Part 4 Spread-Spectrum Modulation

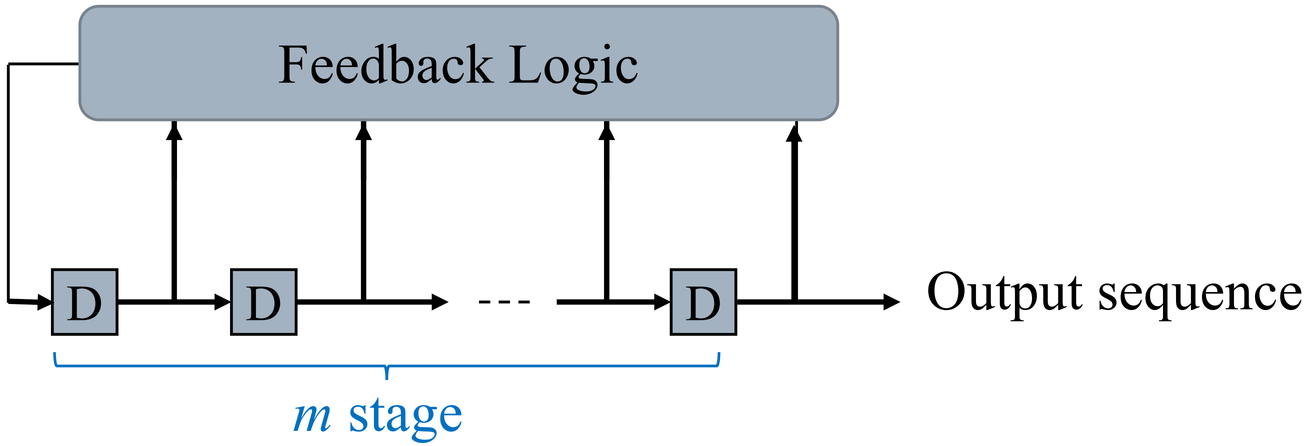
Introduction of Spread-Spectrum Modulation

- Definition of spread-spectrum modulation
  - Weakly sense
    - Occupy a bandwidth that is much larger than the minimum bandwidth ($1/2T$) necessary to transmit a data sequence.
  - Strict sense
    - Spectrum is spreading by means of a pseudo-white or pseudo-noise code.

Pseudo-Noise Sequences

- A (digital) code sequence that mimics the (second-order) statistical behavior of a white noise.
- For example,
  - balance property
  - run property
  - correlation property
- From implementation standpoint, the most convenient way to generate a pseudo-noise sequence is to employ several shift-registers and a feedback through combinational logic.

Pseudo-Noise Sequences

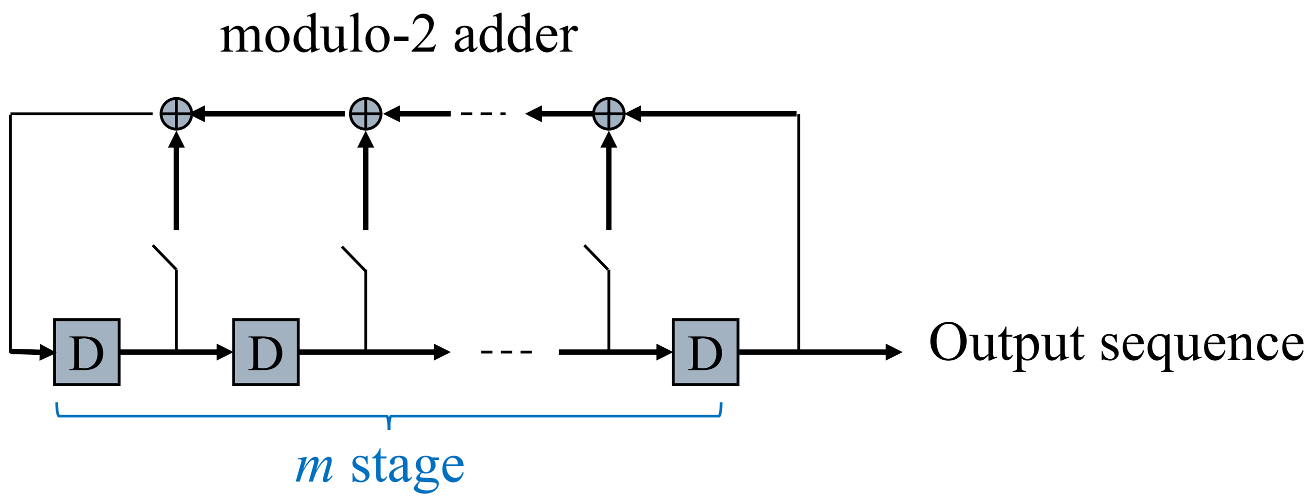


Exemplified block diagram of PN sequence generators

- Feedback shift register becomes "linear" if the feedback logic consists entirely of modulo-2 adders.

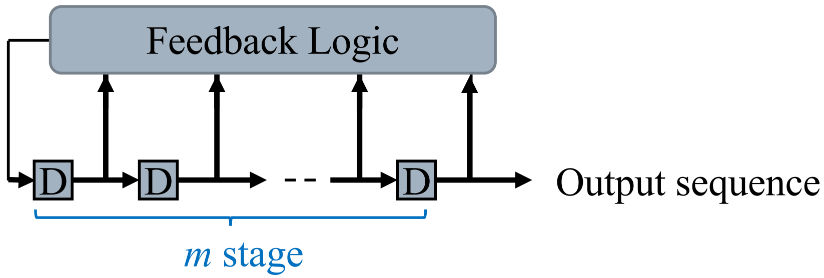
Pseudo-Noise Sequences

- Example of linear feedback shift register



Pseudo-Noise Sequences

- A PN sequence generated by a (possibly non-linear) feedback shift register must eventually become periodic with period at most 2^m, where m is the number of shift registers.
- A PN sequence generated by a linear feedback shift register must eventually become periodic with period at most $2^m - 1$, where m is the number of shift registers.
- A PN sequence whose period reaches its maximum value is named the *maximum-length sequence* or simply *m-sequence*.

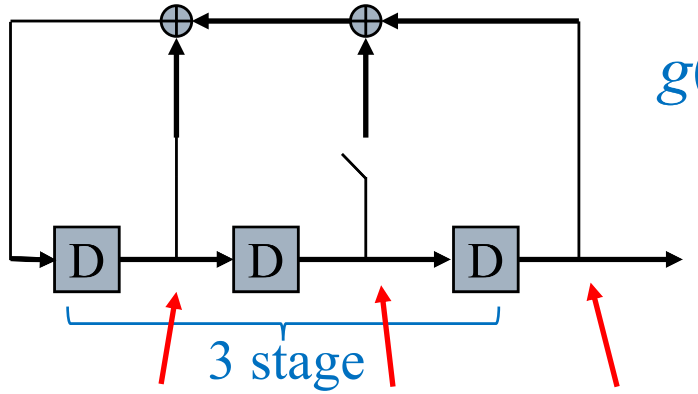


Pseudo-Noise Sequences

- A *maximum-length sequence* generated from a linear shift register satisfies all three properties:
  - Balance property
    - The number of 1s is one more than that of 0s.
  - Run property (total number of runs = 2^{m-1})
    - ½ of the runs is of length 1
    - ¼ of the runs is of length 2
    - …

Pseudo-Noise Sequences

- Example



$g(x) = x^3 + x^2 + 1$

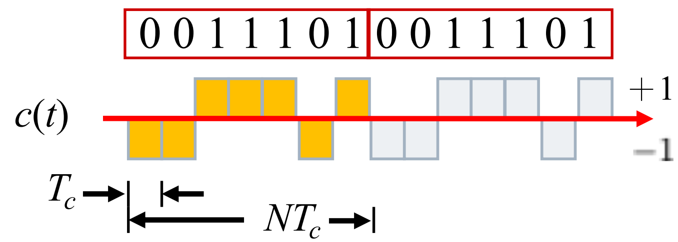
0011101001110100111 01...

1	0	0
1	1	0
1	1	1
0	1	1
1	0	1
0	1	0
0	0	1
1	0	0
	:	

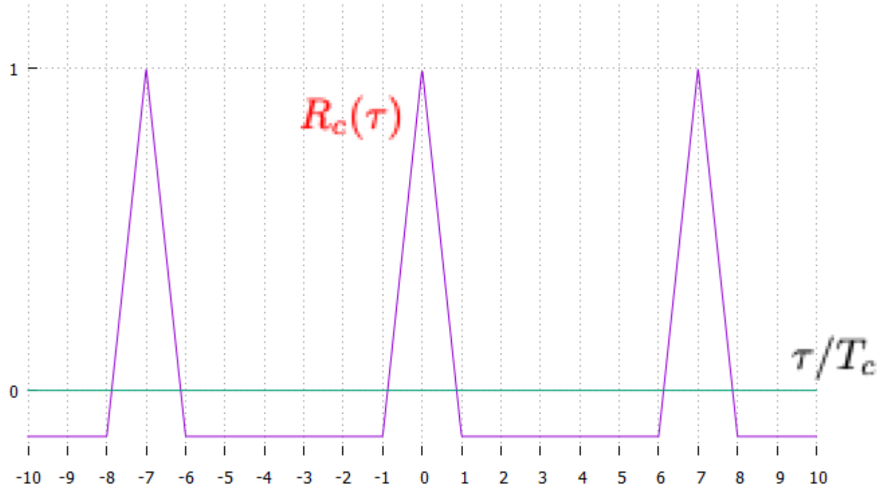
Pseudo-Noise Sequences

- Correlation property
  - Autocorrelation of an ideal discrete white process = $a\cdot\delta[\tau]$, where $\delta[\tau]$ is the Kronecker delta function.

