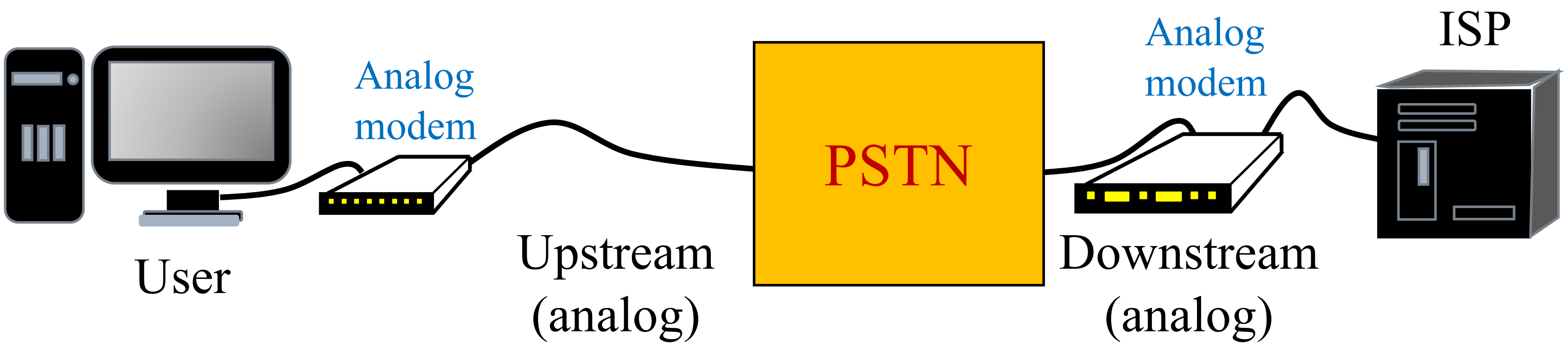
Part 3 Voiceband Modems and DSLs

Voiceband (PSTN) Modems

- Modem to/from an ISP over Public Switched Telephone Network (PSTN)
  - The connection from a home to the central office (coined as local loop) remained analog nowadays.
  - Example Study
    - **Symmetric** modem (1991): V.32

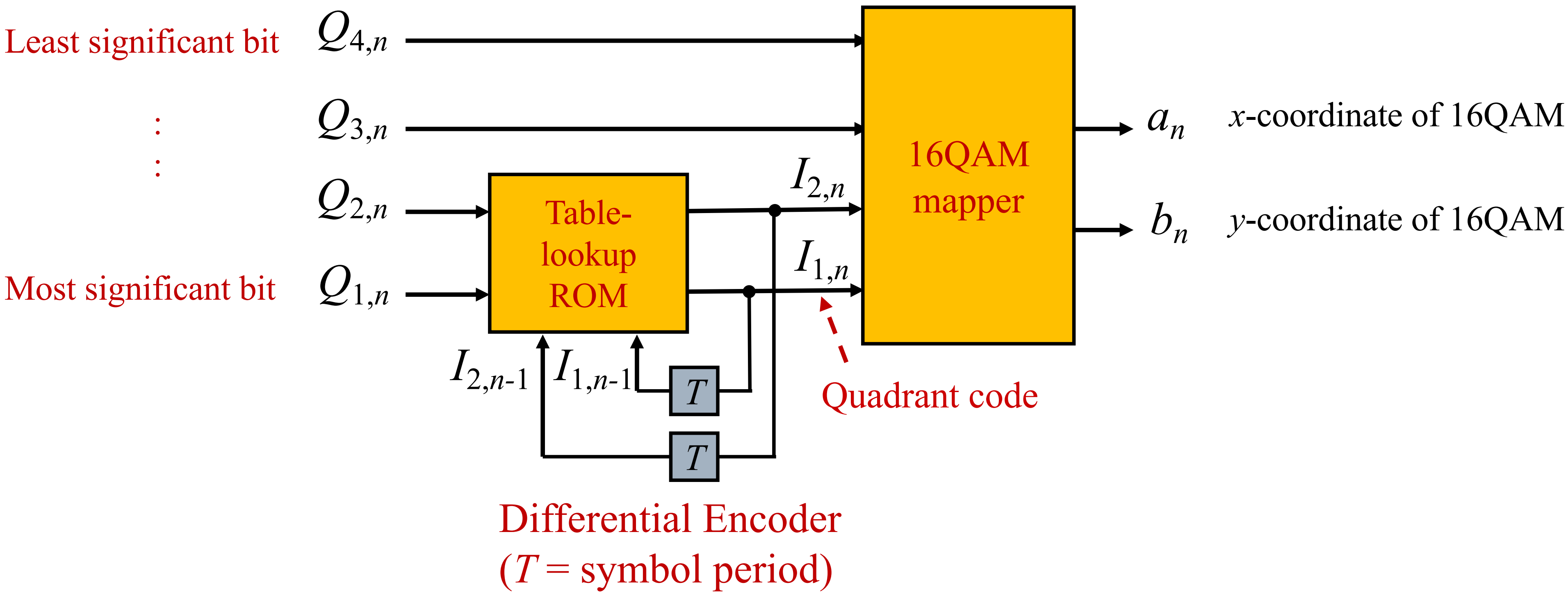


Voiceband (PSTN) Modems

- Two modulation schemes of V.32
  - Nonredundant coding
    - 2 inputs → 2 code bits in QPSK
    - 4 inputs → 4 code bits in 16QAM
  - Trellis coding
    - 4 inputs → 5 code bits

