
$$\frac{\log_2(e)}{2N} \frac{g_j^2/(\Gamma\sigma_j^2)}{1+g_j^2/(\Gamma\sigma_j^2)} - \lambda \begin{cases} = 0, & \text{if } P_j^* > 0 \\ \le 0, & \text{if } P_j^* = 0 \end{cases}$$

- Slide IDC3-37: $\displaystyle\sum_{n=1}^{n} P_j^* = P$ shall be replaced by $\displaystyle\sum_{j=1}^{N} P_j^* = P$.

- Slide IDC3-45: "of length $N+\nu$" shall be replaced by "of length ν".

- Slide IDC3-53:



shall be replaced by


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- Note that the sample problems for the first **four** quizzes will be a key part of the first midterm.

Additional Six Sample Problems for Midterm 1

1. From Slide IDC2-85, we learn that the error rate for binary DPSK is $\frac{1}{2}e^{-E/(2\sigma^2)}$, where $\sigma^2 = N_0/2$.

   Suppose we now have three **independent** channels, each of which uses binary DPSK transmissions. The receiver will use the majority rule to make the final decision, i.e., if two or more "+1" are reported, then "+1" will be the final decision; otherwise, "−1" is the final decision. By this design, the error rate will be

$$P_e(\vec{E}) = \frac{1}{8}e^{-E_1/(2\sigma_1^2)}e^{-E_2/(2\sigma_2^2)}e^{-E_3/(2\sigma_3^2)} + \frac{1}{4}e^{-E_1/(2\sigma_1^2)}e^{-E_2/(2\sigma_2^2)}\left(1 - \frac{1}{2}e^{-E_3/(2\sigma_3^2)}\right)$$
$$+\frac{1}{4}e^{-E_1/(2\sigma_1^2)}e^{-E_3/(2\sigma_3^2)}\left(1 - \frac{1}{2}e^{-E_2/(2\sigma_2^2)}\right) + \frac{1}{4}e^{-E_2/(2\sigma_2^2)}e^{-E_3/(2\sigma_3^2)}\left(1 - \frac{1}{2}e^{-E_1/(2\sigma_1^2)}\right),$$

   where $\vec{E} = (E_1, E_2, E_3)$.

   (a) Subject to $E_1 + E_2 + E_3 = E > 0$, we wish to find the optimal power allocation that minimizes $P_e(\vec{E})$, i.e.,

$$\vec{E}^{\,\text{opt}} = \arg\min_{\vec{E}\in\mathbb{Q}} P_e(\vec{E}),$$

   where

$$\mathbb{Q} \triangleq \big\{(E_1, E_2, E_3) : E_i \in \mathbb{R}^+ \text{ for } i = 1, 2, 3, \text{ and } E_1 + E_2 + E_3 = E\big\}$$

   and $\mathbb{R}^+$ is the set of non-negative real numbers. Define

$$f(\vec{E}, \lambda) = P_e(\vec{E}) + \lambda\big(E - E_1 - E_2 - E_3\big).$$

   Does the following equation hold:

$$\min_{\vec{E}\in\mathbb{Q}} P_e(\vec{E}) = \min_{\vec{E}\in\mathbb{Q}} f(\vec{E}, \lambda)?$$

   Justify your answer.

   (b) Show that $\vec{E}^*$ that minimizes $f(\vec{E}, \lambda)$ over $\vec{E} \in (\mathbb{R}^+)^3$ must satisfy

$$\begin{cases} e^{-E_1^*/(2\sigma_1^2)}\left[1 - \left(1 - e^{-E_2^*/(2\sigma_2^2)}\right)\left(1 - e^{-E_3^*/(2\sigma_3^2)}\right)\right] \begin{cases} = -8\sigma_1^2\lambda, & \text{if } E_1^* > 0; \\ \leq -8\sigma_1^2\lambda, & \text{if } E_1^* = 0, \end{cases} \\ e^{-E_2^*/(2\sigma_2^2)}\left[1 - \left(1 - e^{-E_1^*/(2\sigma_1^2)}\right)\left(1 - e^{-E_3^*/(2\sigma_3^2)}\right)\right] \begin{cases} = -8\sigma_2^2\lambda, & \text{if } E_2^* > 0; \\ \leq -8\sigma_2^2\lambda, & \text{if } E_2^* = 0, \end{cases} \\ e^{-E_3^*/(2\sigma_3^2)}\left[1 - \left(1 - e^{-E_1^*/(2\sigma_1^2)}\right)\left(1 - e^{-E_2^*/(2\sigma_2^2)}\right)\right] \begin{cases} = -8\sigma_3^2\lambda, & \text{if } E_3^* > 0; \\ \leq -8\sigma_3^2\lambda, & \text{if } E_3^* = 0. \end{cases} \end{cases}$$

(c) Let $\sigma_1^2 = \sigma_2^2 = 1$ and $\sigma_3^2 = 3$. Does the best power allocation E_i^* in (b) positive for $i = 1, 2, 3$? Justify your answer.

(d) Continue from (c). Does $E_1^* = E_2^* = \frac{E}{2}$ and $E_3^* = 0$ satisfy the optimality condition in (b) for some λ?

(e) Continue from (c). Does $E_1^* = E_2^* = \frac{E}{2}$ and $E_3^* = 0$ minimize $P_e(\vec{E})$ among all $\vec{E} \in \mathbb{Q}$? Justify your answer.

Solution.

(a) Yes because $P_e(\vec{E}) = f(\vec{E}, \lambda)$ for every $\vec{E} \in \mathbb{Q}$.

(b) Taking the derivatives of $f(\vec{E}, \lambda)$ with respective to E_1 yields

$$\begin{aligned}
\frac{\partial f(\vec{E}, \lambda)}{\partial E_1} &= -\frac{1}{16\sigma_1^2} e^{-E_1/(2\sigma_1^2)} e^{-E_2/(2\sigma_2^2)} e^{-E_3/(2\sigma_3^2)} \\
&\quad -\frac{1}{8\sigma_1^2} e^{-E_1/(2\sigma_1^2)} e^{-E_2/(2\sigma_2^2)} \left(1 - \frac{1}{2} e^{-E_3/(2\sigma_3^2)}\right) \\
&\quad -\frac{1}{8\sigma_1^2} e^{-E_1/(2\sigma_1^2)} e^{-E_3/(2\sigma_3^2)} \left(1 - \frac{1}{2} e^{-E_2/(2\sigma_2^2)}\right) \\
&\quad +\frac{1}{16\sigma_1^2} e^{-E_2/(2\sigma_2^2)} e^{-E_3/(2\sigma_3^2)} e^{-E_1/(2\sigma_1^2)} - \lambda \\
&= -\frac{1}{8\sigma_1^2} e^{-E_1/(2\sigma_1^2)} \left(e^{-E_2/(2\sigma_2^2)} + e^{-E_3/(2\sigma_3^2)} - e^{-E_2/(2\sigma_2^2)} e^{-E_3/(2\sigma_3^2)}\right) - \lambda \\
&= -\frac{1}{8\sigma_1^2} e^{-E_1/(2\sigma_1^2)} \left[1 - \left(1 - e^{-E_2/(2\sigma_2^2)}\right)\left(1 - e^{-E_3/(2\sigma_3^2)}\right)\right] - \lambda
\end{aligned}$$

and

$$\frac{\partial^2 f(\vec{E}, \lambda)}{\partial E_1^2} = \frac{1}{16\sigma_1^4} e^{-E_1/(2\sigma_1^2)} \left[1 - \left(1 - e^{-E_2/(2\sigma_2^2)}\right)\left(1 - e^{-E_3/(2\sigma_3^2)}\right)\right] > 0.$$