$m = 3$



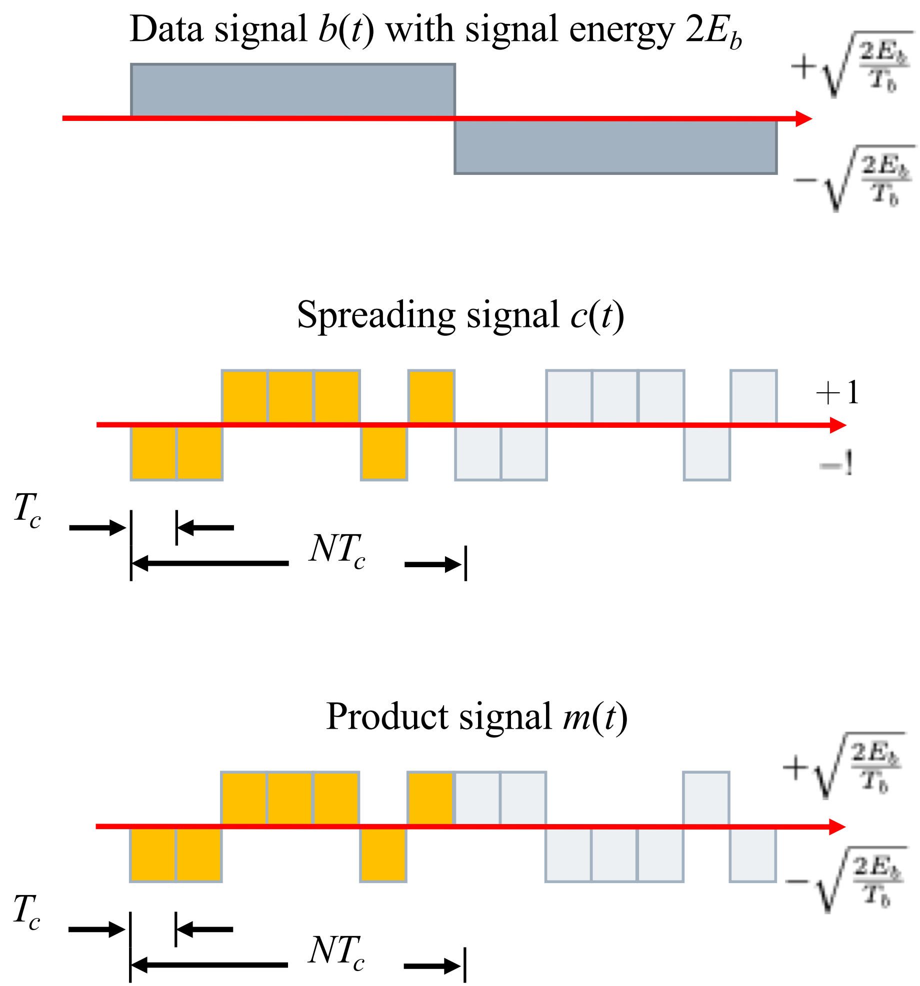
$N = 2^m - 1 = 7$ and $T_b = NT_c$



$$
\begin{aligned}
R_c(\tau) &= \frac{1}{T_b}\int_{-T_b/2}^{T_b/2} c(t)c(t-\tau)dt \\
&= \begin{cases} 1 - \frac{N+1}{NT_c}|\tau|, & |\tau| \le T_c \\ -\frac{1}{N}, & \text{the remainder of the period} \end{cases}
\end{aligned}
$$

Pseudo-Noise Sequences

- Power spectrum view



Suppose $c(t)$ is perfectly white with $c^2(t) = 1$. Then,

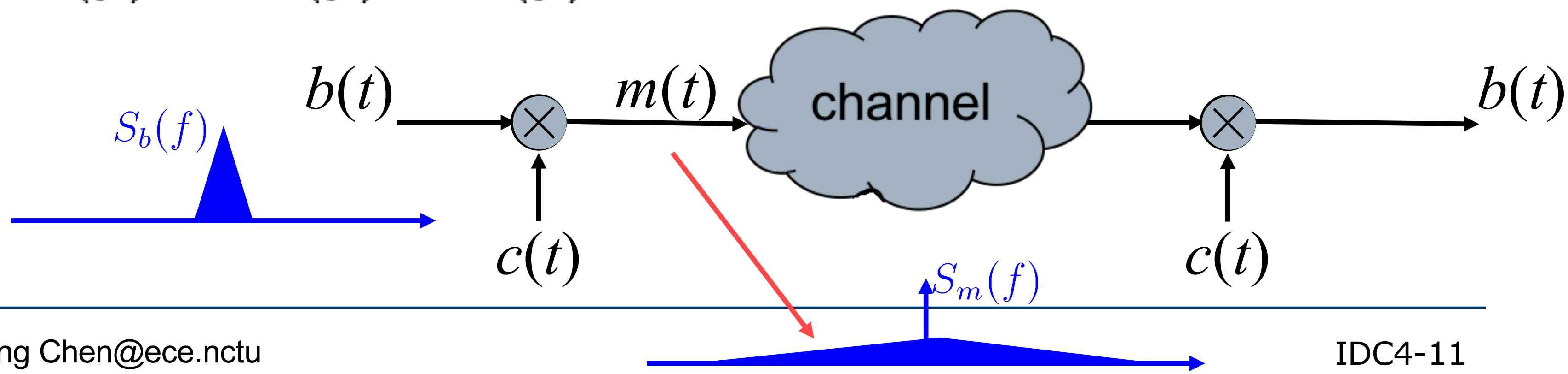
$$m(t) = b(t)c(t) \text{ and } b(t) = m(t)c(t).$$

Question: What will be the power spectrum of $m(t)$?

PSD of Product Signal

$$
\begin{aligned}
R_m(\tau) &= E[m(t)m(t+\tau)] \\
&= E[b(t)b(t+\tau)c(t)c(t+\tau)] \\
&= E[b(t)b(t+\tau)]E[c(t)c(t+\tau)] \\
&\qquad \text{(Independence assumption between } b(t) \text{ and } c(t)) \\
&= R_b(\tau)R_c(\tau)
\end{aligned}
$$

$$\Rightarrow S_m(f) = S_b(f) * S_c(f)$$



Pseudo-Noise Sequences

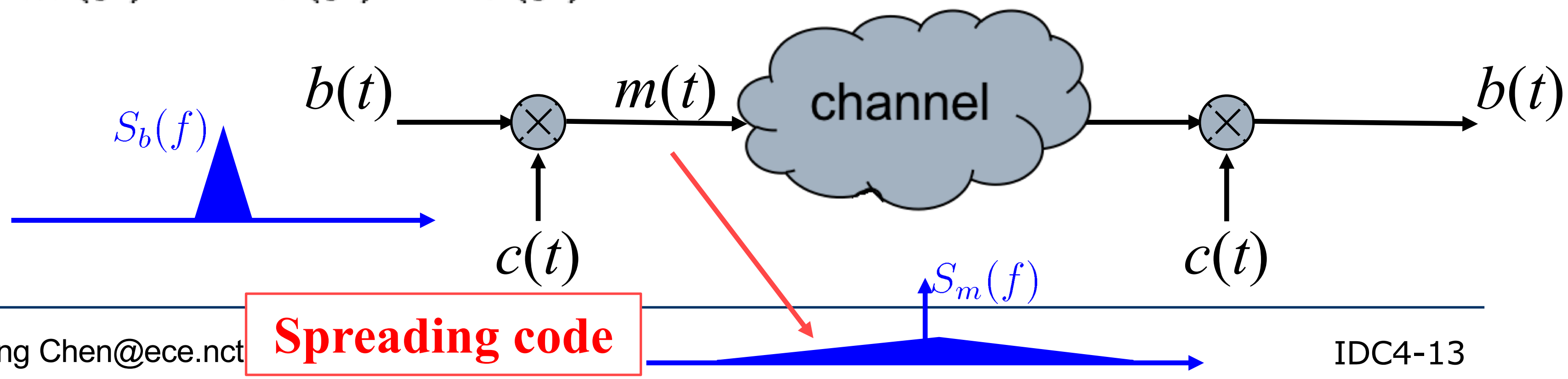
- Please self-study Example 7.2 for m-sequences in textbook.
  - Its understanding will be part of the exam.

