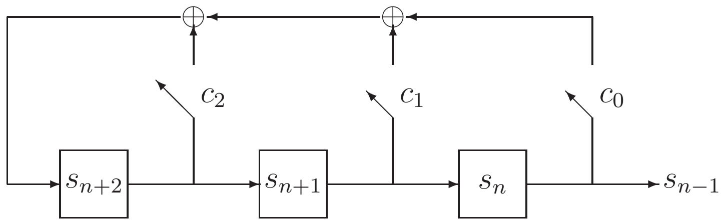
Problems for Midterm 2

1. (a) (6%)



For the pseudo-noise sequence generator above, we have

$$s_{n+3} = c_2 s_{n+2} \oplus c_1 s_{n+1} \oplus c_0 s_n,$$

where each s_n and c_i are in $\{0, 1\}$. Find the two maximum-length shift register sequences (i.e., m-sequences) corresponding to $c_0c_1c_2 = 110$ and 101. Please set $s_0s_1s_2 = 001$ as the initial value.

(b) (3%) Why we cannot set $s_0s_1s_2 = 000$ as the initial value in (a) for the generation of maximum-length shift-register sequences?

(c) (7%) Denote the two sequences in (a) by $a_0a_1a_2a_3a_4a_5a_6$ and $b_0b_1b_2b_3b_4b_5b_6$. Check the cross-correlation between the two m-sequences, defined as

$$C(j) \triangleq \sum_{i=0}^{6} (-1)^{a_i} (-1)^{b_{(i+j) \bmod 7}} \quad \text{for } 0 \le j \le 6.$$

(d) (9%) From (c), we know the cross-correlations are exactly three-valued, i.e., -1, $-t(3) = -5$ and $t(3) - 2 = 3$, where

$$t(m) = \begin{cases} 2^{(m+1)/2} + 1, & m \text{ odd} \\ 2^{(m+2)/2} + 1, & m \text{ even} \end{cases}$$

Thus, the two sequences can be used to generate the Gold sequences. Use the two maximum-length shift-register sequences for $m = 3$ to generate all $2^m + 1 = 9$ Gold sequences.

(e) (5%) List all Gold sequences that satisfy the balance property.

Hint: Balance property $\equiv$ the (absolute value of the) difference between the number of one's and the number of zeros is at most one.

Solution.

(a)

$c_0c_1c_2$	$s_0s_1s_2s_3s_4s_5s_6s_7s_8s_9s_{10}s_{11}s_{12}s_{13}\cdots$
110	$00101110010111\cdots$
101	$00111010011101\cdots$

(b) Because the output sequence will become an all-zero sequence if we set $s_0s_1s_2 = 000$ initially; hence, the (smallest period of the) output sequence cannot reach the maximum period 7 as required by the maximum-length shift register sequence.

(c) With $a_0a_1a_2a_3a_4a_5a_6 = 0010111$ and $b_0b_1b_2b_3b_4b_5b_6 = 0011101$, we have

j	$C(j)$
0	3
1	-1
2	-1
3	-5
4	3
5	3
6	-1

(d) In addition to the two maximum-length shift-register sequences $a_0a_1a_2a_3a_4a_5a_6 = 0010111$ and $b_0b_1b_2b_3b_4b_5b_6 = 0011101$, the other seven Gold sequences are

$a_0a_1a_2a_3a_4a_5a_6$	circular right shift of $b_0b_1b_2b_3b_4b_5b_6$	Gold sequences (Xor the previous two columns)
0010111	0011101	0001010
0010111	1001110	1011001
0010111	0100111	0110000
0010111	1010011	1000100
0010111	1101001	1111110
0010111	1110100	1100011
0010111	0111010	0101101

(e) The two maximum-length shift-register sequences $a_0a_1a_2a_3a_4a_5a_6 = 0010111$ and $b_0b_1b_2b_3b_4b_5b_6 = 0011101$, which are two of the nine Gold sequences, must satisfy the balance property. The remaining Gold sequences that satisfy the balance property are:

$a_0a_1a_2a_3a_4a_5a_6$	circular right shift of $b_0b_1b_2b_3b_4b_5b_6$	Gold sequences (Xor the previous two columns)
0010111	1001110	1011001
0010111	1110100	1100011
0010111	0111010	0101101

2. For a time-flat frequency-flat fading channel, the input signal suffers a multiplicative factor α, and hence the receiver receives

$$r = \alpha s + z,$$

where s is the signal to be transmitted and z is the additive noise. The signal-to-noise ratio of the system becomes $\gamma = \alpha^2 \frac{E_b}{N_0}$. Assume γ is Rayleigh distributed with probability density function

$$f_\gamma(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \quad \text{for } \gamma \geq 0,$$

where $\gamma_0 = E[\gamma]$.