This shows that $f(\vec{E}, \lambda)$ is convex with respect to E_1 and hence the optimal power allocation should satisfy

$$-\frac{1}{8\sigma_1^2} e^{-E_1^*/(2\sigma_1^2)} \left[1 - \left(1 - e^{-E_2/(2\sigma_2^2)}\right)\left(1 - e^{-E_3/(2\sigma_3^2)}\right)\right] - \lambda \begin{cases} = 0, & \text{if } E_1^* > 0; \\ \geq 0, & \text{if } E_1^* = 0, \end{cases}$$

Equivalently,

$$e^{-E_1^*/(2\sigma_1^2)} \left[1 - \left(1 - e^{-E_2/(2\sigma_2^2)}\right)\left(1 - e^{-E_3/(2\sigma_3^2)}\right)\right] \begin{cases} = -8\sigma_1^2\lambda, & \text{if } E_1^* > 0; \\ \leq -8\sigma_1^2\lambda, & \text{if } E_1^* = 0, \end{cases}$$

We can similarly obtain

$$e^{-E_2^*/(2\sigma_2^2)}\left[1-\left(1-e^{-E_1/(2\sigma_1^2)}\right)\left(1-e^{-E_3/(2\sigma_3^2)}\right)\right]\begin{cases}=-8\sigma_2^2\lambda, & \text{if } E_2^*>0;\\ \leq -8\sigma_2^2\lambda, & \text{if } E_2^*=0,\end{cases}$$

and

$$e^{-E_3^*/(2\sigma_3^2)}\left[1-\left(1-e^{-E_1/(2\sigma_1^2)}\right)\left(1-e^{-E_2/(2\sigma_2^2)}\right)\right]\begin{cases}=-8\sigma_3^2\lambda, & \text{if } E_3^*>0;\\ \leq -8\sigma_3^2\lambda, & \text{if } E_3^*=0.\end{cases}$$

(c) Suppose $E_i^*>0$ for $i=1,2,3$. Let $A_i^*=e^{-E_i^*/(2\sigma_i^2)}$. Then, we must have

$$\begin{cases}A_1^*[1-(1-A_2^*)(1-A_3^*)]=-8\sigma_1^2\lambda=-8\lambda\\ A_2^*[1-(1-A_1^*)(1-A_3^*)]=-8\sigma_2^2\lambda=-8\lambda\\ A_3^*[1-(1-A_1^*)(1-A_2^*)]=-8\sigma_3^2\lambda=-24\lambda\end{cases}$$

Then, we shall find a feasible solution of A_1^*, A_2^* and A_3^* lying in $(0,1]$. Let $A_1^*=A_2^*=A^*$ and derive

$$\begin{aligned}-24\lambda=3A^*[1-(1-A^*)(1-A_3^*)]&=A_3^*[1-(1-A^*)^2]\\ \Rightarrow\ \ & A^*(3A^*+A_3^*-2A^*A_3^*)=0\\ \Rightarrow\ \ & (A^*-0.5)(A_3^*-1.5)=0.75\\ \Rightarrow\ \ & 0<A_3^*=\frac{0.75}{A^*-0.5}+1.5\leq 1 \text{ and } 0<A^*<0.5\\ \Rightarrow\ \ & 0>A^*\geq -1 \text{ and } 0<A^*<0.5,\end{aligned}$$

which implies one of E_1^*, E_2^* and E_3^* must be zero.
Note: Taking $A_3^*=1$ (i.e., $E_3^*=0$) yields

$$-24\lambda=3A^*\geq 1-(1-A^*)^2\Rightarrow A^*(A^*+1)\geq 0.$$

Hence, any A^* in $(0,1]$ satisfy the above inequality.

(d) With $E_1^*=E_2^*=\frac{E}{2}$ and $E_3^*=0$, the optimality condition becomes

$$\begin{cases}e^{-E/4}=-8\lambda\\ e^{-E/4}=-8\lambda\\ 1-\left(1-e^{-E/4}\right)^2\leq -24\lambda\end{cases}$$

It is clear that we can set $\lambda=-\frac{1}{8}e^{-E/4}$, and this λ trivially satisfies the third condition because

$$1-\left(1-e^{-E/4}\right)^2\leq 3e^{-E/4}\Leftrightarrow -1\leq e^{E/4}$$

(e) Yes because with $\lambda = -\frac{1}{8}e^{-E/4}$,

$$f(\tfrac{E}{2},\tfrac{E}{2},0) \le \min_{\vec{E}\in\mathbb{Q}} P_e(\vec{E}) = \min_{\vec{E}\in\mathbb{Q}} f(\vec{E},\lambda) \le \min_{\vec{E}\in(\mathbb{R}^+)^3} f(\vec{E},\lambda) = f(\tfrac{E}{2},\tfrac{E}{2},0,\lambda) = f(\tfrac{E}{2},\tfrac{E}{2},0).$$

Note: This is an extension of the derivation in Slides IDC3-35∼IDC3-37, which results in the famous water filling principle. Such derivation is based on the following general Lagrange argument:

For given $\mathbb{Q} \subset \mathbb{R}^+$ and $f(x)$, Lagrange defines $f_\lambda(x)$ such that $f_\lambda(x) = f(x)$ for all $x \in \mathbb{Q}$. Then, Lagrange claims that:

1. $\max_{x\in\mathbb{Q}} f(x) = \max_{x\in\mathbb{Q}} f_\lambda(x) \le \max_{x\in\mathbb{R}^+} f_\lambda(x) = f_\lambda(x^*_\lambda)$
2. If $(\exists\lambda)$ $x^*_\lambda \in \mathbb{Q}$, then $f_\lambda(x^*_\lambda) = f(x^*_\lambda) \le \max_{x\in\mathbb{Q}} f(x) \le f_\lambda(x^*_\lambda)$; hence, $\max_{x\in\mathbb{Q}} f(x) = f_\lambda(x^*_\lambda) = f(x^*_\lambda)$.