PSD of Spread Spectrum

Make the transmitted signal to hide behind the background noise.

$$
\begin{aligned}
R_m(\tau) &= E[m(t)m(t+\tau)] \\
&= E[b(t)b(t+\tau)c(t)c(t+\tau)] \\
&= E[b(t)b(t+\tau)]E[c(t)c(t+\tau)] \\
&\qquad \text{(Independence assumption between } b(t) \text{ and } c(t)) \\
&= R_b(\tau)R_c(\tau)
\end{aligned}
$$

$$\Rightarrow S_m(f) = S_b(f) * S_c(f)$$



- Example
  - Recall Slides IDC1-33 ~ IDC1-36

$$b(t) = \sum_{k=-\infty}^{\infty} I_k \cdot g(t - kT_b)$$

where $I_k = \pm 1$ with equal probability, and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

$$G(f) = \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} e^{-j2\pi f t} dt = \sqrt{\frac{2E_b}{T_b}} T_b \operatorname{sinc}(T_b f) e^{-j\pi f T_b}$$

$$\begin{aligned} \Rightarrow \bar{S}_B(f) = \frac{1}{T_b}|G(f)|^2 &= \frac{1}{T_b}\frac{2E_b}{T_b} T_b^2 \operatorname{sinc}^2(T_b f) \\ &= 2E_b \operatorname{sinc}^2(T_b f) \end{aligned}$$

- Noting that

$$c(t) = \sum_{k=-\infty}^{\infty} c_k \cdot g_c(t - kT_c)$$

where $c_k = \pm 1$ with equal probability, and $\{c_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g_c(t) = \begin{cases} 1, & 0 \le t < T_c \\ 0, & \text{otherwise} \end{cases}$$

we obtain

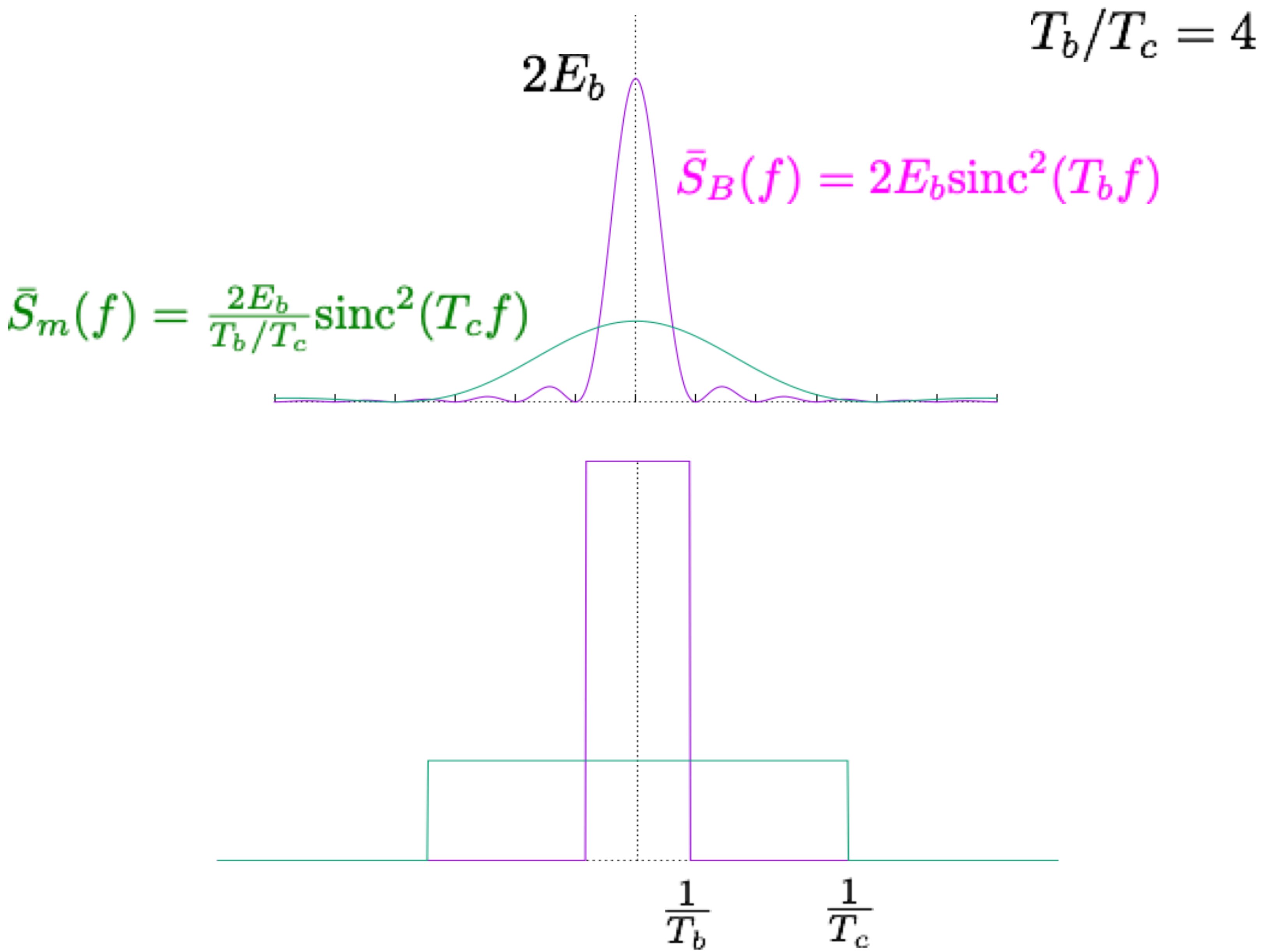
$$\begin{aligned} b(t)c(t) &= \left(\sum_{i=-\infty}^{\infty} I_i \cdot g(t - iT_b)\right)\left(\sum_{k=-\infty}^{\infty} c_k \cdot g_c(t - kT_c)\right) \\ &= \sum_{k=-\infty}^{\infty} \tilde{c}_k \cdot \tilde{g}_c(t - kT_c) \end{aligned}$$

where $\tilde{c}_k = c_k \cdot I_{\lfloor k/(T_b/T_c)\rfloor} = \pm 1$ with equal probability, and $\{\tilde{c}_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } \tilde{g}_c(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}, & 0 \le t < T_c \\ 0, & \text{otherwise} \end{cases}$$

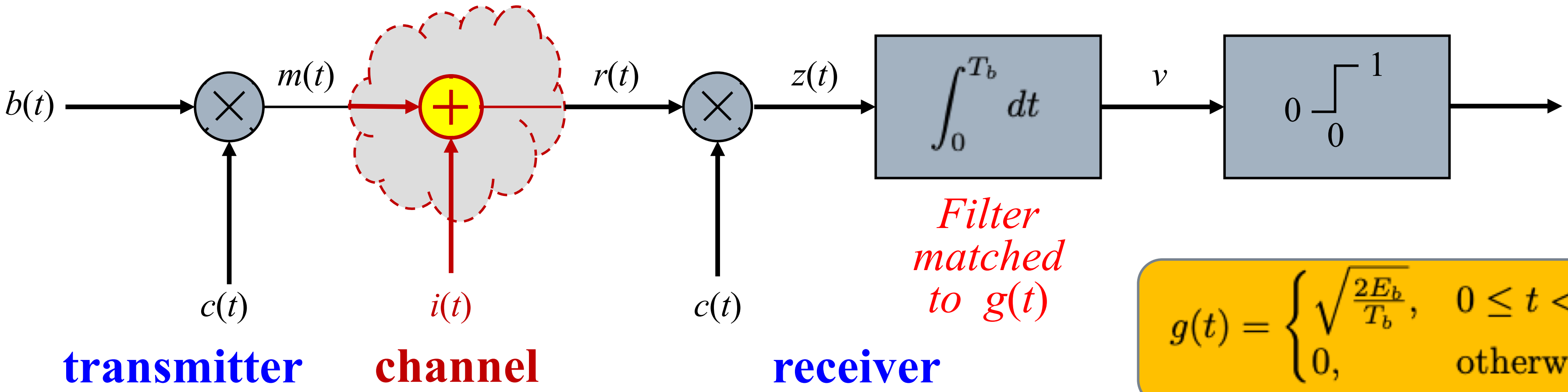
$$\tilde{G}_c(f) = \int_0^{T_c} \sqrt{\frac{2E_b}{T_b}} e^{-j2\pi f t} dt = \sqrt{\frac{2E_b}{T_b}} T_c \text{sinc}(T_c f) e^{-j\pi f T_c}$$

$$\begin{aligned} \Rightarrow \bar{S}_m(f) = \frac{1}{T_c} |\tilde{G}_c(f)|^2 &= \frac{1}{T_c} \frac{2E_b}{T_b} T_c^2 \text{sinc}^2(T_c f) \\ &= \frac{2E_b}{T_b/T_c} \text{sinc}^2(T_c f) \end{aligned}$$



$T_b/T_c = 4$
$2E_b$
$\bar{S}_B(f) = 2E_b \text{sinc}^2(T_b f)$
$\bar{S}_m(f) = \frac{2E_b}{T_b/T_c}\text{sinc}^2(T_c f)$
$\frac{1}{T_b}$
$\frac{1}{T_c}$

