1. Define the inner product of two signals as

$$\langle \phi_1(t), \phi_2(t) \rangle \triangleq \int_0^T \phi_1(t)\phi_2^*(t)dt.$$

(a) (8%) Given that $2(f_1 + f_2)T$ is an integer, what is the smallest non-zero value of $|f_1 - f_2|$ such that

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t) \text{ and } \phi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)$$

are orthogonal?

(b) (8%) Show that the projections of a **zero-mean** white noise process $w(t)$ onto two orthonormal basis $\phi_1(t)$ and $\phi_2(t)$ are uncorrelated.

Hint: $R_w(\tau) = E[w(t+\tau)w^*(t)] = \frac{N_0}{2}\delta(\tau)$. Note that in this sub-problem, $w(t)$, $\phi_1(t)$ and $\phi_2(t)$ are generally complex-valued functions.

Hint: By definition, w_1 and w_2 are uncorrelated if $E[w_1 w_2^*] = E[w_1]E[w_2^*]$.

Solution.

(a)

$$\begin{aligned}
&\langle \phi_1(t), \phi_2(t) \rangle \\
&= \frac{2}{T}\int_0^T \cos(2\pi f_1 t)\cos(2\pi f_2 t)\,dt \\
&= \frac{2}{T}\int_0^T \frac{\cos(2\pi(f_1+f_2)t) + \cos(2\pi(f_1-f_2)t)}{2}dt \\
&= \underbrace{\frac{\sin(2\pi(f_1+f_2)T)}{2\pi(f_1+f_2)T}}_{=0 \text{ because } 2(f_1+f_2)T \text{ integer}} + \frac{\sin(2\pi(f_1-f_2)T)}{2\pi(f_1-f_2)T} \\
&= \frac{\sin(2\pi(f_1-f_2)T)}{2\pi(f_1-f_2)T}
\end{aligned}$$

Hence, the smallest non-zero value of $|f_1 - f_2|$ such that

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t) \text{ and } \phi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)$$

are orthogonal is $2|f_1 - f_2|T = 1$, which implies $|f_1 - f_2| = \frac{1}{2T}$.

(b) Let $w_1 = \langle w(t), \phi_1(t) \rangle$ and $w_2 = \langle w(t), \phi_2(t) \rangle$. By definition, w_1 and w_2 are uncorrelated if $E[w_1 w_2^*] = E[w_1]E[w_2^*] = 0$, where the last equality holds because $w(t)$ is a zero-mean process. Hence, we derive

$$\begin{aligned} E[w_1 w_2] &= E\left[\langle w(t), \phi_1(t)\rangle \cdot (\langle w(t), \phi_2(t)\rangle)^*\right] \\ &= E\left[\left(\int_0^T w(t)\phi_1^*(t)dt\right) \cdot \left(\int_0^T w(s)\phi_2^*(s)ds\right)^*\right] \\ &= \int_0^T \int_0^T E\left[w(t)w^*(s)\right] \phi_1^*(t)\phi_2(s)dtds \\ &= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\phi_1^*(t)\phi_2(s)dtds \\ &= \frac{N_0}{2}\int_0^T \phi_1^*(t)\phi_2(t)dt \\ &= \frac{N_0}{2}\langle \phi_2(t), \phi_1(t)\rangle \\ &= 0 \end{aligned}$$

2. (12%) Consider a binary FSK signaling scheme defined by

$$s(t) = \sum_{n=-\infty}^{\infty} g(t - nT_b)\cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h\left(\frac{t - nT_b}{T_b}\right)\right), \tag{1}$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < T_b; \\ 0, & \text{otherwsie.} \end{cases}$$

We have learned that phase-continuity is guaranteed by the formulation in (1) if $I_n \in \{\pm 1\}$.

Is phase-continuity still valid if we allow $I_n \in \{\pm 1, \pm 3\}$? Justify your answer.

Hint: Discontinuity can only occur at $t = nT_b$; thus, we require

$$\lim_{t \uparrow nT_b} s(t) = \lim_{t \downarrow nT_b} s(t)$$

for arbitrary I_{n-1} and I_n and for arbitrary integer n. Note that $t \uparrow nT_b$ means that t approaches nT_b from below, and $t \downarrow nT_b$ means that t approaches nT_b from above.

