Problems for Quiz 2

Name:____________________ Student ID:____________________ Score:________________

1. Denote the Hilbert transform operation and the Fourier transform operation by $\mathcal{H}\{\cdot\}$ and $\mathcal{F}\{\cdot\}$, respectively.

   (a) (25%) Let $\hat{p}(t) = \mathcal{H}\{p(t)\}$, $P(f) = \mathcal{F}\{p(t)\}$ and $\hat{P}(f) = \mathcal{F}\{\hat{p}(t)\}$. With $P(2000) = 1 + j$ and $P(4000) = 1 - j$, what are the values of $j\hat{P}(2000)$ and $j\hat{P}(-4000)$?
   Hint: $p(t)$ is real; hence, $P(-f) = P^*(f)$.

   (b) (25%) Let $H(f) = \mathcal{F}\{h(t)\}$ and $\hat{G}(f) = \mathcal{F}\{\mathcal{H}\{p(t) \star h(t)\}\}$, where

$$H(f) = \begin{cases} 1, & 1000 < |f| < 3000 \\ 0, & \text{otherwise} \end{cases}$$

   What are the values of $j\hat{G}(2000)$ and $j\hat{G}(-4000)$?

Solution.

(a) From

$$j\hat{P}(f) = \begin{cases} P(f), & f > 0 \\ -P(f) = -P^*(-f), & f < 0 \end{cases}$$

we have

$$j\hat{P}(2000) = P(2000) = 1 + j$$

and

$$j\hat{P}(-4000) = -P^*(4000) = -(1-j)^* = -1 - j.$$

(b) Since $j\hat{G}(f) = j\mathcal{F}\{\mathcal{H}\{p(t) \star h(t)\}\} = j\mathcal{F}\{\hat{p}(t) \star h(t)\} = j\hat{P}(f)H(f)$, we have

$$j\hat{G}(2000) = j\hat{P}(2000)H(2000) = 1 + j$$

and

$$j\hat{G}(-4000) = j\hat{P}(-4000)H(-4000) = 0.$$

2. (50%) Suppose the channel model follows $x(t) = s(t) + w(t)$. After the reception of $x(t)$, the detector performs "projection" onto $\phi(t)$-axis, i.e.,

$$\underbrace{\langle x(t), \phi(t)\rangle}_{x} = \underbrace{\langle s(t), \phi(t)\rangle}_{s} + \underbrace{\langle w(t), \phi(t)\rangle}_{w},$$

where s is equal to $-\sqrt{E_b}$ or $\sqrt{E_b}$ with equal probability, and w is zero-mean Gaussian distributed with variance $\frac{N_0}{2}$. Under the maximum-likelihood decision, i.e.,

$$\Pr(x|s = -\sqrt{E_b}) \mathop{\lesseqgtr}_{\hat{s} = -\sqrt{E_b}}^{\hat{s} = \sqrt{E_b}} \Pr(x|s = \sqrt{E_b}).$$

what will $\hat{s}$ be if $x = -2$? Does the optimal decision depend on the value of E_b or N_0? No justification for your answers is needed.

Solution. The maximum-likelihood decision is equivalent to

$$x \underset{\hat{s}=\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0.$$

Thus, $\hat{s} = -\sqrt{E_b}$ if $x = -2$. This decision has nothing to do with either E_b or N_0.

Note: Equal prior probability, alone, does not ensure the irrelevance of the optimal decision to either E_b or N_0. For example, if $s = -2$ and $s = 1$ with equal probability, then the optimal decision will be a function of both E_b and N_0. Thus, we need a very balanced constellation design in both "values" and "their probabilities" in order to have a robust optimal decision rule.