A Notion of Spread Spectrum



$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

$$r(t) = m(t) + i(t) = c(t)b(t) + i(t)$$

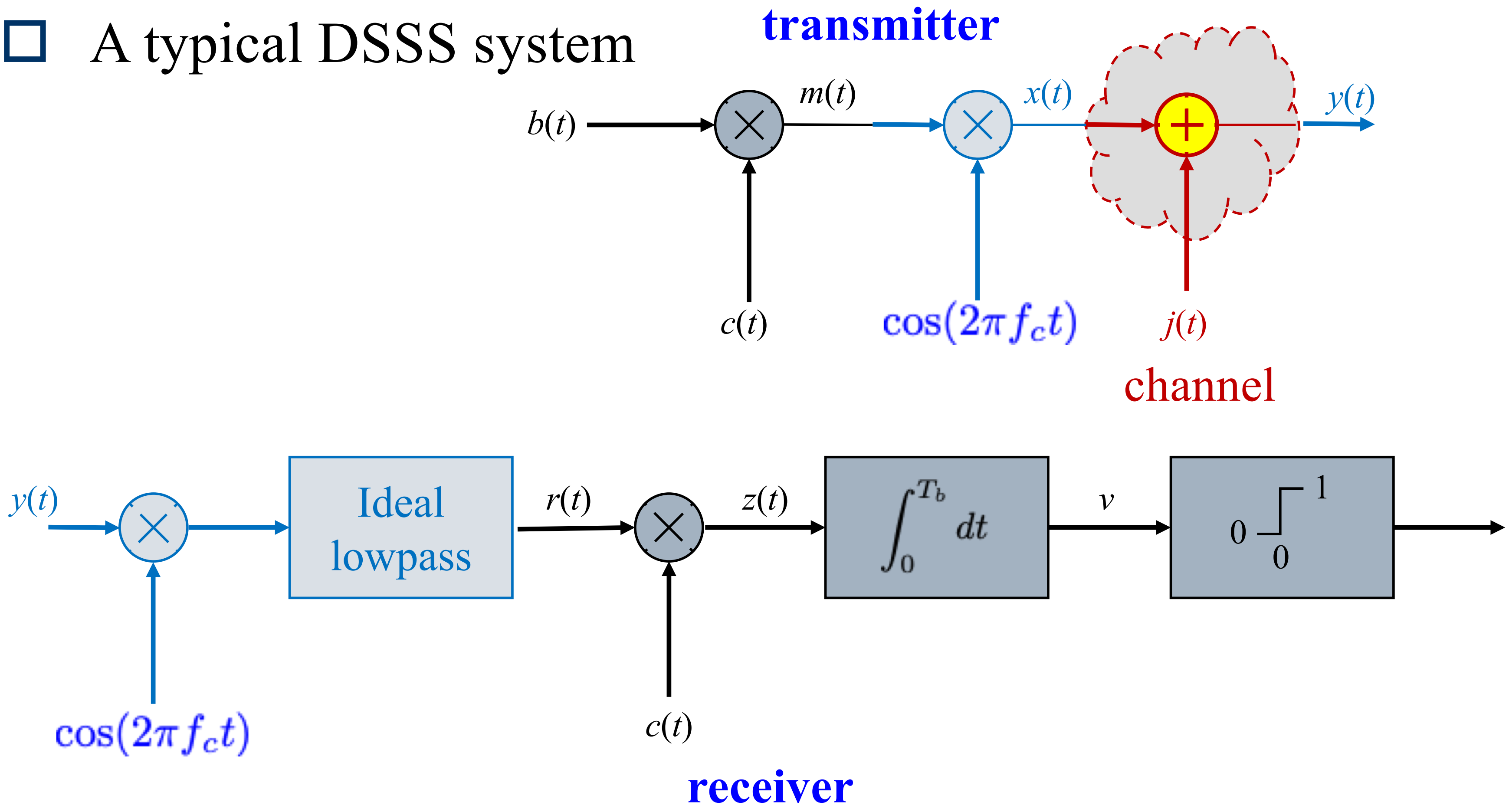
$$z(t) = c(t)r(t) = c^2(t)b(t) + c(t)i(t) = b(t) + c(t)i(t)$$

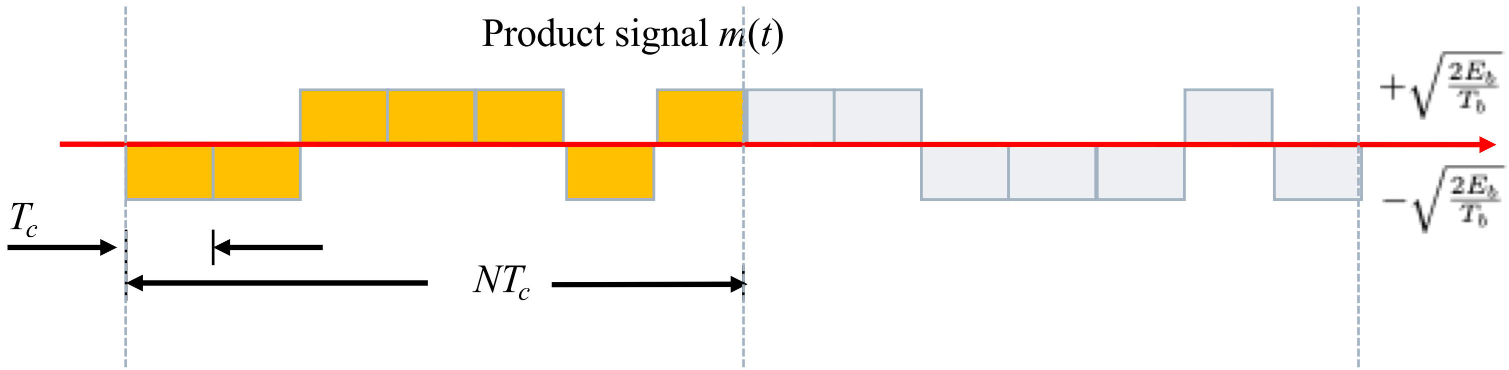
$$\boxed{c^2(t) = 1}$$

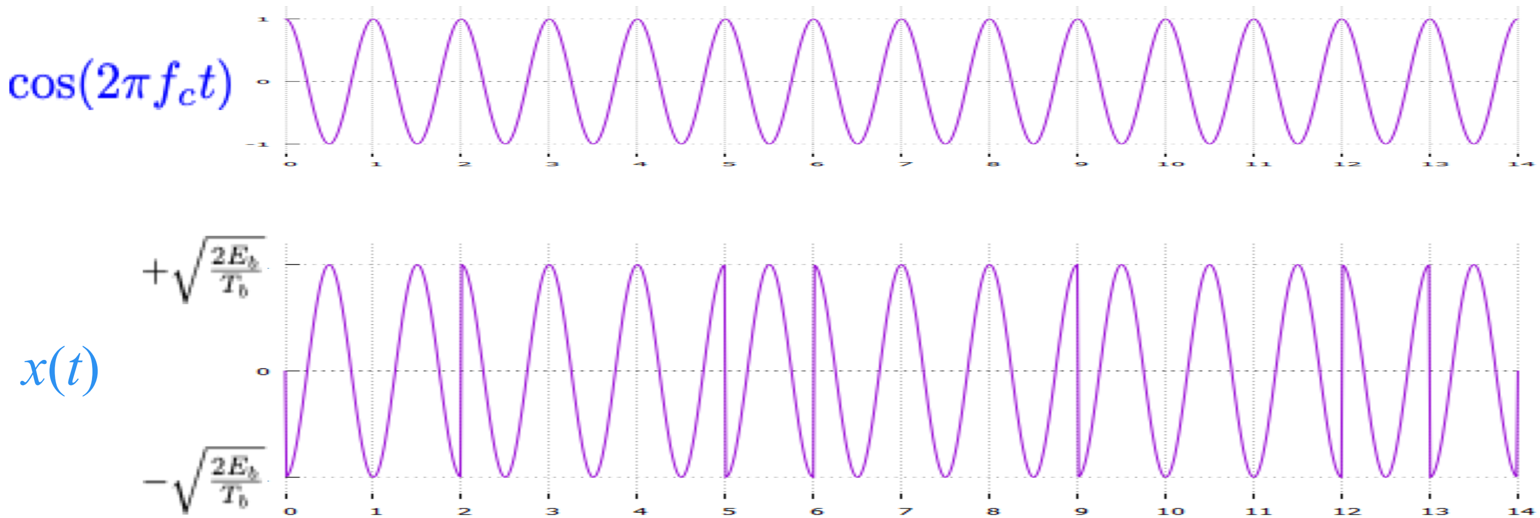
Direct-Sequence Spread Spectrum with Coherent Binary Phase-Shift Keying

In text, $i(t)$ is the baseband interference, while $j(t)$ is the passband interference.

- A typical DSSS system





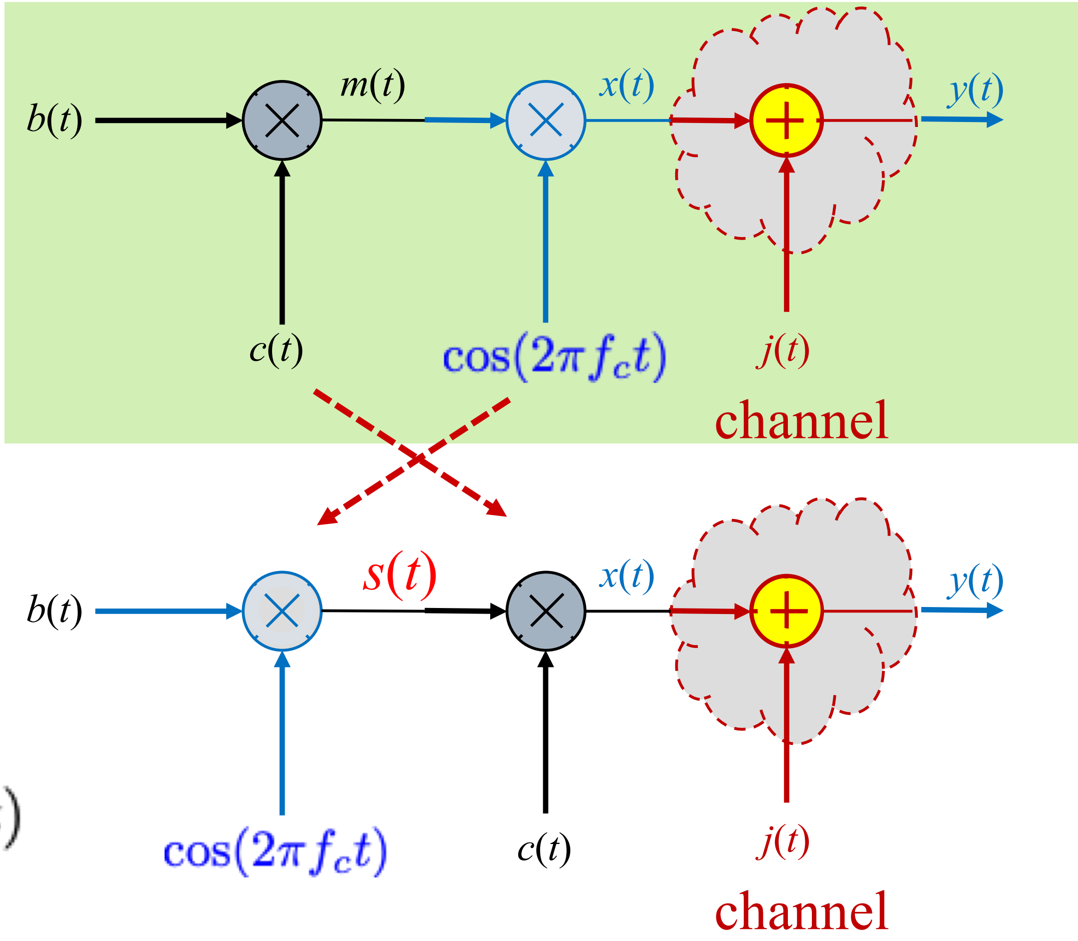


$$x(t) = m(t)\cos(2\pi f_c t) = b(t)c(t)\cos(2\pi f_c t)$$

(Here, f_c=1/T_c.)