Solution. For phase-continuity, we observe that discontinuity can only occur at $t = \ell T_b$; thus, we require $\lim_{t \uparrow \ell T_b} s(t) = \lim_{t \downarrow \ell T_b} s(t)$. Derive

$$\begin{aligned} &\lim_{t \uparrow \ell T_b} s(t) \\ &= \lim_{t \uparrow \ell T_b} \sum_{n=-\infty}^{\infty} g(t - nT_b)\cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h\left(\frac{t - nT_b}{T_b}\right)\right) \\ &= \cos\left(2\pi f_c \ell T_b + \sum_{k=-\infty}^{\ell-2} I_k \pi h + I_{\ell-1}\pi h\left(\frac{\ell T_b - (\ell-1)T_b}{T_b}\right)\right) \quad \text{(i.e., } n = \ell - 1\text{)} \\ &= \cos\left(2\pi f_c \ell T_b + \sum_{k=-\infty}^{\ell-1} I_k \pi h\right), \end{aligned}$$

and

$$
\begin{aligned}
&\lim_{t \downarrow \ell T_b} s(t) = s(\ell T_b) \\
&= \sum_{n=-\infty}^{\infty} g(t - nT_b) \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t - nT_b}{T_b}\right)\right)\Bigg|_{t=\ell T_b} \\
&= \cos\left(2\pi f_c \ell T_b + \sum_{k=-\infty}^{\ell-1} I_k \pi h\right) \quad (\text{i.e., } n = \ell).
\end{aligned}
$$

Thus, the FSK scheme in (1) fulfills phase-continuity no matter what I_n is.

3. For digital communications, we can ignore completely the waveforms and work on the system design over the projections (i.e., over the signal constellation). Suppose a N-dimensional constellation is constructed. A system designer chooses two constellation points $\mathbf{s}_1$ and $\mathbf{s}_2$ for binary transmission over the AWGN channel. The additive noise vector $\mathbf{n}$ has the pdf

$$
\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{\|\mathbf{n}\|^2}{2\sigma^2}\right\}.
$$

(a) (8%) Complete the optimal decision rule below by filling in the quantity inside parentheses for the received vector $\mathbf{x} = \mathbf{s} + \mathbf{n}$,

$$
\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2 \rangle \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} \left(\qquad \qquad \right)
$$

where $\mathbf{s}$ is either $\mathbf{s}_1$ or $\mathbf{s}_2$ with equal probability.
Hint: MAP decision rule:

$$
\hat{\mathbf{s}}_{\text{MAP}} = \max_{m=1,2} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{\|\mathbf{x} - \mathbf{s}_m\|^2}{2\sigma^2}\right\} = \min_{m=1,2} \|\mathbf{x} - \mathbf{s}_m\|^2.
$$

(b) (8%) Suppose

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix}
$$

are four dimensional vectors, and the receiver can only observe

$$
\ell_1^2 \triangleq x_1^2 + x_2^2 \quad \text{and} \quad \ell_2^2 \triangleq x_3^2 + x_4^2.
$$

The receiver thus adopts the decision rule as:

$$
\ell_1^2 \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} \ell_2^2
$$

Complete the derivation of the probability of erroneous decision given $\mathbf{s} = \mathbf{s}_1$ below:

$$\begin{aligned}
\Pr[\ell_1^2 \le \ell_2^2 | \mathbf{s} = \mathbf{s_1}] &= \Pr\left(x_1^2 + x_2^2 \le x_3^2 + x_4^2\right) \quad \text{with } \begin{cases} x_1 \sim \mathcal{N}(a_1, \sigma^2) \\ x_2 \sim \mathcal{N}(a_2, \sigma^2) \\ x_3 \sim \mathcal{N}(0, \sigma^2) \\ x_4 \sim \mathcal{N}(0, \sigma^2) \end{cases} \\
&= \Pr\left(\sqrt{x_1^2 + x_2^2} \le \ell_2\right) \quad \text{with } \begin{cases} x_1 \sim \mathcal{N}(a_1, \sigma^2) \\ x_2 \sim \mathcal{N}(a_2, \sigma^2) \\ \ell_2 \text{ Rayleigh with } E[\ell_2^2] = 2\sigma^2 \end{cases} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1-a_1)^2}{2\sigma^2}} e^{-\frac{(x_2-a_2)^2}{2\sigma^2}} \left(\int_{\sqrt{x_1^2+x_2^2}}^{\infty} \frac{\ell_2}{\sigma^2} e^{-\frac{\ell_2^2}{2\sigma^2}} d\ell_2\right) dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1-a_1)^2}{2\sigma^2}} e^{-\frac{(x_2-a_2)^2}{2\sigma^2}} \left(e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}\right) dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{2x_1^2 - 2a_1x_1 + a_1^2 + 2x_2^2 - 2a_2x_2 + a_2^2}{2\sigma^2}} dx_1 dx_2 \\
&= \cdots
\end{aligned}$$

Hint: The pdf of $\ell_2 \triangleq \sqrt{n_3^2 + n_4^2}$ is $\frac{\ell_2}{\sigma^2} e^{-\frac{\ell_2^2}{2\sigma^2}}$ for $\ell_2 \ge 0$.