2. For discrete multitones described in Slide IDC3-40 with $N = 4$ and $\nu = 1$, the input-output relation can be described as

$$\boldsymbol{x} = \begin{bmatrix} x[3] \\ x[2] \\ x[1] \\ x[0] \end{bmatrix} = \begin{bmatrix} 1 & h_1 & 0 & 0 \\ 0 & 1 & h_1 & 0 \\ 0 & 0 & 1 & h_1 \\ h_1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s[3] \\ s[2] \\ s[1] \\ s[0] \end{bmatrix} + \begin{bmatrix} w[3] \\ w[2] \\ w[1] \\ w[0] \end{bmatrix} = \mathbb{H}_{\text{circulant}}\boldsymbol{s} + \boldsymbol{w}.$$

(a) Find the diagonal matrix Λ such that $\mathbb{H}_{\text{circulant}} = \mathbb{Q}^\dagger \Lambda \mathbb{Q}$, where

$$\mathbb{Q} = \frac{1}{2}\begin{bmatrix} e^{-j\frac{9}{2}\pi} & e^{-j3\pi} & e^{-j\frac{3}{2}\pi} & 1 \\ e^{-j3\pi} & e^{-j2\pi} & e^{-j\pi} & 1 \\ e^{-j\frac{3}{2}\pi} & e^{-j\pi} & e^{-j\frac{1}{2}\pi} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -j & -1 & j & 1 \\ -1 & 1 & -1 & 1 \\ j & -1 & -j & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hint: $\mathbb{Q}\mathbb{H}_{\text{circulant}} = \Lambda\mathbb{Q}$ and note that Λ is a function of h_1.

(b) Now if we transform the system to

$$\boldsymbol{X} \quad \left(= \mathbb{Q}\boldsymbol{x} = \Lambda\mathbb{Q}\boldsymbol{s} + \mathbb{Q}\boldsymbol{w} \right) \quad = \Lambda\boldsymbol{S} + \boldsymbol{W}.$$

Determine the four transmission symbols $\boldsymbol{s}$ corresponding to

$$\boldsymbol{S} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ -1 \\ -1 \\ +1 \end{bmatrix}.$$

Hint: $\boldsymbol{s} = \mathbb{Q}^\dagger \boldsymbol{S}$.

Solution.

(a) From $\mathbb{QH}_{\text{circulant}} = \Lambda\mathbb{Q}$, we obtain

$$\begin{bmatrix} -j & -1 & j & 1 \\ -1 & 1 & -1 & 1 \\ j & -1 & -j & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & h_1 & 0 & 0 \\ 0 & 1 & h_1 & 0 \\ 0 & 0 & 1 & h_1 \\ h_1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{bmatrix} \begin{bmatrix} -j & -1 & j & 1 \\ -1 & 1 & -1 & 1 \\ j & -1 & -j & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \lambda_3 = 1 + jh_1 \\ \lambda_2 = 1 - h_1 \\ \lambda_1 = 1 - jh_1 \\ \lambda_0 = 1 + h_1 \end{cases}$$

Note: From (a), it is apparent that λ_3, λ_2, λ_1 and λ_0 are all functions of h_1. Thus, we need not to separately estimate each of them, nor to estimate all of them. As an example, we can fix $S[2] = -1$ as a pilot tone for the estimation of h_1 (i.e., λ_2), and use $S[3]$, $S[1]$ and $S[0]$ to carry information. This provides another advantage of discrete multitones.

Please note that N is usually much larger than ν (e.g., $N = 4 > \nu = 1$ in this problem; a specific example is for WLAN, $NT_s = 3.2\mu s$, while $\nu T_s = 0.8\mu s$ for long guard-interval (GI) and $\nu T_s = 0.4\mu s$ for short GI), and λ_1, λ_2, $\ldots$, λ_N are only a function of h_1, $\ldots$, h_ν. So, in principle, it suffices to place ν pilot tones for the determination of h_1, $\ldots$, h_ν if the noise is nullified.

(b) By

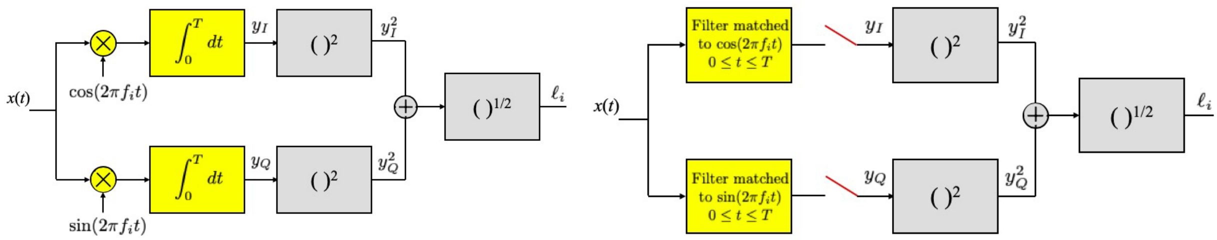
$$\boldsymbol{s} = \frac{1}{2} \begin{bmatrix} j & -1 & -j & 1 \\ -1 & 1 & -1 & 1 \\ -j & -1 & j & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \boldsymbol{S},$$

we obtain

$\boldsymbol{S}$	$\boldsymbol{s}$	$\boldsymbol{S}$	$\boldsymbol{s}$	$\boldsymbol{S}$	$\boldsymbol{s}$	$\boldsymbol{S}$	$\boldsymbol{s}$
$\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \end{bmatrix}$	$\begin{bmatrix} 1-j \\ 0 \\ 1+j \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

Note: After performing discrete Fourier transform (DFT), the equally distributed transmission powers of the four components can be "concentrated" into one comment. For example, $\boldsymbol{S} = [-1, -1, \ldots, -1]$ (of length N) will be transformed to $\boldsymbol{s} = [0, 0, 0, -N]$. This causes a problem in the transmission of $\boldsymbol{s}$ because the transmitter must have the capability to emit a very large signal with amplitude N. How to reduce the peak-to-average power ratio of the DFT signal, therefore, becomes a practically important research topic.

3. Below are two non-coherent receivers. Are the two (y_I, y_Q) pairs respectively obtained by the two quadratic receivers below identical? Justify your answer.