Voiceband (PSTN) Modems

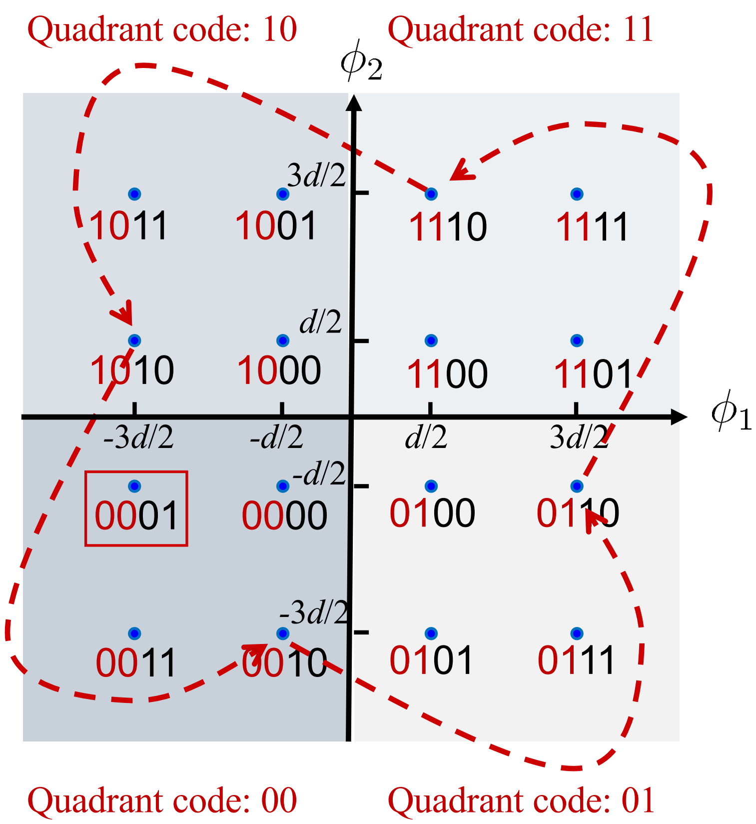
- V.32 16-QAM = Hybrid amplitude/phase modulation scheme



$Q_{1,n}$	$Q_{2,n}$	Phase change
0	0	90
0	1	0
1	0	180
1	1	270

Voiceband (PSTN) Modems

- Example $Q_{1,n}Q_{2,n} = 10$
  $Q_{3,n}Q_{4,n} = 01$
  $I_{1,n-1}I_{2,n-1} = 11$



Previous quadrant code $= I_{1,n-1}I_{2,n-1} = 11$
$\Rightarrow$ First Quadrant
$Q_{1,n}Q_{2,n} = 10$
$\Rightarrow$ Rotate 180 degree counterclockwisely
$\Rightarrow I_{1,n}I_{2,n} = 00$ (See the Table above.)
$Q_{3,n}Q_{4,n} = 01$
$\Rightarrow a_n = -3$ and $b_n = -1$

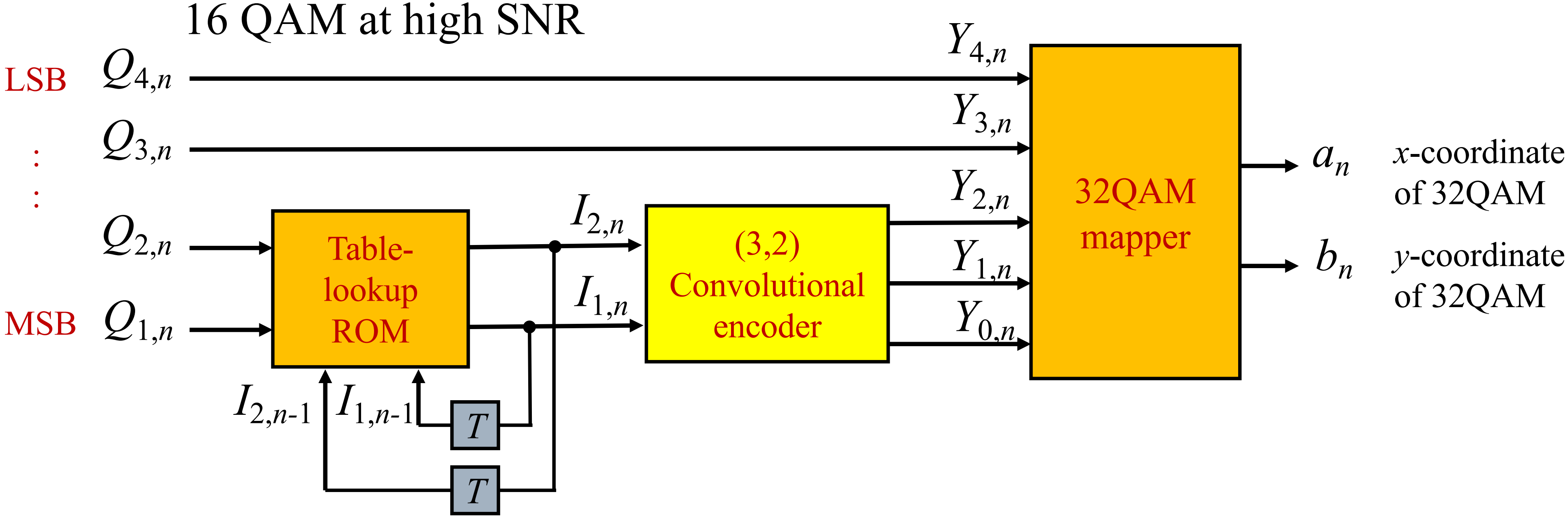
Operational example at the receiver

$I_{1,n-1}I_{2,n-1}$	$Q_{1,n}Q_{2,n}$	$Q_{3,n}Q_{4,n}$	constant 90° error	$\hat{I}_{1,n}\hat{I}_{2,n}$	$\hat{Q}_{3,n}\hat{Q}_{4,n}$	$\hat{Q}_{1,n}\hat{Q}_{2,n}$
11	10	01	$(-3,-1) \rightarrow (1,-3)$	01		
00	10	10	$(1,3) \rightarrow (-3,1)$	10	10	$\rightarrow$ 10
11						

Voiceband (PSTN) Modems

- V.32 Trellis coding = Hybrid amplitude/phase modulation scheme : 4 dB coding gain over 16 QAM at high SNR

90-degree rotational invariant with respect to $Q_{3,n} = Y_{3,n}$ and $Q_{4,n} = Y_{4,n}$.