(a) (10%) Show that the average bit error rate (BER) for binary DPSK is $\frac{1}{2(1+\gamma_0)}$, where its BER without fading is equal to $\frac{1}{2}e^{-\gamma}$.

Hint:

$$\text{BER} = \int_0^\infty \frac{1}{2} e^{-\gamma} \left(\frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \right) \mathrm{d}\gamma$$

(b) (10%) With L-diversity, we have

$$\begin{aligned} r_1 &= \alpha_1 s + z_1 \\ r_2 &= \alpha_2 s + z_2 \\ &\vdots \\ r_L &= \alpha_L s + z_L \end{aligned}$$

where $\{z_k\}_{k=1}^L$ are zero-mean i.i.d. with variance σ^2 and $\{\alpha_k\}_{k=1}^L$ are assumed to be perfectly estimated. Find the optimal **linear** combiner

$$r = \sum_{k=1}^{L} w_k \cdot r_k$$

that maximizes the output signal-to-noise ratio (SNR).

Hint: Please particular note that

$$E[z_i z_j] = \begin{cases} \sigma^2, & i = j \\ 0, & i \neq j \end{cases}$$

Find the weights $\{w_k\}_{k=1}^L$ such that the output SNR is maximized by using the Cauchy-Schwartz inequality.

Solution.

(a)

$$\begin{aligned}
\text{BER} &= \int_0^\infty \frac{1}{2} e^{-\gamma} \left(\frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \right) \mathrm{d}\gamma \\
&= \int_0^\infty \frac{1}{2\gamma_0} e^{-\gamma(1+1/\gamma_0)} \mathrm{d}\gamma \quad (x = \gamma(1+1/\gamma_0)) \\
&= \int_0^\infty \frac{1}{2(1+\gamma_0)} e^{-x} dx \\
&= \frac{1}{2(1+\gamma_0)} \left(-e^{-x}\right) \Big|_0^\infty \\
&= \frac{1}{2(1+\gamma_0)}
\end{aligned}$$

(b) We derive

$$\begin{aligned}
r &= \sum_{k=1}^{L} w_k r_k = \sum_{k=1}^{L} w_k (s + z_k) \\
&= \sum_{k=1}^{L} w_k \alpha_k s + \sum_{k=1}^{L} w_k z_k \\
&= s \sum_{k=1}^{L} w_k \alpha_k + \sum_{k=1}^{L} w_k z_k .
\end{aligned}$$

Under the assumption that $\{\alpha_k\}_{k=1}^{L}$ can be perfectly estimated by the receiver, the output SNR satisfies

$$\text{SNR} = \frac{\left(\sum_{k=1}^{L} w_k \alpha_k\right)^2 E[s^2]}{\sum_{k=1}^{L} w_k^2 E[z_k^2]} \le \frac{\left(\sum_{k=1}^{L} w_k^2\right)\left(\sum_{k=1}^{L} \alpha_k^2\right) E[s^2]}{\sigma^2 \sum_{k=1}^{L} w_k^2} = \left(\sum_{k=1}^{L} \alpha_k^2\right) \frac{E[s^2]}{\sigma^2},$$

where the inequality follows the Cauchy-Schwartz inequality. The optimal $\{w_k\}_{k=1}^{L}$ are thus the ones that equate the Cauchy-Schwartz inequality, i.e.,

$$w_k = C \cdot \alpha_k$$

for some constant C.

3. (a) (10%) In an antenna arrays system, suppose each of the two users equips one antenna, and the base station has four antennas. Thus, the base station receives

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{=\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\boldsymbol{v}},$$

where

$$\begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

are perfectly pre-measured, and $\{v_k\}_{k=1}^4$ are zero-mean i.i.d. with variance σ^2. Let m_1 be the message of interest. The receiver then uses a **linear** combiner with weights $\{w_k\}_{k=1}^4$ to determine m_1. Find the weights that maximize the output SNR, subject to that $\boldsymbol{w}$ is orthogonal to $\boldsymbol{c}_2$, where

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

Hint: By using a dot to denote the inner product,

$$\boldsymbol{c}_1 = \underbrace{\left(\boldsymbol{c}_1 \cdot \frac{\boldsymbol{c}_2}{\|\boldsymbol{c}_2\|}\right)\frac{\boldsymbol{c}_2}{\|\boldsymbol{c}_2\|}}_{\boldsymbol{u}} + \underbrace{\left(\boldsymbol{c}_1 - \left(\boldsymbol{c}_1 \cdot \frac{\boldsymbol{c}_2}{\|\boldsymbol{c}_2\|}\right)\frac{\boldsymbol{c}_2}{\|\boldsymbol{c}_2\|}\right)}_{\boldsymbol{u}^\perp} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

where $\boldsymbol{u}^\perp$ is orthogonal to $\boldsymbol{c}_2$.