Solution. For the structure on the left, it is clear that

$$y_I = \int_0^T x(t)\cos(2\pi f_c t)dt \quad \text{and} \quad y_Q = \int_0^T x(t)\sin(2\pi f_c t)dt.$$

For the structure on the right, the impulse response of the two filters are

$$h_I(\tau) = \cos(2\pi f_i(T-\tau)) \cdot \mathbf{1}\{0 \le \tau \le T\} \text{ and } h_Q(\tau) = \sin(2\pi f_i(T-\tau)) \cdot \mathbf{1}\{0 \le \tau \le T\},$$

which implies the two inputs of the samplers should be equal to

$$\int_{-\infty}^{\infty} x(\tau)h_I(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\cos(2\pi f_i(T-(t-\tau))) \cdot \mathbf{1}\{0 \le t-\tau \le T\}d\tau$$

and

$$\int_{-\infty}^{\infty} x(\tau)h_Q(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\sin(2\pi f_i(T-(t-\tau))) \cdot \mathbf{1}\{0 \le t-\tau \le T\}d\tau.$$

Accordingly, sampling at time $t = T$ gives

$$y_I = \int_0^T x(\tau)\cos(2\pi f_i(T-(T-\tau)))d\tau = \int_0^T x(\tau)\cos(2\pi f_i \tau)d\tau$$

and

$$y_Q = \int_0^T x(\tau)\sin(2\pi f_i(T-(T-\tau)))d\tau = \int_0^T x(\tau)\sin(2\pi f_i \tau)d\tau.$$

Thus, the two (y_I, y_Q) pairs respectively obtained by the two receivers are identical.

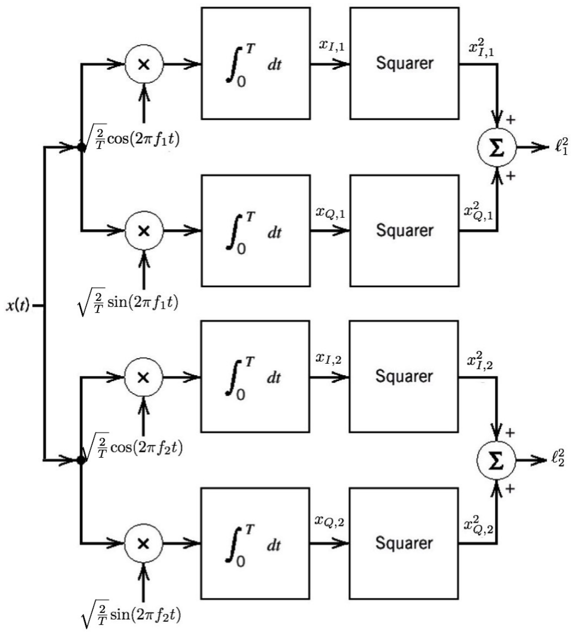
4. (a) Suppose the transmitter uses binary FSK signaling scheme, where f_1 and f_2 denote the two frequencies for information bits 0 and 1, respectively. Let f_1 and f_2 be a multiple of $1/T$, and let $x(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_1 t + \theta) + w(t)$, where $w(t)$ is a zero-mean additive white Gaussian noise process with one-sided power spectrum density N_0.

For the quadratic receiver below, represent $x_{I,1}$, $x_{Q,1}$, $x_{I,2}$ and x_{Q_2} in terms of E, θ, w_1, w_2, w_3 and w_4, where

$$w_1 = \int_0^T w(t)\cdot\sqrt{\frac{2}{T}}\cos(2\pi f_1 t)dt, \quad w_2 = \int_0^T w(t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_1 t)dt$$

and

$$w_3 = \int_0^T w(t)\cdot\sqrt{\frac{2}{T}}\cos(2\pi f_2 t)dt, \quad w_4 = \int_0^T w(t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_2 t)dt.$$



(b) Prove that w_1 and w_2 are independent.

Hint: By definition of zero-mean Gaussian random process $w(t)$, the projections w_1 and w_2 are two dimensional zero-mean Gaussian random variables. Thus, w_1 and w_2 are independent if, and only if, $E[w_1 w_2] = E[w_1]E[w_2] = 0$.

Solution.

(a) ($x_{Q,1}$ and $x_{Q,2}$ are corrected, as indicating in color red.)

$$
\begin{aligned}
x_{I,1} &= \int_0^T \left(\sqrt{\frac{2E}{T}}\cos(2\pi f_1 t+\theta)+w(t)\right)\sqrt{\frac{2}{T}}\cos(2\pi f_1 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T 2\cos(2\pi f_1 t+\theta)\cos(2\pi f_1 t)dt + \int_0^T w(t)\sqrt{\frac{2}{T}}\cos(2\pi f_1 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T [\cos(4\pi f_1 t+\theta)+\cos(\theta)]dt + w_1 \\
&= \sqrt{E}\cos(\theta)+w_1
\end{aligned}
$$

$$
\begin{aligned}
x_{Q,1} &= \int_0^T \left(\sqrt{\frac{2E}{T}}\cos(2\pi f_1 t+\theta)+w(t)\right)\sqrt{\frac{2}{T}}\sin(2\pi f_1 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T 2\cos(2\pi f_1 t+\theta)\sin(2\pi f_1 t)dt + \int_0^T w(t)\sqrt{\frac{2}{T}}\sin(2\pi f_1 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T [\sin(4\pi f_1 t+\theta)-\sin(\theta)]dt + w_2 \\
&= -\sqrt{E}\sin(\theta)+w_2
\end{aligned}
$$

$$
\begin{aligned}
x_{I,2} &= \int_0^T \left(\sqrt{\frac{2E}{T}}\cos(2\pi f_1 t+\theta)+w(t)\right)\sqrt{\frac{2}{T}}\cos(2\pi f_2 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T 2\cos(2\pi f_1 t+\theta)\cos(2\pi f_2 t)dt + \int_0^T w(t)\sqrt{\frac{2}{T}}\cos(2\pi f_2 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T [\cos(2\pi(f_1+f_2)t+\theta)+\cos(2\pi(f_1-f_2)t+\theta)]dt + w_3 \\
&= w_3
\end{aligned}
$$

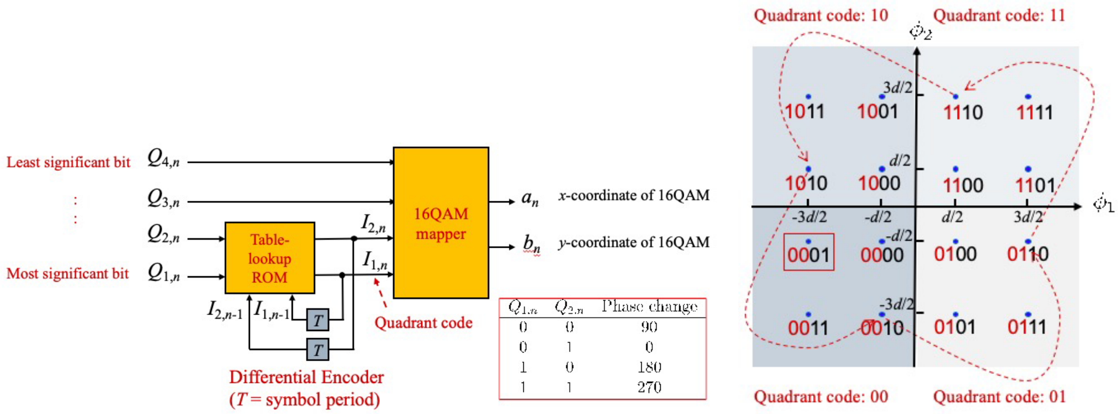
$$
\begin{aligned}
x_{Q,2} &= \int_0^T \left(\sqrt{\frac{2E}{T}}\cos(2\pi f_1 t+\theta)+w(t)\right)\sqrt{\frac{2}{T}}\sin(2\pi f_2 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T 2\cos(2\pi f_1 t+\theta)\sin(2\pi f_2 t)dt + \int_0^T w(t)\sqrt{\frac{2}{T}}\sin(2\pi f_2 t)dt \\
&= \frac{\sqrt{E}}{T}\int_0^T [\sin(2\pi(f_2+f_1)t+\theta)+\sin(2\pi(f_2-f_1)t-\theta)]dt + w_4 \\
&= w_4
\end{aligned}
$$