Solution.

(a) $\hat{\mathbf{s}}_{\text{MAP}} = \mathbf{s}_1$ if $\|\mathbf{x} - \mathbf{s}_1\|^2 \le \|\mathbf{x} - \mathbf{s}_2\|^2$, and $\hat{\mathbf{s}}_{\text{MAP}} = \mathbf{s}_2$, otherwise. Derive

$$\begin{aligned}
&\|\mathbf{x} - \mathbf{s}_1\|^2 \le \|\mathbf{x} - \mathbf{s}_2\|^2 \\
&\quad\Leftrightarrow\quad \|\mathbf{x}\|^2 + \|\mathbf{s}_1\|^2 - 2\langle \mathbf{x}, \mathbf{s}_1\rangle \le \|\mathbf{x}\|^2 + \|\mathbf{s}_2\|^2 - 2\langle \mathbf{x}, \mathbf{s}_2\rangle \\
&\quad\Leftrightarrow\quad \|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2 \le 2\langle \mathbf{x}, \mathbf{s}_1\rangle - 2\langle \mathbf{x}, \mathbf{s}_2\rangle = 2\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2\rangle
\end{aligned}$$

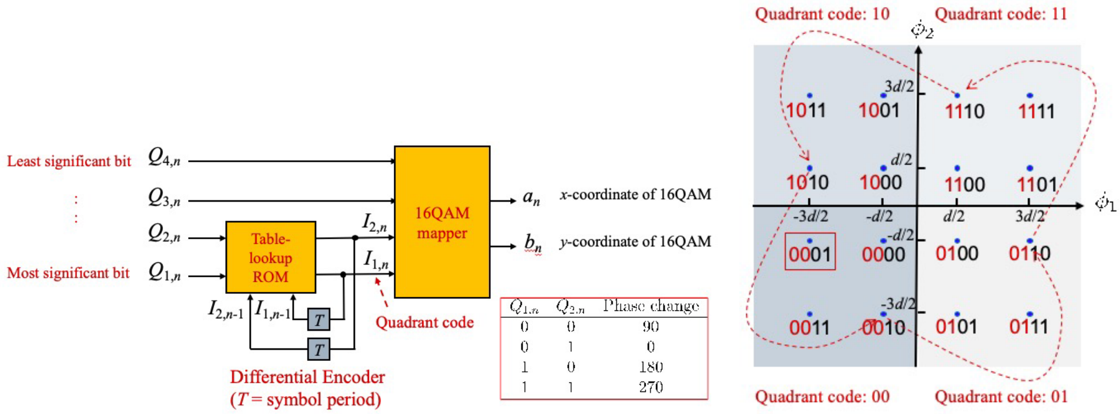
Consequently, the optimal decision rule is

$$\langle \mathbf{x}, \mathbf{s}_1 - \mathbf{s}_2\rangle \underset{\mathbf{s}_1 \text{ is transmitted}}{\overset{\mathbf{s}_2 \text{ is trasmitted}}{\lessgtr}} \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2}$$

(b)

$$\begin{aligned}
\Pr[\ell_1^2 \le \ell_2^2 | \mathbf{s} = \mathbf{s_1}] &= \cdots \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{2x_1^2 - 2a_1x_1 + a_1^2 + 2x_2^2 - 2a_2x_2 + a_2^2}{2\sigma^2}} dx_1 dx_2 \\
&= \frac{1}{2} e^{-\frac{(a_1^2 + a_2^2)}{4\sigma^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\frac{(x_1 - \frac{1}{2}a_1)^2}{\sigma^2}} e^{-\frac{(x_2 - \frac{1}{2}a_2)^2}{\sigma^2}} dx_1 dx_2 \\
&= \frac{1}{2} e^{-\frac{(a_1^2 + a_2^2)}{4\sigma^2}} \quad \left(= \frac{1}{2} e^{-\frac{\|\mathbf{s}_1\|^2}{4\sigma^2}}\right).
\end{aligned}$$