(b) (8%) The recovery technique of m_1 in (a) is based on the conventional notion of "orthogonality." By noting that m_1 and m_2 are actually discrete, orthogonality might not be necessary for the recovery of m_1. If

$$\begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

how we can recover m_1 from $\boldsymbol{x}$, provided $m_1 \in \{-2, 2\}$ and $m_2 \in \{-1, 1\}$? Justify your answer.

Solution.

(a) The receiver performs

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{x} &= \boldsymbol{w} \cdot \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \boldsymbol{w} \cdot \boldsymbol{v} = \begin{bmatrix} \boldsymbol{w} \cdot \boldsymbol{c}_1 & 0 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \boldsymbol{w} \cdot \boldsymbol{v} \\ &= m_1 \boldsymbol{w} \cdot \boldsymbol{c}_1 + \boldsymbol{w} \cdot \boldsymbol{v} \end{aligned}$$

The output SNR is thus given by

$$\text{SNR} = \frac{E[|m_1 \boldsymbol{w} \cdot \boldsymbol{c}_1|^2]}{E[|\boldsymbol{w} \cdot \boldsymbol{v}|^2]} = \frac{|\boldsymbol{w} \cdot \boldsymbol{c}_1|^2 \, E[|m_1|^2]}{\|\boldsymbol{w}\|^2 \sigma^2}.$$

Note that $\boldsymbol{w}$ must be orthogonal to the space $\mathcal{I}$ spanned by $\boldsymbol{c}_2$. Decompose

$$\boldsymbol{c}_1 = \boldsymbol{u} + \boldsymbol{u}^{\perp},$$

where $\boldsymbol{u}$ is on the space $\mathcal{I}$ (so, $\boldsymbol{w} \cdot \boldsymbol{u} = 0$) and $\boldsymbol{u}^{\perp}$ is orthogonal to the space $\mathcal{I}$. The output SNR can the be rewritten as

$$\text{SNR} = \frac{\boldsymbol{w} \cdot (\boldsymbol{u}^{\perp} + \boldsymbol{u}) E[|m_1|^2]}{\|\boldsymbol{w}\|^2 \sigma^2} = \frac{\boldsymbol{w} \cdot \boldsymbol{u}^{\perp} E[|m_1|^2]}{\|\boldsymbol{w}\|^2 \sigma^2} \le \frac{\|\boldsymbol{w}\|^2 \|\boldsymbol{u}^{\perp}\|^2 E[|m_1|^2]}{\|\boldsymbol{w}\|^2 \sigma^2} = \frac{\|\boldsymbol{u}^{\perp}\|^2 E[|m_1|^2]}{\sigma^2}$$

where we use the Cauchy-Schwartz inequality. The optimal $\boldsymbol{w}$ is thus proportional to $\boldsymbol{u}^{\perp}$, which is

$$\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

(b) The receiver knows $x_1 = x_3 = m_1 + m_2 \in \{-2-1, -2+1, 2-1, 2+1\} = \{-3, -1, 1, 3\}$. Thus, when $x_1 = x_3 \in \{-3, -1\}$, the receiver knows $m_1 = -2$; and when $x_1 = x_3 \in \{1, 3\}$, the receiver is certain that $m_1 = 2$.

4. (a) (8%) Give a binary prefix code $\{00, 01, \ldots\}$, of which the first two codewords are 00 and 01 and the longest codeword is 3, and which contains six codewords of lengths equating the Kraft-McMillan inequality.

   Hint: The binary tree that corresponds to a prefix code, whose codeword lengths equate the Kraft-MaMillan inequality, must be saturated.

   (b) (8%) Let the six symbols correspond to codeword $\{00, 01, \ldots\}$ be $ABCDEF$. Decode 0010101010100000010100.

   Hint: You shall give **your** correspondence between binary codewords and symbols before performing the decoding.

Solution.

(a) $\{00, 01, 100, 101, 110, 111\}$.

(b) Let $\{00, 01, 100, 101, 110, 111\}$ correspond to $\{A, B, C, D, E, F\}$. Then,

$$0010101010100000010100$$

can be decoded as $00, 101, 01, 01, 01, 00, 00, 01, 01, 00 = ADBBBAABBA$.

5. Prove that the entropy of a discrete random variable X satisfies the following inequalities. Also, give the necessary and sufficient condition under which equality holds.

   (a) (8%) $H(X) \geq 0$

   (b) (8%) $H(X) \leq \log_2(K)$, where K is the size of the support of X (i.e., X only takes on K possible values).

   Solution. See Slides IDC 6-17 and 6-18.