(b)

$$
\begin{aligned}
E[w_1 w_2] &= E\left[\left(\int_0^T w(t)\cdot\sqrt{\frac{2}{T}}\cos(2\pi f_1 t)dt\right)\left(\int_0^T w(s)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_1 s)ds\right)\right] \\
&= \frac{2}{T}\int_0^T\int_0^T E\left[w(t)w(s)\right]\cos(2\pi f_1 t)\sin(2\pi f_1 s)\right]dtds \\
&= \frac{2}{T}\int_0^T\int_0^T \frac{N_0}{2}\delta(t-s)\cos(2\pi f_1 t)\sin(2\pi f_1 s)dtds \\
&= \frac{N_0}{T}\int_0^T \cos(2\pi f_1 s)\sin(2\pi f_1 s)ds \\
&= \frac{N_0}{2T}\int_0^T \sin(4\pi f_1 s)ds \\
&= 0
\end{aligned}
$$

Note: We can similarly prove w_1, w_2, w_3 and w_4 are independent.

5. Below is the functional diagram of the V.32 16-QAM Hybrid amplitude/phase modulation scheme:



(a) Assume $I_{1,-1} = I_{2,-1} = 1$. Give the sequence of 16QAM symbols (indicated by their coordinates) corresponding to

$$(Q_{1,0}Q_{2,0}Q_{3,0}Q_{4,0}\; Q_{1,1}Q_{2,1}Q_{3,1}Q_{4,1}\; Q_{1,2}Q_{2,2}Q_{3,2}Q_{4,2}) = (1001\,1010\,0000)$$

(b) Suppose there is a 30° phase difference between the transmitter and the receiver, i.e.,

$$\begin{bmatrix} a_{\text{receive}} \\ b_{\text{receive}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} a_{\text{transmit}} \\ b_{\text{transmit}} \end{bmatrix}.$$

By using the nearest Euclidean distance criterion, recover the transmitted information sequence $(Q_{1,0}Q_{2,0}Q_{3,0}Q_{4,0}\, Q_{1,1}Q_{2,1}Q_{3,1}Q_{4,1}\, Q_{1,2}Q_{2,2}Q_{3,2}Q_{4,2})$ from the rotated 16QAM symbols.

Hint: The nearest Euclidean distance decision can be made separately on x-axis and y-axis over the 16QAM constellation with thresholds -2, 0 and 2.

(c) Re-do (b) if the phase difference between the transmitter and the receiver is changed to 90° phase difference, i.e.,

$$\begin{bmatrix} a_{\text{receive}} \\ b_{\text{receive}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{\text{transmit}} \\ b_{\text{transmit}} \end{bmatrix}.$$

Solution.

(a)

n	$I_{1,n-1}I_{2,n-1}$	$Q_{1,n}Q_{2,n}$	phase change	$I_{1,n}I_{2,n}$	$Q_{3,n}Q_{4,n}$	16QAM symbol n
0	11	10	180	00	01	$(-3,-1)$
1	00	10	180	11	10	$(1,3)$
2	11	00	90	10	00	$(-1,1)$

(b)

$$\begin{bmatrix} \frac{-3\sqrt{3}+1}{2} \\ \frac{-3-\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} \to \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} \frac{\sqrt{3}-3}{2} \\ \frac{1+3\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

and

$$\begin{bmatrix} \frac{-\sqrt{3}+1}{2} \\ \frac{-1+\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore,

n	16QAM symbol n	$I_{1,n}I_{2,n}$	$Q_{3,n}Q_{4,n}$	$I_{1,n-1}I_{2,n-1}$	phase change	$Q_{1,n}Q_{2,n}$
0	$(-3,-3)$	00	11	11	180	10
1	$(-1,3)$	10	01	00	270	11
2	$(-1,1)$	10	00	10	0	01

and

$$(\hat{Q}_{1,0}\hat{Q}_{2,0}\hat{Q}_{3,0}\hat{Q}_{4,0}\hat{Q}_{1,1}\hat{Q}_{2,1}\hat{Q}_{3,1}\hat{Q}_{4,1}\hat{Q}_{1,1}\hat{Q}_{2,1}\hat{Q}_{3,1}\hat{Q}_{4,1}) = (1011\,1101\,0100)$$

where the red-color numbers indicate the errors during transmission.

Note: So, a 30° phase difference, if not being calibrated to zero, causes many transmission errors in both $Q_{1,n}Q_{2,n}$ and $Q_{3,n}Q_{4,n}$ positions.

(c)

$$\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}-3\\-1\end{bmatrix}\to\begin{bmatrix}1\\-3\end{bmatrix},\quad \begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\to\begin{bmatrix}-3\\1\end{bmatrix},$$

and

$$\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\to\begin{bmatrix}-1\\-1\end{bmatrix}.$$

Therefore,

n	16QAM symbol n	$I_{1,n}I_{2,n}$	$Q_{3,n}Q_{4,n}$	$I_{1,n-1}I_{2,n-1}$	phase change	$Q_{1,n}Q_{2,n}$
0	$(1,-3)$	01	01	11	270	11
1	$(-3,1)$	10	10	01	180	10
2	$(-1,-1)$	00	00	10	90	00