Differential Encoder
(T = symbol period)

$Q_{1,n}$	$Q_{2,n}$	Phase change
0	0	0
0	1	180
1	0	270
1	1	90

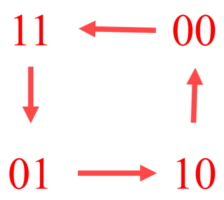


TABLE 2/V.32
Differential encoding for use with trellis coded alternative at 9600 bps

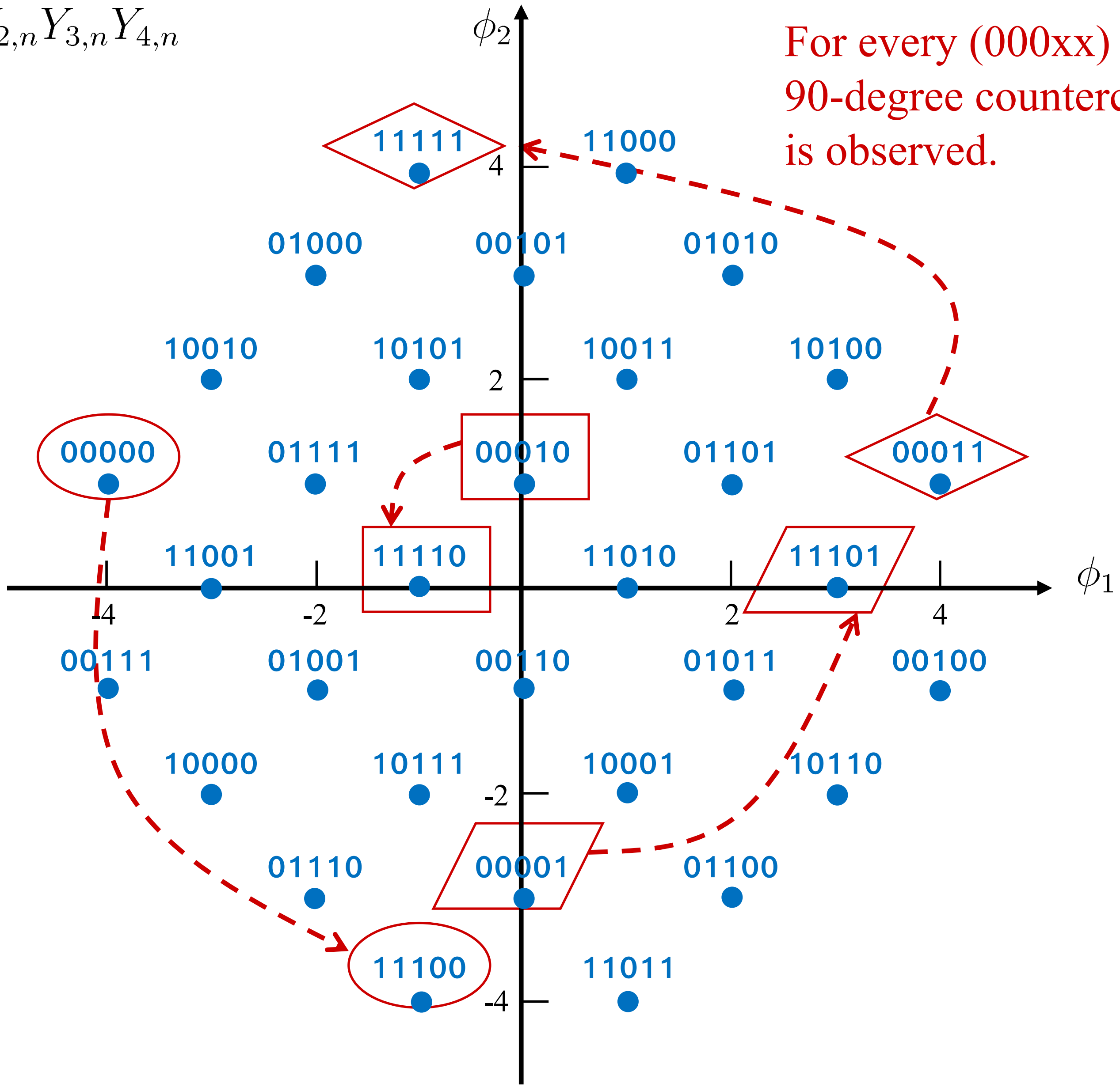
Input		Previous Output		Output	
$Q_{1,n}$	$Q_{2,n}$	$I_{1,n-1}$	$I_{2,n-1}$	$I_{1,n-1}$	$I_{2,n-1}$
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	1	1
0	1	0	0	0	1
0	1	0	1	0	0
0	1	1	0	1	1
0	1	1	1	1	0
1	0	0	0	1	0
1	0	0	1	1	1
1	0	1	0	0	1
1	0	1	1	0	0
1	1	0	0	1	1
1	1	0	1	1	0
1	1	1	0	0	0
1	1	1	1	0	1

Voiceband (PSTN) Modems

- Rotationally invariant
  - "Rotation" of the constellation points must be "invariant' with respect to $Q_{3,n} = Y_{3,n}$ and $Q_{4,n} = Y_{4,n}$.