4. Below is the functional diagram of the V.32 16-QAM Hybrid amplitude/phase modulation scheme:



(a) (8%) Assume $I_{1,-1} = I_{2,-1} = 1$. Give the sequence of 16QAM symbols (indicated by their coordinates) corresponding to

$$(Q_{1,0}Q_{2,0}Q_{3,0}Q_{4,0}\; Q_{1,1}Q_{2,1}Q_{3,1}Q_{4,1}\; Q_{1,2}Q_{2,2}Q_{3,2}Q_{4,2}) = (1001\,1010\,0000)$$

(b) (8%) Suppose there is a 90° phase difference between the transmitter and the receiver, i.e.,

$$\begin{bmatrix} a_{\text{receive}} \\ b_{\text{receive}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{\text{transmit}} \\ b_{\text{transmit}} \end{bmatrix}.$$

By using the nearest Euclidean distance criterion, recover the transmitted information sequence $(Q_{1,0}Q_{2,0}Q_{3,0}Q_{4,0}\; Q_{1,1}Q_{2,1}Q_{3,1}Q_{4,1}\; Q_{1,2}Q_{2,2}Q_{3,2}Q_{4,2})$ from the rotated 16QAM symbols.

Hint: The nearest Euclidean distance decision can be made separately on x-axis and y-axis over the 16QAM constellation with thresholds -2, 0 and 2.

Solution.

(a)

n	$I_{1,n-1}I_{2,n-1}$	$Q_{1,n}Q_{2,n}$	phase change	$I_{1,n}I_{2,n}$	$Q_{3,n}Q_{4,n}$	16QAM symbol n
0	11	10	180	00	01	$(-3,-1)$
1	00	10	180	11	10	$(1,3)$
2	11	00	90	10	00	$(-1,1)$

(b)

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Therefore,

n	16QAM symbol n	$I_{1,n}I_{2,n}$	$Q_{3,n}Q_{4,n}$	$I_{1,n-1}I_{2,n-1}$	phase change	$Q_{1,n}Q_{2,n}$
0	$(1,-3)$	01	01	11	270	11
1	$(-3,1)$	10	10	01	180	10
2	$(-1,-1)$	00	00	10	90	00

and

$$(\hat{Q}_{1,0}\hat{Q}_{2,0}\hat{Q}_{3,0}\hat{Q}_{4,0}\hat{Q}_{1,1}\hat{Q}_{2,1}\hat{Q}_{3,1}\hat{Q}_{4,1}\hat{Q}_{1,1}\hat{Q}_{2,1}\hat{Q}_{3,1}\hat{Q}_{4,1}) = (1101\,1010\,0000)$$

where red-color numbers indicate the errors during transmission.

5. Suppose we now have three **independent** channels with transmission powers P_1, P_2 and P_3, respectively. The transmission rates attainable are governed by the Shannon capacity formula as:

$$\frac{1}{2}\log_2\left(1+\frac{P_i}{\sigma_i^2}\right) \text{ for } i = 1, 2, 3,$$

where σ_i^2 is the noise power of the ith channel. The sum rate is thus equal to

$$R(\vec{P}) = \sum_{i=1}^{3} \frac{1}{2}\log_2\left(1+\frac{P_i}{\sigma_i^2}\right),$$

where $\vec{P} = (P_1, P_2, P_3)$.

(a) (8%) Subject to $P_1 + P_2 + P_3 = P > 0$, we wish to find the optimal power allocation that maximizes $R(\vec{P})$, i.e.,

$$\vec{P}^{\,\text{opt}} = \arg\max_{\vec{P}\in\mathbb{Q}} R(\vec{P}),$$

where

$$\mathbb{Q} \triangleq \left\{(P_1, P_2, P_3) : P_i \in \mathbb{R}^+ \text{ for } i = 1, 2, 3, \text{ and } P_1 + P_2 + P_3 = P\right\}$$

and $\mathbb{R}^+$ is the set of non-negative real numbers. Define

$$f(\vec{P}, \lambda) = R(\vec{P}) + \lambda\big(P - P_1 - P_2 - P_3\big).$$

Does the following equation hold:

$$\max_{\vec{P}\in\mathbb{Q}} R(\vec{P}) = \max_{\vec{P}\in\mathbb{Q}} f(\vec{P}, \lambda)?$$

Justify your answer.