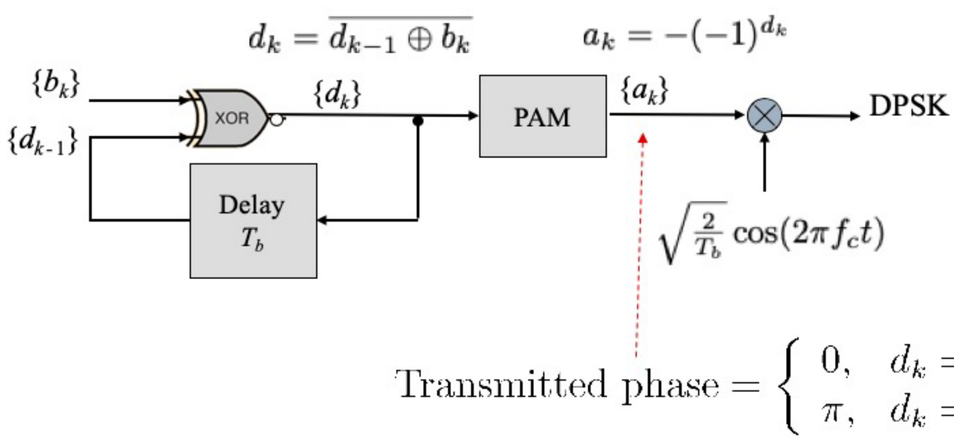
and

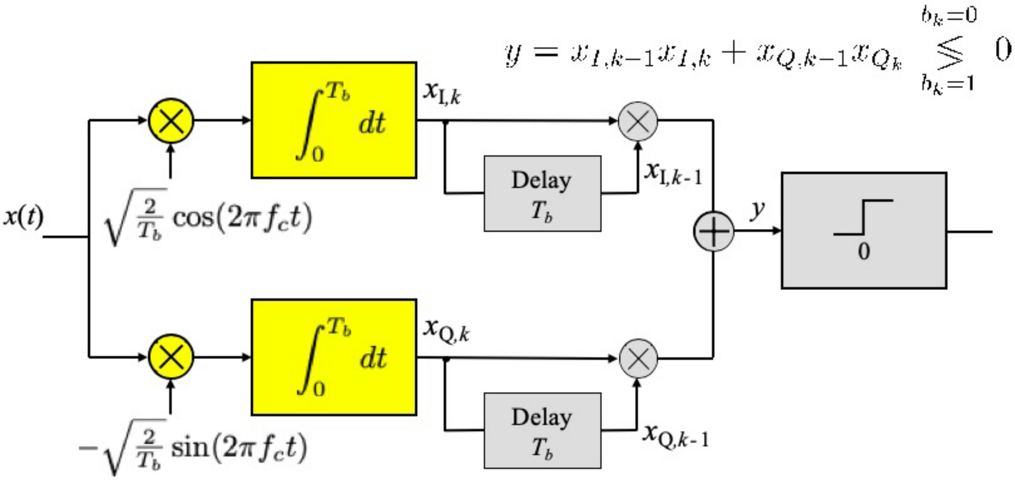
$$(\hat{Q}_{1,0}\hat{Q}_{2,0}\hat{Q}_{3,0}\hat{Q}_{4,0}\hat{Q}_{1,1}\hat{Q}_{2,1}\hat{Q}_{3,1}\hat{Q}_{4,1}\hat{Q}_{1,1}\hat{Q}_{2,1}\hat{Q}_{3,1}\hat{Q}_{4,1})=(1101\,1010\,0000)$$

where red-color numbers indicate the errors during transmission.

Note: So, a 90° phase difference, if not being calibrated to zero, causes no transmission errors in $Q_{3,n}Q_{4,n}$ positions, and also causes no transmission errors in $Q_{1,n}Q_{2,n}$ positions **except** the very first one (unless we properly adjust $I_{1,-1}I_{2,-1}$ to 10).

6. The functional blocks of the transmitter and the receiver of a DPSK signaling scheme is given as follows:





(a) Suppose $x(t) = -(-1)^{d_k}\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta) + w(t)$, where θ is an unknown phase difference between the transmitter oscillator and receiver oscillator, and $w(t)$ is a zero-mean additive Gaussian noise process with one-sided power spectrum density N_0. Given that f_c is a multiple of $1/T_b$, show that

$$\begin{cases} x_{I,k} = -(-1)^{d_k}\sqrt{E_b}\cos(\theta) + w_{I,k}; \\ x_{Q,k} = -(-1)^{d_k}\sqrt{E_b}\sin(\theta) + w_{I,k}, \end{cases}$$

where

$$w_{I,k} = \sqrt{\frac{2}{T_b}}\int_0^{T_b} w(t)\cos(2\pi f_c t)dt \quad \text{and} \quad w_{Q,k} = -\sqrt{\frac{2}{T_b}}\int_0^{T_b} w(t)\sin(2\pi f_c t)dt.$$

If $f_c = (2\ell - 1)/(4T_b)$ for some integer $\ell \geq 1$, does the above relation of x_{I_k} and x_{Q_k} remain valid?

(b) If $f_c = (2\ell - 1)/(4T_b)$ for some integer $\ell \geq 1$, are the two random variables $w_{I,k}$ and $w_{Q,k}$ independent of each other?

(c) (Just for your reference and not a part of the midterm) It is clear from the receiver functional diagram that the receiver makes decision based on the rule:

$$x_{I,k-1}x_{I,k} + x_{Q,k-1}x_{Q,k} \underset{b_k = 1}{\overset{b_k = 0}{\lessgtr}} 0$$

Is it a maximum-likelihood (ML) decision rule based on the expectation over a uniformly distributed θ over $[-\pi, \pi)$, provided f_c is a multiple of $1/(2T_b)$? Justify your answer.

Hint: For a known θ, a maximum-likelihood decision rule should be derived based on

$$\hat{b}_k = \arg\max_{0 \leq b_k \leq 1} \Pr(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k}|b_k).$$

Solution.

(a)