Direct-Sequence Spread Spectrum with Coherent Binary Phase-Shift Keying

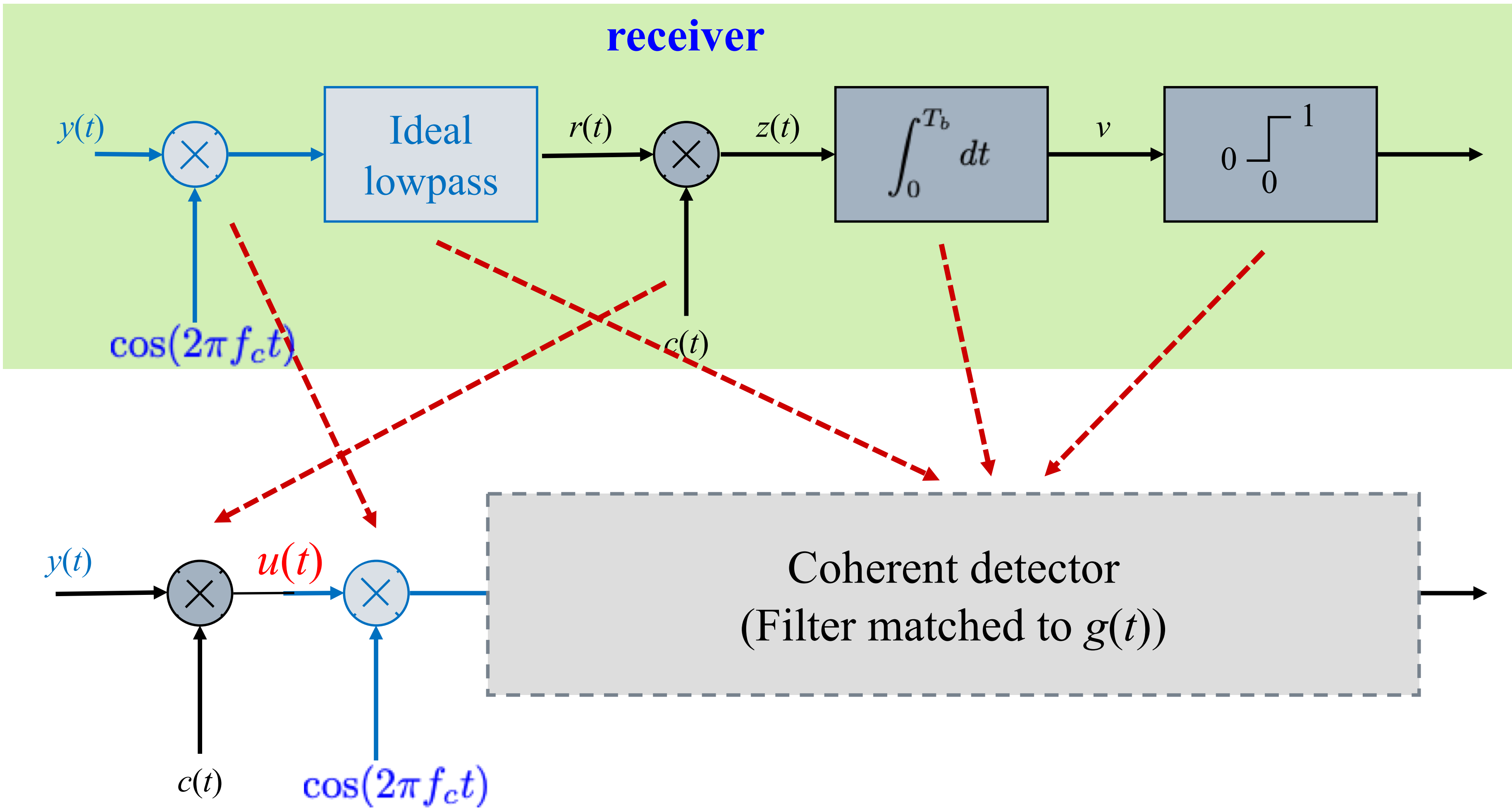
- DSSS transmitter
  - For analytical convenience



Let $s(t) = b(t)\cos(2\pi f_c t)$.

Then, $x(t)=c(t)s(t)$.

$$\begin{aligned} \Rightarrow y(t) &= x(t) + j(t) \\ &= c(t)s(t) + j(t) \end{aligned}$$



$$\Rightarrow u(t) = c(t)y(t) = c^2(t)s(t) + c(t)j(t) = s(t) + c(t)j(t).$$

Signal-Space Dimensionality and Processing Gain

Assume "coherent detection". In other words, perfect synchronization and no phase mismatch.

- SNR before spreading

$$y(t) = c(t)s(t) + j(t).$$

- SNR after spreading

$$u(t) = s(t) + c(t)j(t).$$

- Orthonormal basis used at the receiver end

$$\phi_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\cos(2\pi f_c t), & kT_c \le t < (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$

where $k = 0, 1, \cdots, N-1$.

SNR before spreading (SNR_I)

$$\begin{aligned} y(t) &= c(t)s(t) + j(t) \quad \text{for } 0 \le t < T_b \\ &= \pm\sqrt{\frac{2E_b}{T_b}}c(t)\cos(2\pi f_c t) + j(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}}\sqrt{\frac{T_c}{2}}\sum_{k=0}^{N-1} c_k\phi_k(t) + j(t) \end{aligned}$$

$c_k \in \{\pm 1\}$ and $j(t)$ white

$$\Rightarrow \boldsymbol{y} = \pm\sqrt{\frac{E_b}{N}}\begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix}_{N\times 1} + \begin{bmatrix} j_0 \\ \vdots \\ j_{N-1} \end{bmatrix}_{N\times 1}$$

$j_k = \int_0^{T_b} j(t)\phi_k(t)dt$

$$\Rightarrow \mathrm{SNR}_I = \frac{(E_b/N)\cdot E[c_0^2 + \cdots + c_{N-1}^2]}{E\left[j_0^2 + \cdots + j_{N-1}^2\right]} = \frac{(E_b/N)N}{NE[j_0^2]} = \frac{E_b}{NE[j_0^2]}$$

SNR after spreading (SNR_O)

$$
\begin{aligned}
u(t) &= s(t) + c(t)j(t) \quad \text{for } 0 \le t < T_b \\
&= \pm\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) + c(t)j(t) \\
&= \pm\sqrt{E_b}\phi(t) + c(t)j(t)
\end{aligned}
$$

$$
\phi(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}
$$

$$
\Rightarrow u = \pm\sqrt{E_b} + j, \text{ where } j = \int_0^{T_b} c(t)j(t)\phi(t)\,dt
$$

$$
\begin{aligned}
j &= \int_0^{T_b} c(t) j(t) \phi(t) dt \\
&= \int_0^{T_b} \left(\sum_{\ell=0}^{N-1} c_\ell\, g_c(t - \ell T_c) \right) j(t) \phi(t) dt \\
&= \sum_{\ell=0}^{N-1} c_\ell \int_0^{T_b} g_c(t - \ell T_c) j(t) \phi(t) dt \\
&= \sqrt{\frac{T_c}{T_b}} \sum_{\ell=0}^{N-1} c_\ell \int_{\ell T_c}^{(\ell+1)T_c} j(t) \sqrt{\frac{2}{T_c}} \cos(2\pi f_c t) dt \\
&= \sqrt{\frac{T_c}{T_b}} \sum_{\ell=0}^{N-1} c_\ell \int_0^{T_b} j(t) \phi_\ell(t) dt \\
&= \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} c_\ell j_\ell
\end{aligned}
$$

$$
\boxed{g_c(t) = \begin{cases} 1, & 0 \le t < T_c \\ 0, & \text{otherwise} \end{cases}}
$$

$$
\boxed{\phi(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t), & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}}
$$

$$
\begin{aligned}
E[j^2] &= \frac{1}{N} E\left[\left(\sum_{m=0}^{N-1} c_m j_m\right)\left(\sum_{k=0}^{N-1} c_k j_k\right)\right] \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} E[c_m c_k] E[j_m j_k] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} E[j_k^2] \\
&= E[j_0^2]
\end{aligned}
$$

$$
\Rightarrow \mathrm{SNR}_O = \frac{E_b}{E[j_0^2]} = N \cdot \mathrm{SNR}_I
$$

$\boxed{N = \frac{T_b}{T_c} \text{ is hence named the processing gain of spread spectrum technique.}}$

Phase Mismatch in SNR$_I$

Assume "coherent detection". In other words, perfect time synchronization but **with phase mismatch**.