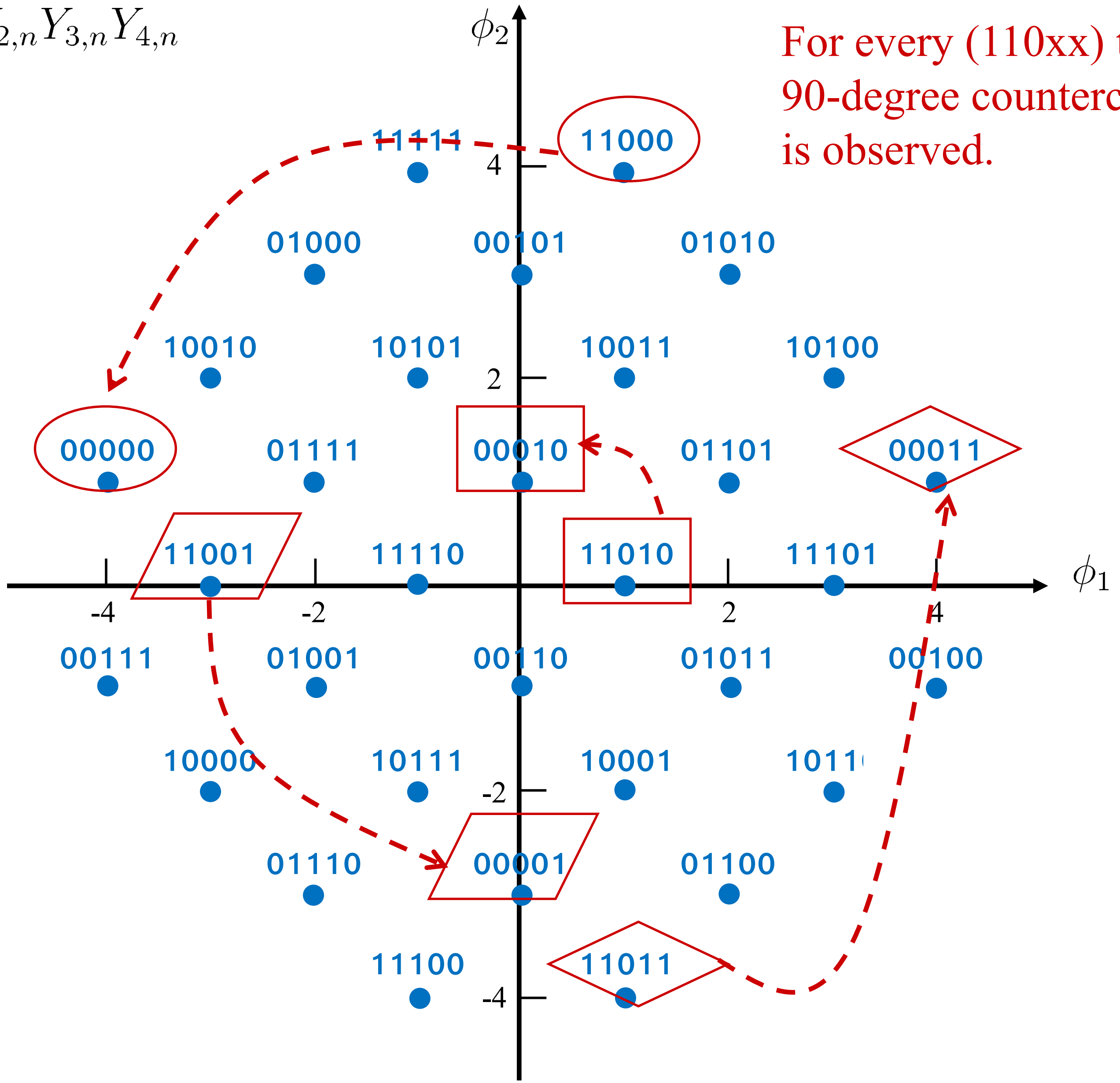
the rotational angle between signal points corresponding to
$$abc00 \text{ and } def00$$
is exactly the same as that of
$$abc01 \text{ and } def01$$
$$abc10 \text{ and } def10$$
$$abc11 \text{ and } def11,$$
where $a, b, c, d, e, f \in \{0, 1\}$.

$Y_{0,n}Y_{1,n}Y_{2,n}Y_{3,n}Y_{4,n}$



For every (000xx) to (111xx), 90-degree counterclockwise rotation is observed.

$Y_{0,n}Y_{1,n}Y_{2,n}Y_{3,n}Y_{4,n}$



For every (110xx) to (000xx), 90-degree counterclockwise rotation is observed.