$$\begin{aligned}
x_{I,k} &= \int_0^{T_b}\left[-(-1)^{d_k}\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t+\theta)+w(t)\right]\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)dt \\
&= -(-1)^{d_k}\frac{2\sqrt{E_b}}{T_b}\int_0^{T_b}\cos(2\pi f_c t+\theta)\cos(2\pi f_c t)dt+\sqrt{\frac{2}{T_b}}\int_0^{T_b}w(t)\cos(2\pi f_c t)dt \\
&= -(-1)^{d_k}\frac{\sqrt{E_b}}{T_b}\int_0^{T_b}[\cos(4\pi f_c t+\theta)+\cos(\theta)]dt+w_{I,k} \\
&= -(-1)^{d_k}\frac{\sqrt{E_b}}{T_b}\int_0^{T_b}\cos(4\pi f_c t+\theta)dt-(-1)^{d_k}\sqrt{E_b}\cos(\theta)+w_{I,k} \\
&= -(-1)^{d_k}\frac{\sqrt{E_b}}{4\pi f_c T_b}\sin(4\pi f_c t+\theta)\Big|_0^{T_b}-(-1)^{d_k}\sqrt{E_b}\cos(\theta)+w_{I,k} \\
&= -(-1)^{d_k}\frac{\sqrt{E_b}}{4\pi f_c T_b}[\sin(4\pi f_c T_b+\theta)-\sin(\theta)]-(-1)^{d_k}\sqrt{E_b}\cos(\theta)+w_{I,k}, \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
x_{Q,k} &= \int_0^{T_b}\left[-(-1)^{d_k}\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t+\theta)+w(t)\right]\left(-\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)\right)dt \\
&= (-1)^{d_k}\frac{2\sqrt{E_b}}{T_b}\int_0^{T_b}\sin(2\pi f_c t)\cos(2\pi f_c t+\theta)dt-\sqrt{\frac{2}{T_b}}\int_0^{T_b}w(t)\sin(2\pi f_c t)dt \\
&= (-1)^{d_k}\frac{\sqrt{E_b}}{T_b}\int_0^{T_b}[\sin(4\pi f_c t+\theta)+\sin(-\theta)]dt+w_{Q,k} \\
&= (-1)^{d_k}\frac{\sqrt{E_b}}{T_b}\int_0^{T_b}\sin(4\pi f_c t+\theta)dt-(-1)^{d_k}\sqrt{E_b}\sin(\theta)+w_{Q,k} \\
&= -(-1)^{d_k}\frac{\sqrt{E_b}}{4\pi f_c T_b}\cos(4\pi f_c t+\theta)\Big|_0^{T_b}-(-1)^{d_k}\sqrt{E_b}\sin(\theta)+w_{Q,k} \\
&= -(-1)^{d_k}\frac{\sqrt{E_b}}{4\pi f_c T_b}[\cos(4\pi f_c T_b+\theta)-\cos(\theta)]-(-1)^{d_k}\sqrt{E_b}\sin(\theta)+w_{Q,k}. \qquad (2)
\end{aligned}$$

Both of the first terms in (1) and (2) equal zero as long as $2f_cT_b$ is an integer. However, if $f_c=(2\ell-1)/(4T_b)$, then

$$\begin{cases} x_{I,k}=-(-1)^{d_k}\sqrt{E_b}\cos(\theta)+w_{I,k}; \\ x_{Q,k}=-(-1)^{d_k}\sqrt{E_b}\sin(\theta)+w_{I,k}, \end{cases}$$

is no longer true.

Note: If $\theta = 0$, then the first term in (1) can be zero when $4\pi f_c T_b$ is a multiple of π (i.e., $4f_c T_b$ is an integer). On the other hand, if $\theta = \frac{\pi}{2}$, then the first term in (2) can be zero when $4\pi f_c T_b$ is a multiple of π (i.e., $4f_c T_b$ is an integer). However, θ can only be either 0 or $\frac{\pi}{2}$ (but not both), so we require $2f_c T_b$ to be an integer.

(b) Since

$$E[w_{I,k}] = E\left[\sqrt{\frac{2}{T_b}}\int_0^{T_b} w(t)\cos(2\pi f_c t)dt\right] = \sqrt{\frac{2}{T_b}}\int_0^{T_b} E[w(t)]\cos(2\pi f_c t)dt = 0$$

and

$$E[w_{Q,k}] = E\left[-\sqrt{\frac{2}{T_b}}\int_0^{T_b} w(t)\sin(2\pi f_c t)dt\right] = -\sqrt{\frac{2}{T_b}}\int_0^{T_b} E[w(t)]\sin(2\pi f_c t)dt = 0,$$

and since they are joint Gaussian distributed, we have that $w_{I,k}$ and $w_{Q,k}$ are independent if, and only if, $E[w_{I,k}w_{Q,k}] = 0$. We then derive

$$\begin{aligned}
E[w_{I,k}w_{Q,k}] &= E\left[\left(\sqrt{\frac{2}{T_b}}\int_0^{T_b} w(t)\cos(2\pi f_c t)dt\right)\left(-\sqrt{\frac{2}{T_b}}\int_0^{T_b} w(s)\sin(2\pi f_c s)ds\right)\right] \\
&= -\frac{2}{T_b}\int_0^{T_b}\int_0^{T_b} E[w(t)w(s)]\cos(2\pi f_c t)\sin(2\pi f_c s)dtds \\
&= -\frac{2}{T_b}\int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-s)\cos(2\pi f_c t)\sin(2\pi f_c s)dtds \\
&= -\frac{1}{T_b N_0}\int_0^{T_b}\cos(2\pi f_c t)\sin(2\pi f_c t)dt \\
&= -\frac{1}{2T_b N_0}\int_0^{T_b}\sin(4\pi f_c t)dt \\
&= \frac{1}{8T_b N_0 \pi f_c}\cos(4\pi f_c t)\Big|_0^{T_b} \\
&= \frac{1}{8T_b N_0 \pi f_c}[\cos(4\pi f_c T_b) - 1],
\end{aligned}$$

which equals zero as long as $2f_c T_b$ is an integer. However, if $f_c = (2\ell - 1)/(4T_b)$ for some integer $\ell \geq 1$, then $w_{I,k}$ and $w_{Q,k}$ are no longer independent.

Note: You shall learn from this problem that if f_c is not a multiple of $1/(2T_b)$, not only the first terms of the "projections" in (1) and (2) do not equal zeros (nevertheless, they could be very small when f_c is much larger than $1/(2T_b)$) but also the resultant additive noises are no longer independent. The latter could also deteriorate the system performance.

(c) Knowning

$$\begin{cases} -(-1)^{b_k} = [-(-1)^{d_k}][-(-1)^{d_{k-1}}] \\ x_{I,k-1} = -(-1)^{d_{k-1}}\sqrt{E_b}\cos(\theta) + w_{I,k-1}; \\ x_{Q,k-1} = -(-1)^{d_{k-1}}\sqrt{E_b}\sin(\theta) + w_{I,k-1}; \\ x_{I,k} = -(-1)^{d_k}\sqrt{E_b}\cos(\theta) + w_{I,k}; \\ x_{Q,k} = -(-1)^{d_k}\sqrt{E_b}\sin(\theta) + w_{I,k}, \end{cases}$$

we derive

$$\begin{aligned} & f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | b_k = 0) \\ &= f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | (d_{k-1}, d_k) = (0,1) \text{ or } (1,0)) \\ &= \frac{1}{2} f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | (d_{k-1}, d_k) = (0,1)) \\ &\quad + \frac{1}{2} f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | (d_{k-1}, d_k) = (1,0)), \end{aligned}$$

where the last equality is due to that subject to $B \cup C = \emptyset$,

$$\begin{aligned} \Pr(A|B \cup C) &= \frac{\Pr(A \cap (B \cup C))}{\Pr(B \cup C)} \\ &= \frac{\Pr((A \cap B) \cup (A \cap C))}{\Pr(B \cup C)} \\ &= \frac{\Pr(A \cap B) + \Pr(A \cap C)}{\Pr(B) + \Pr(C)} \\ &= \frac{\Pr(B)}{\Pr(B) + \Pr(C)} \Pr(A|B) + \frac{\Pr(C)}{\Pr(B) + \Pr(C)} \Pr(A|C). \end{aligned}$$