(b) (8%) Instead of working on $\max_{\vec{P}\in\mathbb{Q}} f(\vec{P}, \lambda)$, we turn to the determination of the power allocation that maximizes its upper bound $\max_{\vec{P}\in(\mathbb{R}^+)^3} f(\vec{P}, \lambda)$. The necessary condition for an optimizer $\vec{P}^*$ to maximize $f(\vec{P}, \lambda)$ over $\vec{P} \in (\mathbb{R}^+)^3$ must satisfy

$$\frac{\partial f(\vec{P}, \lambda)}{\partial P_i} \begin{cases} = 0, & \text{if } P_i^* > 0 \\ \leq 0, & \text{if } P_i^* = 0 \end{cases}$$

Explain why we need to introduce the inequality condition $\frac{\partial f(\vec{P},\lambda)}{\partial P_i} \leq 0$ when $P_i^* = 0$. Hint: $f(\vec{P}, \lambda)$ is concave with respect to each P_i, and you may draw a picture to explain what happens when $P_i^* = 0$.

(c) (8%) From (b), we know that the optimal power allocation that achieves $\max_{\vec{P}\in(\mathbb{R}^+)^3} f(\vec{P},\lambda)$ satisfies

$$\sigma_i^2 + P_i^* \begin{cases} = K = \frac{\log_2(e)}{6\lambda}, & \text{if } P_i^* > 0 \\ \geq K & \text{if } P_i^* = 0. \end{cases} \tag{2}$$

Let $\sigma_1^2 = \sigma_2^2 = 1$ and $\sigma_3^2 = 3$, and let $P = 2$. Argue that $P_1^* = P_2^* = 1$ and $P_3^* = 0$ maximizes $R(\vec{P})$ among all $\vec{P} \in \mathbb{Q}$.

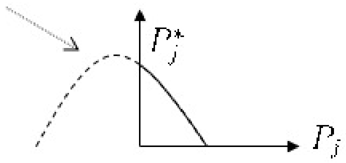
You shall give the value of K (equivalently, λ) if you think $(P_1^*, P_2^*, P_3^*) = (1, 1, 0)$ satisfies (2).

Hint: $\max_{\vec{P}\in\mathbb{Q}} R(\vec{P}) = \max_{\vec{P}\in\mathbb{Q}} f(\vec{P},\lambda) \leq \max_{\vec{P}\in(\mathbb{R}^+)^3} f(\vec{P},\lambda)$

Solution.

(a) Yes because $R(\vec{P}) = f(\vec{P},\lambda)$ for every $\vec{P} \in \mathbb{Q}$ (regardless of λ chosen).

(b) Since $R(\vec{P})$ is concave with respect to each P_i, its optimizer may occur at an $P_i < 0$ as shown in the below figure.



In such case, a feasible maximizer is $P_i^* = 0$ and the derivative at this point is non-positive.

(c) With $K = 2$ $(= \frac{\log_2(e)}{6\lambda})$, we have

$$\begin{cases} \sigma_1^2 + P_1^* = 1 + 1 = 2, & \text{if } P_1^* = 1 > 0 \\ \sigma_2^2 + P_2^* = 1 + 1 = 2, & \text{if } P_2^* = 1 > 0 \\ \sigma_3^2 + P_3^* = 3 + 0 \geq 2 & \text{if } P_3^* = 0. \end{cases}$$

Therefore, $\left(\text{with } \lambda = \frac{\log_2(e)}{12},\right)$

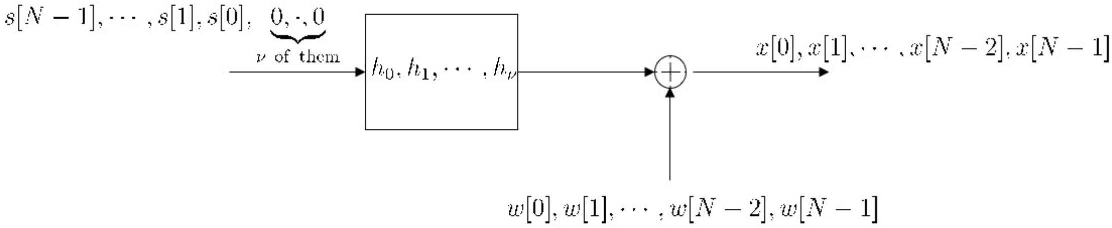
$$\max_{\vec{P}\in\mathbb{Q}} R(\vec{P}) = \max_{\vec{P}\in\mathbb{Q}} f(\vec{P},\lambda) \leq \max_{\vec{P}\in(\mathbb{R}^+)^3} f(\vec{P},\lambda) = f((1,1,0),\lambda).$$

Since $f((1,1,0),\lambda) = R(1,1,0)$ as $\vec{P} = (1,1,0) \in \mathbb{Q}$, we obtain

$$\begin{aligned} R(1,1,0) &\leq \max_{\vec{P}\in\mathbb{Q}} R(\vec{P}) \left(= \max_{\vec{P}\in\mathbb{Q}} f(\vec{P},\lambda) \leq \max_{\vec{P}\in(\mathbb{R}^+)^3} f(\vec{P},\lambda) = f((1,1,0),\lambda) \right) \\ &= R(1,1,0). \end{aligned}$$

Consequently, $R(1,1,0) = \max_{\vec{P}\in\mathbb{Q}} R(\vec{P})$.