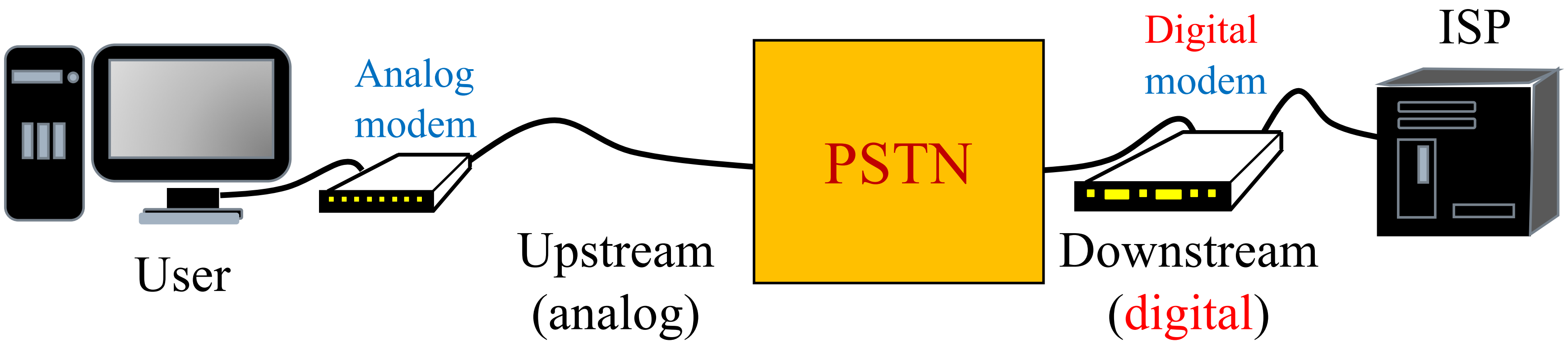
Voiceband (PSTN) Modems

- At the receiver end, a convoludional decoder is performed first, followed by the differential decoder.
- In presence of a (fixed) phase error, e.g., 90 degree, as long as the phase transition between two consecutive signal points remains the same, the differential decoder can correctly determine $Q_{1,n}$ and $Q_{2,n}$ based on the output of the convolutional decoder.

$Q_{1,n}$	$Q_{2,n}$	Phase change
0	0	0
0	1	180
1	0	270
1	1	90

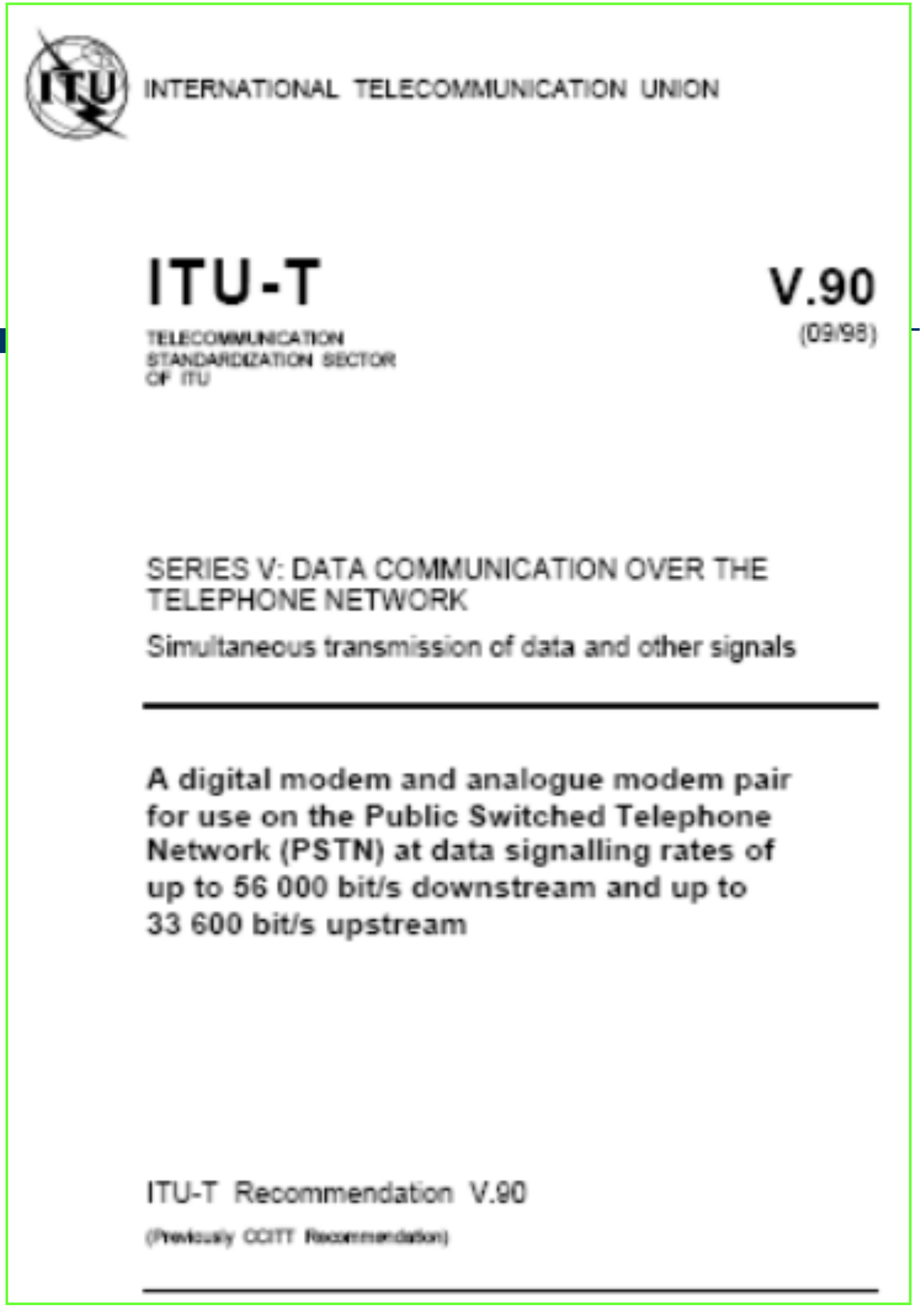
Voiceband (PSTN) Modems

- Modem to/from an ISP over PSTN
  - The connection from a home to the central office (coined as local loop) remained analog nowadays.
  - Example Study
    - **Asymmetric** modem
      - The communication between **PSTN** and **ISP** becomes digital.