Similarly,

$$\begin{aligned} & f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | b_k = 1) \\ &= \frac{1}{2} f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | (d_{k-1}, d_k) = (0,0)) \\ &\quad + \frac{1}{2} f(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k} | (d_{k-1}, d_k) = (1,1)), \end{aligned}$$

As a result, for a known θ,

$$
\begin{aligned}
\hat{b}_k &= \arg \max_{0 \le b_k \le 1} \Pr(x_{I,k-1}, x_{Q,k-1}, x_{I,k}, x_{Q,k}|b_k) \\
&= \arg\max \left\{ e^{-\frac{(x_{I,k-1}+\sqrt{E_b}\cos(\theta))^2+(x_{Q,k-1}+\sqrt{E_b}\sin(\theta))^2+(x_{I,k}-\sqrt{E_b}\cos(\theta))^2+x_{Q,k}-\sqrt{E_b}\sin(\theta))^2}{2\sigma^2}} \right. \\
&\quad +e^{-\frac{(x_{I,k-1}-\sqrt{E_b}\cos(\theta))^2+(x_{Q,k-1}-\sqrt{E_b}\sin(\theta))^2+(x_{I,k}+\sqrt{E_b}\cos(\theta))^2+x_{Q,k}+\sqrt{E_b}\sin(\theta))^2}{2\sigma^2}}, \\
&\quad e^{-\frac{(x_{I,k-1}+\sqrt{E_b}\cos(\theta))^2+(x_{Q,k-1}+\sqrt{E_b}\sin(\theta))^2+(x_{I,k}+\sqrt{E_b}\cos(\theta))^2+x_{Q,k}+\sqrt{E_b}\sin(\theta))^2}{2\sigma^2}} \\
&\quad \left. +e^{-\frac{(x_{I,k-1}-\sqrt{E_b}\cos(\theta))^2+(x_{Q,k-1}-\sqrt{E_b}\sin(\theta))^2+(x_{I,k}-\sqrt{E_b}\cos(\theta))^2+x_{Q,k}-\sqrt{E_b}\sin(\theta))^2}{2\sigma^2}} \right\} \\
&= \arg\max \left\{ e^{\frac{-x_{I,k-1}\sqrt{E_b}\cos(\theta)-x_{Q,k-1}\sqrt{E_b}\sin(\theta)+x_{I,k}\sqrt{E_b}\cos(\theta)+x_{Q,k}\sqrt{E_b}\sin(\theta)}{\sigma^2}} \right. \\
&\quad +e^{\frac{x_{I,k-1}\sqrt{E_b}\cos(\theta)+x_{Q,k-1}\sqrt{E_b}\sin(\theta)-x_{I,k}\sqrt{E_b}\cos(\theta)-x_{Q,k}\sqrt{E_b}\sin(\theta)}{\sigma^2}}, \\
&\quad e^{\frac{-x_{I,k-1}\sqrt{E_b}\cos(\theta)-x_{Q,k-1}\sqrt{E_b}\sin(\theta)-x_{I,k}\sqrt{E_b}\cos(\theta)-x_{Q,k}\sqrt{E_b}\sin(\theta)}{\sigma^2}} \\
&\quad \left. +e^{\frac{x_{I,k-1}\sqrt{E_b}\cos(\theta)+x_{Q,k-1}\sqrt{E_b}\sin(\theta)+x_{I,k}\sqrt{E_b}\cos(\theta)+x_{Q,k}\sqrt{E_b}\sin(\theta)}{\sigma^2}} \right\} \\
&= \arg\max \left\{ e^{-\frac{v_I\cos(\theta)+v_Q\sin(\theta)}{\sigma^2/\sqrt{E_b}}} + e^{\frac{v_I\cos(\theta)+v_Q\sin(\theta)}{\sigma^2/\sqrt{E_b}}}, e^{-\frac{u_I\cos(\theta)+u_Q\sin(\theta)}{\sigma^2/\sqrt{E_b}}} + e^{\frac{u_I\cos(\theta)+u_Q\sin(\theta)}{\sigma^2/\sqrt{E_b}}} \right\} \\
&= \arg\max \left\{ \cosh\left(\frac{v_I\cos(\theta)+v_Q\sin(\theta)}{\sigma^2/\sqrt{E_b}} \right), \cosh\left(\frac{u_I\cos(\theta)+u_Q\sin(\theta)}{\sigma^2/\sqrt{E_b}} \right) \right\} \\
&= \arg\max \left\{ \cosh\left(\frac{\sqrt{E_b(v_I^2+v_Q^2)}}{\sigma^2}\cos(\theta-\phi_v) \right), \cosh\left(\frac{\sqrt{E_b(u_I^2+u_Q^2)}}{\sigma^2}\cos(\theta-\phi_u) \right) \right\},
\end{aligned}
$$

where

$$
\begin{cases}
u_I = x_{I,k-1} + x_{I,k}; \\
u_Q = x_{Q,k-1} + x_{Q,k}; \\
v_I = x_{I,k-1} - x_{I,k}; \\
v_Q = x_{Q,k-1} - x_{Q,k},
\end{cases}
\quad \text{and} \quad
\begin{cases}
\phi_v = \arctan(v_Q/v_I); \\
\phi_u = \arctan(u_Q/u_I).
\end{cases}
$$

Taking the expectation with respect to a uniformly distributed θ over $\theta \in [-\pi, \pi)$

yields

$$\begin{aligned}
\hat{b}_k &= \arg\max \left\{ E_\theta \left[\cosh \left(\frac{\sqrt{E_b(v_I^2+v_Q^2)}}{\sigma^2} \cos(\theta-\phi_v) \right) \right], \right. \\
&\qquad \left. E_\theta \left[\cosh \left(\frac{\sqrt{E_b(u_I^2+u_Q^2)}}{\sigma^2} \cos(\theta-\phi_u) \right) \right] \right\} \\
&= \arg\max \left\{ E_\theta \left[\cosh \left(\frac{\sqrt{E_b(v_I^2+v_Q^2)}}{\sigma^2} \cos(\theta) \right) \right], \right. \\
&\qquad \left. E_\theta \left[\cosh \left(\frac{\sqrt{E_b(u_I^2+u_Q^2)}}{\sigma^2} \cos(\theta) \right) \right] \right\} \\
&= \arg\max \{ \underbrace{v_I^2+v_Q^2}_{\hat{b}_k=0}, \underbrace{u_I^2+u_Q^2}_{\hat{b}_k=1} \},
\end{aligned}$$

which from Slide IDC2-84 confirms that

$$x_{I,k-1}x_{I,k} + x_{Q,k-1}x_{Q,k} \underset{b_k=1}{\overset{b_k=0}{\lessgtr}} 0$$

is an "expectation-based" maximum-likelihood (ML) decision rule.