6. The block diagram of a discrete transmission using guard period to avoid ISI is pictured as follows.



(a) (8%) Suppose $\nu = 2$, $N = 4$ and $s[0] = 1 + j$, $s[1] = 1 - j$, $s[2] = 1 + j$, $s[3] = -1 - j$. Give the values of $s[-1]$ and $s[-2]$ such that output $x[0]$, $x[1]$, $x[2]$, $x[3]$ become a circular convolution of input $s[0]$, $s[1]$, $s[2]$, $s[3]$ and channel impulse response $h[0] = 1$, $h[1] = h_1$, $h[2] = h_2$, i.e.,

$$\boldsymbol{x} = \begin{bmatrix} x[3] \\ x[2] \\ x[1] \\ x[0] \end{bmatrix} = \begin{bmatrix} 1 & h_1 & h_2 & 0 \\ 0 & 1 & h_1 & h_2 \\ h_2 & 0 & 1 & h_1 \\ h_1 & h_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} s[3] \\ s[2] \\ s[1] \\ s[0] \end{bmatrix} + \begin{bmatrix} w[3] \\ w[2] \\ w[1] \\ w[0] \end{bmatrix} = \mathbb{H}_{\text{circulant}} \boldsymbol{s} + \boldsymbol{w}.$$

(b) (8%) A circulant matrix satisfies $\mathbb{Q}\mathbb{H}_{\text{circulant}} = \Lambda \mathbb{Q}$, where

$$\Lambda = \begin{bmatrix} \lambda_{N-1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{N-2} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{N-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_0 \end{bmatrix}$$

and

$$\mathbb{Q} = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{-j\frac{2\pi}{N}(N-1)(N-1)} & \cdots & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}(N-1)} & 1 \\ e^{-j\frac{2\pi}{N}(N-1)(N-2)} & \cdots & e^{-j\frac{2\pi}{N}2(N-2)} & e^{-j\frac{2\pi}{N}(N-2)} & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ e^{-j\frac{2\pi}{N}(N-1)} & \cdots & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}} & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

Prove that

$$\mathbb{Q} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_\nu \\ h_{\nu-1} \\ \vdots \\ h_0 \end{bmatrix}}_{\vec{h}} = \frac{1}{\sqrt{N}} \begin{bmatrix} \lambda_{N-1} \\ \lambda_{N-2} \\ \vdots \\ \lambda_0 \end{bmatrix}.$$

In other words, the eigenvalues λ_{N-1}, λ_{N-2}, $\ldots$, λ_0 are the N-point discrete Fourier transform of h_ν, $h_{\nu-1}$, $\ldots$, h_0.

Hint: $\vec{h}$ is the **last** column of $\mathbb{H}_{\text{circulant}}$, where

$$\mathbb{H}_{\text{circulant}} = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{\nu-1} & h_\nu & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_{\nu-2} & h_{\nu-1} & h_\nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_\nu \\ h_\nu & 0 & 0 & \cdots & 0 & 0 & h_0 & \cdots & h_{\nu-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & h_3 & \cdots & h_\nu & 0 & 0 & \cdots & h_0 \end{bmatrix}.$$

Solution.

(a) $s[-1] = s[3] = -1 - j$ and $s[-2] = s[2] = 1 + j$.

(b) From $\mathbb{Q}\mathbb{H}_{\text{circulant}} = \Lambda\mathbb{Q}$, and noting that the last column of $\sqrt{N}\mathbb{Q}$ is an all-one column, we obtain

$$\mathbb{Q} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_\nu \\ h_{\nu-1} \\ \vdots \\ h_0 \end{bmatrix}}_{\vec{h}} = \frac{1}{\sqrt{N}} \Lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The proof is completed by noting that

$$\Lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_{N-1} \\ \lambda_{N-2} \\ \vdots \\ \lambda_0 \end{bmatrix}.$$