Voiceband (PSTN) Modems

- A digital modem and analogue modem pair for use on the Public Switched Telephone Network (PSTN) at data signalling rates of up to 56 000 bit/s downstream and up to 33 600 bit/s upstream


INTERNATIONAL TELECOMMUNICATION UNION

ITU-T

TELECOMMUNICATION STANDARDIZATION SECTOR OF ITU

V.90

(09/98)

SERIES V: DATA COMMUNICATION OVER THE TELEPHONE NETWORK

Simultaneous transmission of data and other signals

A digital modem and analogue modem pair for use on the Public Switched Telephone Network (PSTN) at data signalling rates of up to 56 000 bit/s downstream and up to 33 600 bit/s upstream

ITU-T Recommendation V.90

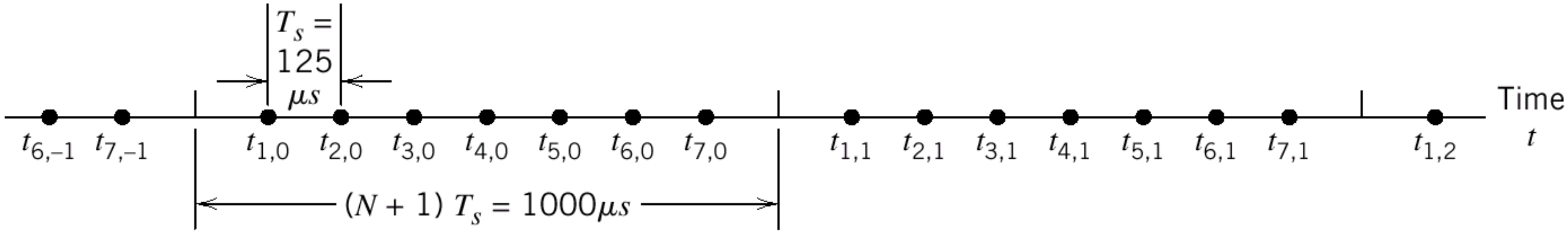
(Previously CCITT Recommendation)

Voiceband (PSTN) Modems

- Typical realization of digital modem
  - With PCM sampling rate = 8 KHz, and 256 levels per PCM sample, the PCM should result in 64 Kbps in theory.
  - However, since the conventional **PCM transmit filter** has a **bandwidth of about 3.5 KHz**, the theoretical speed of 64 Kbps cannot be achieved.
    - One solution: Use the recurrent non-uniform equivalent form of the sampling theorem.

Voiceband (PSTN) Modems

- The below non-uniform sampling is equivalent to the sampling rate of 7 KHz.



$$t_{k,\ell} = t_k + 8\ell T_s = (k-1)T_s + 8\ell T_s \text{ for } k = 1, 2, \cdots, 7,\ \ell = 0, \pm 1, \pm 2, \cdots$$

Voiceband (PSTN) Modems

- The bandlimited signal is now interpolated as:

$$s(t) = \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{7} s(t_{k,\ell}) \psi_k(t - 8\ell T_s)$$

$\boxed{t_k = (k-1)T_s}$

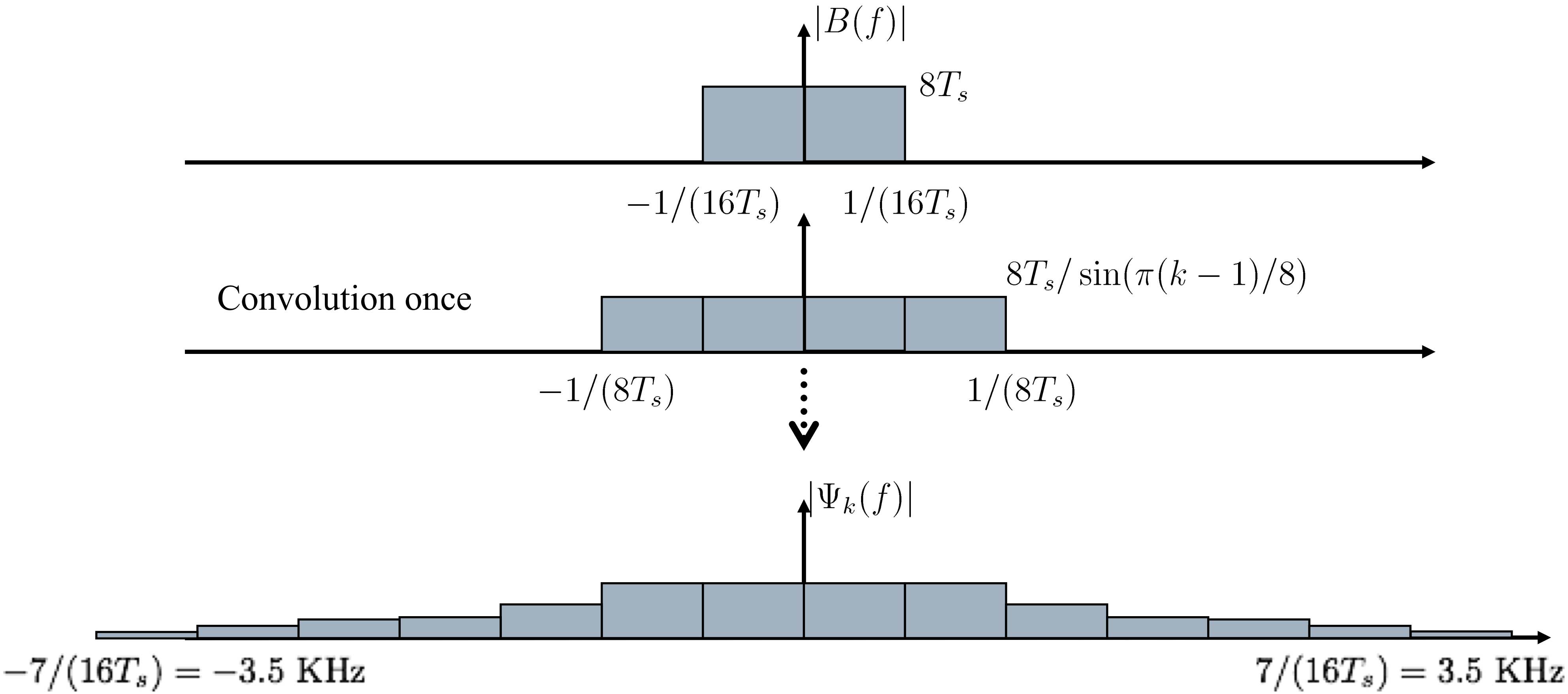
$$\text{where } \psi_k(t) = \operatorname{sinc}\left(\frac{t-t_k}{8T_s}\right) \prod_{q=1, q\neq k}^{7} \frac{\sin\left(\frac{\pi}{8T_s}(t-t_q)\right)}{\sin\left(\frac{\pi}{8T_s}(t_k-t_q)\right)}.$$