- SNR before spreading

$$y(t) = c(t)s(t) + j(t)$$

- SNR after spreading

$$u(t) = s(t) + c(t)j(t)$$

- Orthonormal basis used at the receiver end

$$\begin{cases} \phi_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\cos(2\pi f_c t), & kT_c \le t < (k+1)T_c \\ 0, & \text{otherwise} \end{cases} \\ \hat{\phi}_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\sin(2\pi f_c t), & kT_c \le t < (k+1)T_c \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

where $k = 0, 1, \ldots, N-1.$

SNR before spreading (SNR_I)

$$
\begin{aligned}
y(t) &= c(t)s(t) + j(t) \quad \text{for } 0 \le t < T_b \\
&= \pm\sqrt{\frac{2E_b}{T_b}}c(t)\cos(2\pi f_c t + \theta) + j(t) \\
&= \pm\sqrt{\frac{2E_b}{T_b}}\sqrt{\frac{T_c}{2}}\sum_{k=0}^{N-1} c_k\left(\phi_k(t)\cos(\theta) - \hat{\phi}_k(t)\sin(\theta)\right) + j(t)
\end{aligned}
$$

$c_k \in \{\pm 1\}$ and $j(t)$ white

$$
\Rightarrow \boldsymbol{y} = \pm\sqrt{\frac{E_b}{N}}\begin{bmatrix} c_0\cos(\theta) \\ \vdots \\ c_{N-1}\cos(\theta) \\ -c_0\sin(\theta) \\ \vdots \\ -c_{N-1}\sin(\theta) \end{bmatrix}_{2N\times 1} + \begin{bmatrix} j_0 \\ \vdots \\ j_{N-1} \\ \hat{j}_0 \\ \vdots \\ \hat{j}_{N-1} \end{bmatrix}_{2N\times 1}
$$

$$
\begin{cases} j_k = \displaystyle\int_0^{T_b} j(t)\phi_k(t)dt \\ \hat{j}_k = \displaystyle\int_0^{T_b} j(t)\hat{\phi}_k(t)dt \end{cases}
$$

$$\Rightarrow \mathrm{SNR}_I = \frac{(E_b/N)\cdot E[c_0^2+\cdots+c_{N-1}^2]}{E[j_0^2+\cdots+j_{N-1}^2+\hat{j}_0^2+\cdots+\hat{j}_{N-1}^2]}$$

$$= \frac{(E_b/N)N}{2NE[j_0^2]} = \frac{E_b}{2NE[j_0^2]}$$

SNR after spreading (SNR_O)

$$
\begin{aligned}
u(t) &= s(t) + c(t)j(t) \quad \text{for } 0 \le t < T_b \\
&= \pm\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta) + c(t)j(t) \\
&= \pm\sqrt{E_b}[\phi(t)\cos(\theta) - \hat{\phi}(t)\sin(\theta)] + c(t)j(t)
\end{aligned}
$$

$$
\begin{cases}
\phi(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases} \\
\hat{\phi}(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t), & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}
\end{cases}
$$

$$
\boldsymbol{u} = \pm\sqrt{E_b}\begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix} + \begin{bmatrix} j \\ \hat{j} \end{bmatrix}, \text{ where } \begin{bmatrix} j \\ \hat{j} \end{bmatrix} = \begin{bmatrix} \int_0^{T_b} c(t)j(t)\phi(t)dt \\ \int_0^{T_b} c(t)j(t)\hat{\phi}(t)dt \end{bmatrix}
$$

Same as the derivation in Slide IDC4-26, we derive

$$j = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} c_m j_m$$

$$\hat{j} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} c_m \hat{j}_m$$

$$E[j^2] = E[j_0^2] \text{ and } E[\hat{j}^2] = E[\hat{j}_0^2] = E[j_0^2]$$

$$\begin{aligned} \Rightarrow \text{SNR}_O &= \frac{E_b}{2E[j_0^2]} \\ &= N \cdot \text{SNR}_I \end{aligned}$$

Final Notes

- In the previous two derivations, we compute the SNRs before and after processing under the same system setting, and obtain consistently for both situations:

$$\text{SNR}_O = N \cdot \text{SNR}_I$$

- We thus refer N as the processing gain.
- (7.36) in textbook, however, computes the SNR_O under perfect synchronization, and **defines** the input SNR as

$$\text{SNR}_I = \frac{E_b/T_b}{J} = \frac{E_b}{2NE[j_0^2]}$$

where J is the noise power given in the next slide.

- As a result, the textbook obtains $\text{SNR}_O = 2N \cdot \text{SNR}_I$ (cf. (7.37)).

Jamming Margin

- Since the signal we transmit must be a linear combination of

$$\{\phi_k(t)\}_{k=0}^{N-1} \text{ and } \{\hat{\phi}_k(t)\}_{k=0}^{N-1}.$$

- The noise energy that can affect the performance is only from the projection of $j(t)$ onto these basis.

$$E_j = \sum_{k=0}^{N-1} E[j_k^2] + \sum_{k=0}^{N-1} E[\hat{j}_k^2] = 2N\, E[j_0^2]$$

- The noise power J is thus given by

$$J = \frac{E_j}{T_b} = \frac{2NE[j_0^2]}{T_b} = \frac{2E[j_0^2]}{T_c} \quad \left(\text{Thus, } E[j_0^2] = \frac{JT_c}{2}.\right)$$

Jamming Margin

- Without phase mismatch, Slide IDC4-27 obtains

$$\text{SNR}_O = \frac{E_b}{E[j_0^2]}$$

- In Slide IDC1-30, the BER for coherent BPSK transmission (without spread spectrum) is derived under the assumption that the energies of the signal and the noise sample are respectively E_b and $N_0/2$.
- Consequently, the two results in the same BER if

$$E[j_0^2] = \frac{JT_c}{2} = \frac{N_0}{2}$$

Coherent Phase-Shift Keying (PSK) – Error Probability

- Error probability of Binary PSK
  - Based on the decision rule $x \underset{\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0$

$$
\begin{aligned}
P(\text{Error}) &= P\left(-\sqrt{E_b}\text{ transmitted}\right) P\left(x > 0 \middle| -\sqrt{E_b}\text{ transmitted}\right) \\
&\quad P\left(+\sqrt{E_b}\text{ transmitted}\right) P\left(x > 0 \middle| +\sqrt{E_b}\text{ transmitted}\right) \\
&= \frac{1}{2}\int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+\sqrt{E_b})^2/2\sigma^2}dx + \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2}dx \\
&= \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+\sqrt{E_b})^2/2\sigma^2}dx = \Phi\left(\frac{0-\sqrt{E_b}}{\sigma}\right) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}
$$

$$\boxed{\Phi(-x) = \frac{1}{2}\text{eric}\left(\frac{x}{\sqrt{2}}\right)}$$

Jamming Margin

- With $N_0 = T_c J$ and $P = E_b/T_b$, i.e., P is the average signal power, we establish

$$\frac{E_b}{N_0} = \left(\frac{T_b}{T_c}\right)\left(\frac{P}{J}\right) \text{ or equivalently, } \frac{J}{P} = \frac{\text{PG}}{E_b/N_0}$$

  - J/P is termed *the jamming margin* (required for a specific error rate).
  - PG denotes the processing gain.
- The above analysis gives an intuitive way to obtain the jamming margin for a given processing and for a demanded error rate.

$$
\begin{aligned}
(\text{Jamming margin})_{\text{dB}} =& (\text{Processing gain})_{\text{dB}} \\
& -(\text{required } (E_b/N_0)_{\text{dB}} \text{ for a given } P_e)
\end{aligned}
$$

- Example
  - Without spreading, (E_b/N_0) required for $P_e = 10^{-5}$ for coherent BPSK transmission is 9.6 dB.
  - Choose PG = 4095.
  - Then, Jamming margin for $P_e = 10^{-5}$ is

$$
\begin{aligned}
(\text{Jamming margin})_{\text{dB}} &= 10 \log_{10}(4095) - 9.6 = 36.1 - 9.6 \\
&= 26.5 \text{ dB} = 10 \log_{10}(446.68)
\end{aligned}
$$

  - Information bits can be detected subject to the required error rate, even if the interference power level is 446.68 times larger than the signal power (in the price of the transmission speed is 4095 times slower).

Frequency-Hopping Spread Spectrum

- Basic characterization of frequency hopping
  - Slow-frequency hopping

    Symbol rate $R_s >$ hop rate R_h
    (Usually, an interger multiple of)

  - Fast-frequency hopping

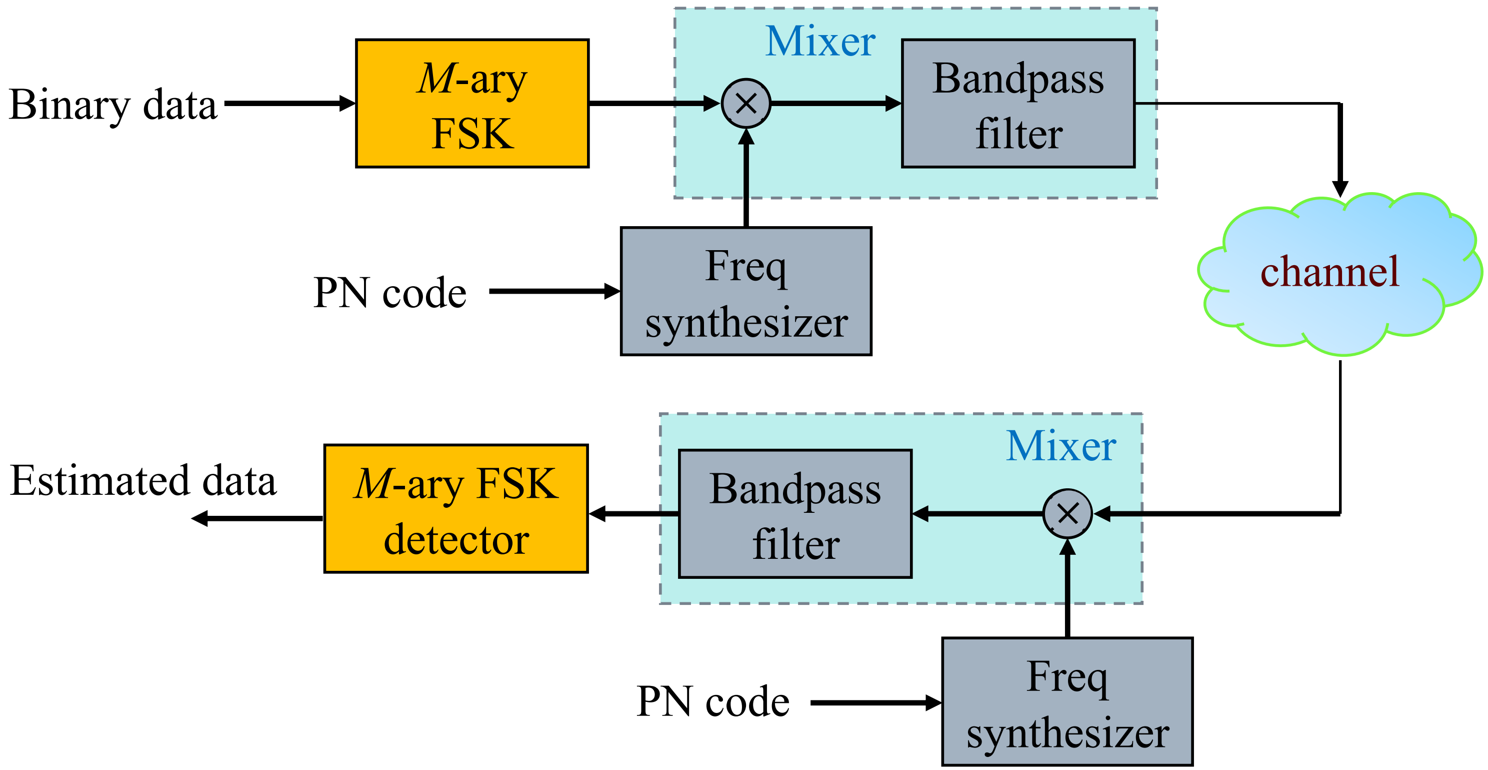
    Symbol rate $R_s <$ hop rate R_h
    (Usually, an interger multiple of)

  - Chip rate (The smallest unit = Chip)

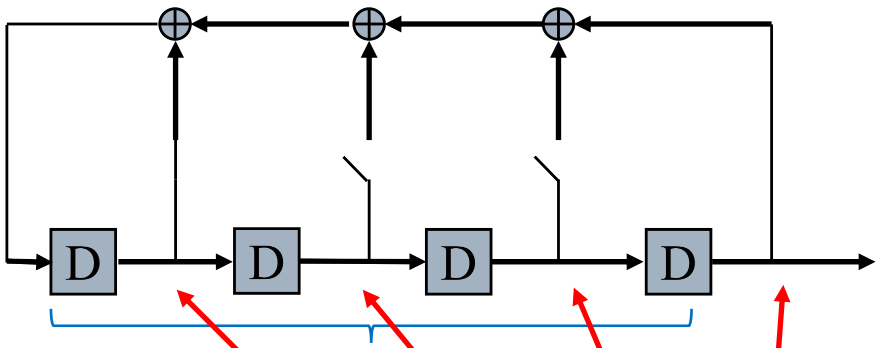
    $$R_c = \max\{R_h, R_s\}$$

Frequency-Hopping Spread Spectrum

- A common modulation scheme for FH systems is the *M-ary frequency-shift keying*



Pseudo-Noise Sequences of Length 15



$$g(x) = x^4 + x^3 + 1$$

00011110101100100111101011001

4 stage

1	0	0	0
1	1	0	0
1	1	1	0
1	1	1	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	1	0
1	1	0	1
0	1	1	0
0	0	1	1
1	0	0	1
0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0

:

(Chip = The smallest unit)

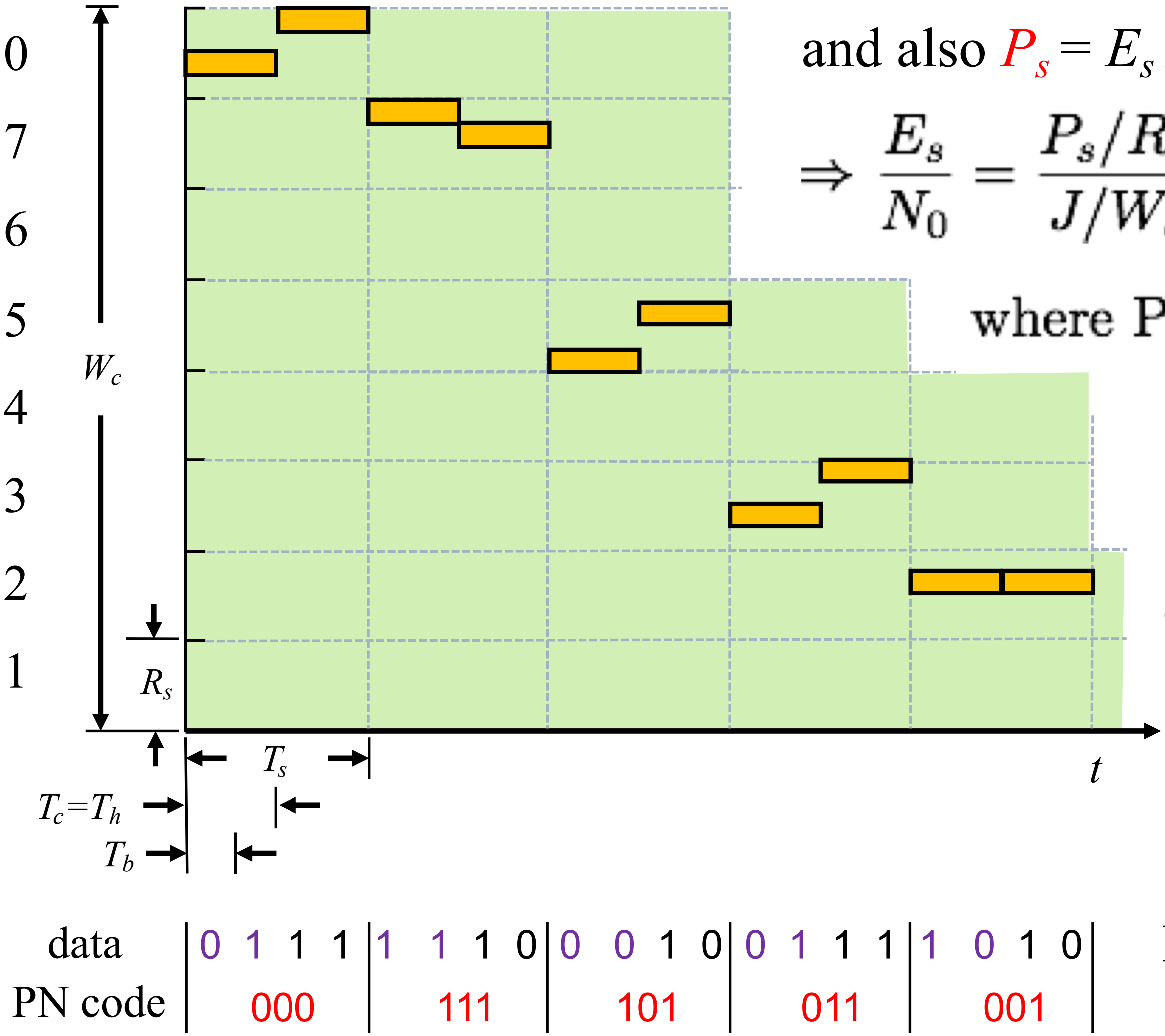
$R_c = \max\{R_h, R_s\} = R_s = \frac{R_b}{2}$ for 4-ary FSK

Slide IDC4-35 indicates $N_0 = T_c J = J / W_c$

and also $P_s = E_s / T_s = R_s E_s$

$$\Rightarrow \frac{E_s}{N_0} = \frac{P_s/R_s}{J/W_c} = \left(\frac{W_c}{R_s}\right)\left(\frac{P_s}{J}\right) = \text{PG}\left(\frac{P_s}{J}\right)$$

$$\text{where PG} = \frac{W_c}{R_s} = \frac{1/T_c}{1/T_s} = \frac{T_s}{T_c} = 8.$$