- Properties of the seven standard pulse functions

$$\left.\begin{array}{l} \text{1. } \psi_k(t_k) = 1 \text{ but not peak at } t = t_k. \\ \text{2. } \psi_k(t) = 0 \text{ at } t = t_k + 8\ell T_s \text{ for } \ell = \pm 1, \pm 2, \cdots \\ \text{3. } \psi_k(t) = 0 \text{ at } t = t_q \text{ for } 1 \leq q \leq 8 \text{ and } q \neq k \end{array}\right\} \text{No ISI}$$

$\Psi_k(f)$ is $B(f) = 8T_s \cdot \Pi(8T_s f)\, e^{-j2\pi(k-1)T_s f}$ convolved with six sine waves of the form

$$\frac{e^{-j\pi(q-1)/8}\,\delta\left(f - \frac{1}{16T_s}\right) - e^{j\pi(q-1)/8}\,\delta\left(f + \frac{1}{16T_s}\right)}{\sin\left(\frac{\pi}{8}(k-q)\right)}.$$



Voiceband (PSTN) Modems

- The above realization resolves the A/D problem, subject to the 3.5 KHz bandwidth constraint of the transmit filter.
- How about D/A? Is 64 Kbps achievable for the digital modem?
  - Still, 56 Kbps is the feasible rate.
    - The reconstructed voice as below is susceptible to (quantization) residual ISI.

$$s(t) = \sum_\ell a(c_\ell) g(t - \ell T_s),$$

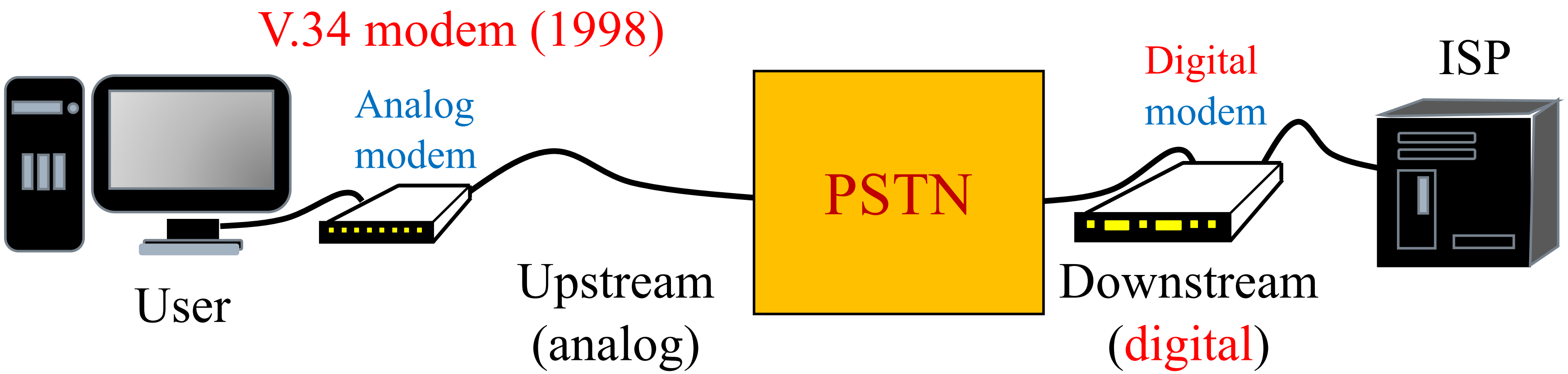
where $g(\cdot)$ interpolation function bandlimited to $1/(2T_s)$, c_ℓ is the ℓth octet, and $a(\cdot)$ is the companding law (e.g., μ law).

Voiceband (PSTN) Modems

- Most importantly, LSBs in each octet are robbed from the data stream for various purposes internal to the PSTN (which is called "bit-robbing", and which makes worse the susceptibility to residual ISI).