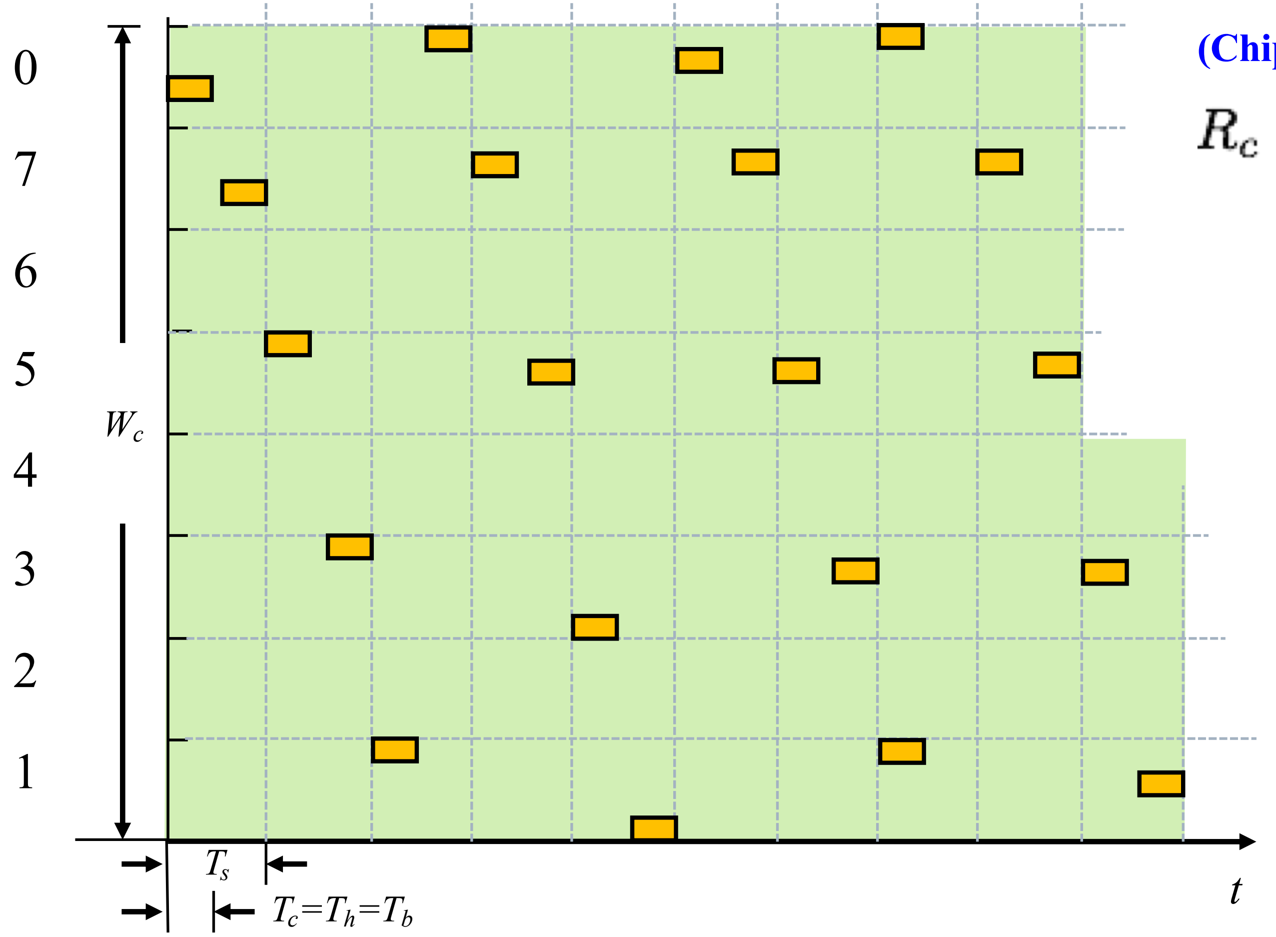
4-ary FSK $\begin{cases} R_c = \frac{1}{T_c} \\ R_b = \frac{1}{T_b} \\ R_h = \frac{1}{T_h} \\ R_s = \frac{1}{T_s} \end{cases}$

Period of PN sequence=15

Length of PN segment per hop=3

$\boxed{\text{So there are } 2^3 \text{ hop frequencies.}}$

Slow frequency hopping



(Chip = The smallest unit)

$$R_c = \max\{R_h, R_s\} = R_h = R_b$$

4-ary FSK $\begin{cases} R_c = \frac{1}{T_c} \\ R_b = \frac{1}{T_b} \\ R_h = \frac{1}{T_h} \\ R_s = \frac{1}{T_s} \end{cases}$

Period of PN sequence=15

Length of PN segment per hop=3

So there are 2^3 hop frequencies.

data _0_1_1_1_1_1_1_0_0_0_1_0_0_1_1_1_1_0_1_0_

PN code 000111101011001 000111101011001 000111101011001 000111101011001

Fast-frequency hopping

Frequency-Hopping Spread Spectrum

- Fast-frequency hopping is popular in military use because the transmitted signal hops to a new frequency before the jammer is able to sense and jam it.
- Two detection rules are generally used in fast-frequency hopping
  - Make decision separately for each chip, and do majority vote based on these chip-based decisions (Simple)
  - Make maximum-likelihood decision based on all chip receptions (Optimal)

Maximum-Length and Gold Sequences

- Code-division multiplexing (CDM)
  - Each user is assigned a different spreading code.

    For simplicity, assume $s_1(t), s_2(t) \in \{\pm\sqrt{E_b}\}$ (i.e., $f_c = 0$)

    Let $c_i(t) = \sum_{k=0}^{N-1} c_{k,i}\, g(t - kT_c)$, where $c_{k,i} \in \{\pm 1\}$, and $g(t) = 1$ for $0 \le t < T_c$, and zero, otherwise.

    Multiplexing $x_1(t) + x_2(t) = c_1(t)s_1(t) + c_2(t)s_2(t)$

    $\Rightarrow c_1(t)[x_1(t) + x_2(t)] = s_1(t) + c_1(t)c_2(t)s_2(t)$

    $$\Rightarrow \text{Decision is made based on } \int_0^{T_b} s_1(t)dt + \int_0^{T_b} c_1(t)c_2(t)s_2(t)dt$$
    $$= s_1 T_b + s_2 \int_0^{T_b} c_1(t)c_2(t)dt$$

$\boxed{s_1, s_2 \in \{\pm\sqrt{E_b}\}}$

Maximum-Length and Gold Sequences

- So, if

$$\int_0^{T_b} c_1(t)c_2(t)dt = \sum_{k=0}^{N-1} c_{k,1}c_{k,2} = 0 \qquad \boxed{\text{cross-correlation}}$$

then *signal one* (i.e., s_1) can be exactly reconstructed.

- In practice, it may not be easy to have a big number of PN sequences satisfying the above equality. Instead, we desire

$$\left|\sum_{k=0}^{N-1} c_{k,1}c_{k,2}\right| \text{ small}$$

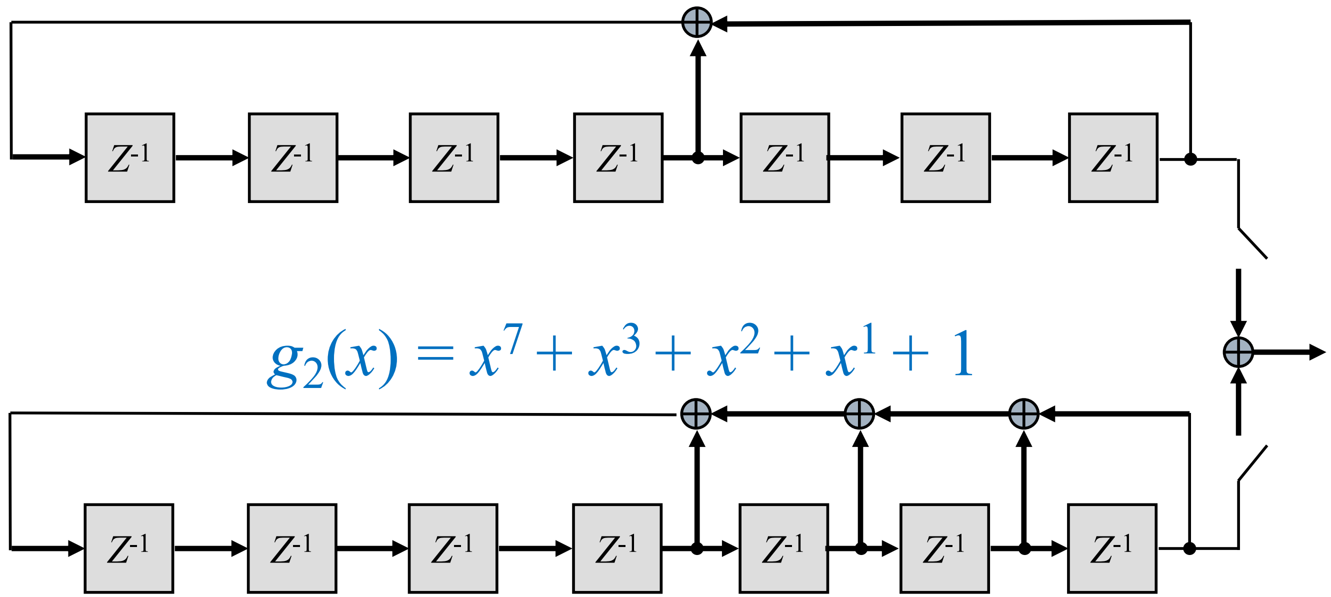
Maximum-Length and Gold Sequences

- 2^m+1 Gold sequences

$$m = 7$$

$$g_1(x) = x^7 + x^3 + 1$$

$$g_2(x) = x^7 + x^3 + x^2 + x^1 + 1$$



Maximum-Length and Gold Sequences

- Gold sequences
  - $g_1(x)$ and $g_2(x)$ are two maximum-length shift-register sequences of period $2^m - 1$, whose "cross-correlation" lies in:

$$\{-1, -t(m), t(m) - 2\},$$

$$\text{where } t(m) = \begin{cases} 2^{(m+1)/2} + 1, & m \text{ odd} \\ 2^{(m+2)/2} + 1, & m \text{ even} \end{cases}$$

  - Then, the structure in previous slide can give us $2^m - 1$ sequences (by setting different initial value in the shift registers).
  - Together with the two original m-sequences, we have $2^m + 1$ sequences.

Maximum-Length and Gold Sequences

- Gold's theorem
  - The cross-correlation between any pair in the 2^m+1 sequences also lies in

$$\{-1, -t(m), t(m)-2\},$$

$$\text{where } t(m) = \begin{cases} 2^{(m+1)/2}+1, & m \text{ odd} \\ 2^{(m+2)/2}+1, & m \text{ even} \end{cases}$$

Further discussion on $g_1(x)$ and $g_2(x)$ is defered to Chapter 8.

Maximum-Length and Gold Sequences

- Experimental results on autocorrelation and crosscorrelation properties of the maximum shift register sequences and Gold's sequences can be found in Figures 7.13 and 7.15 in textbook.