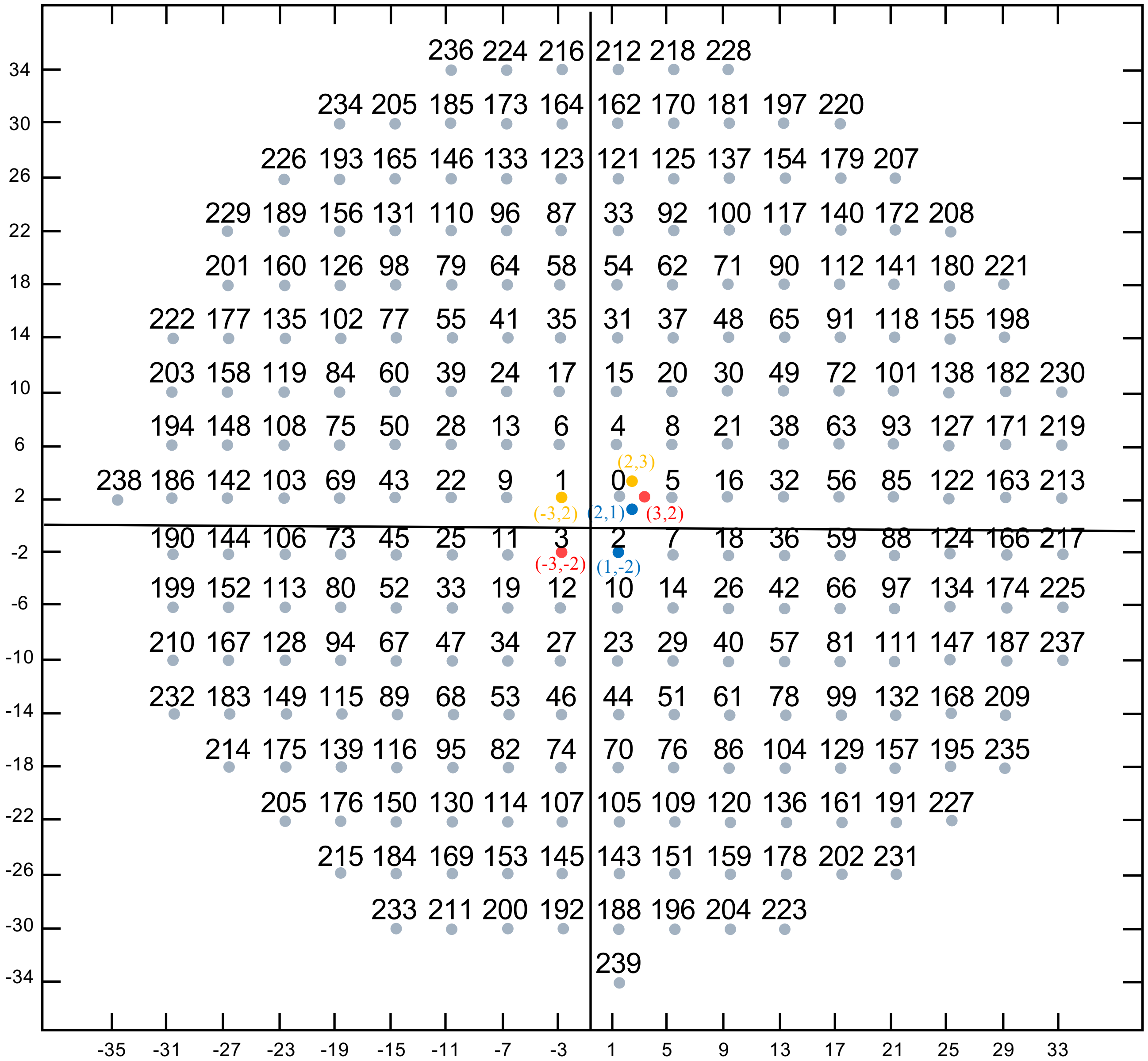
Voiceband (PSTN) Modems

- Modem to/from an ISP over PSTN
  - The connection from a home to the central office (coined as local loop) remained analog nowadays.
  - Example Study
    - **Asymmetric** modem



Voiceband (PSTN) Modems

- Due to the 3.5KHz bandwidth restriction of anti-aliasing and interpolation filters, and the quantization noise, the rate of the analog modem operates at 33.6 kbps with the below features:
  - 960-QAM super-constellation
    - You may view it as a selection of 240 message points, and their rotated versions through 90, 180, and 270 degrees.



Quarter-superconstellation of V.34 modem with 240 signal points. The full superconstellation is obtained by the rotated versions of these points by 90, 180, and 270 degrees.

Voiceband (PSTN) Modems

- Adaptive bandwidth
  - A set of probe tones will be transmitted for measurement of SNR as a function of frequency.
  - Then the appropriate carrier frequency and bandwidth will be appropriately selected based on the measurement results.
- Adaptive bit rates
  - Selection of bit rates subject to bit error rate requirement.

$$10^{-5} \geq \mathrm{BER} \geq 10^{-6}$$

- Tomlinson, M. 1971. A new automatic equalizer employing modulo arithmetic. Electr. Lett., 7: 138-139.
- Harashima, H. and Miyakawa, H. 1972. Matched-transmission technique for channels with intersymbol interference. IEEE Trans. Commun., COM-20: 774-780.

Voiceband (PSTN) Modems

- Trellis coding
  - Compulsory trellis coding provides 3.6 dB coding gain
  - Optional trellis coding provides 4.7 dB coding gain
- Decision feedback equalizer (DFE) (See Sec. 4.10)
  - DFE requires immediate decision that cannot be directly obtained when channel coding technology is introduced.
  - Hence, the feedback section of the DFE is moved to the transmitter, which is made possible through the use of the Tomlinson-Harashima precoding.

Multichannel Modulation

- Shannon information capacity theorem

$$C = B\log_2(1+\mathrm{SNR})\ \mathrm{b/s} = \tfrac{1}{2}\log_2(1+\mathrm{SNR})\ \mathrm{b/transmission}$$

where B is the baseband bandwidth. $\boxed{\text{One transmission takes } \tfrac{1}{2B} \text{ seconds.}}$

- Alternatively, we can write:

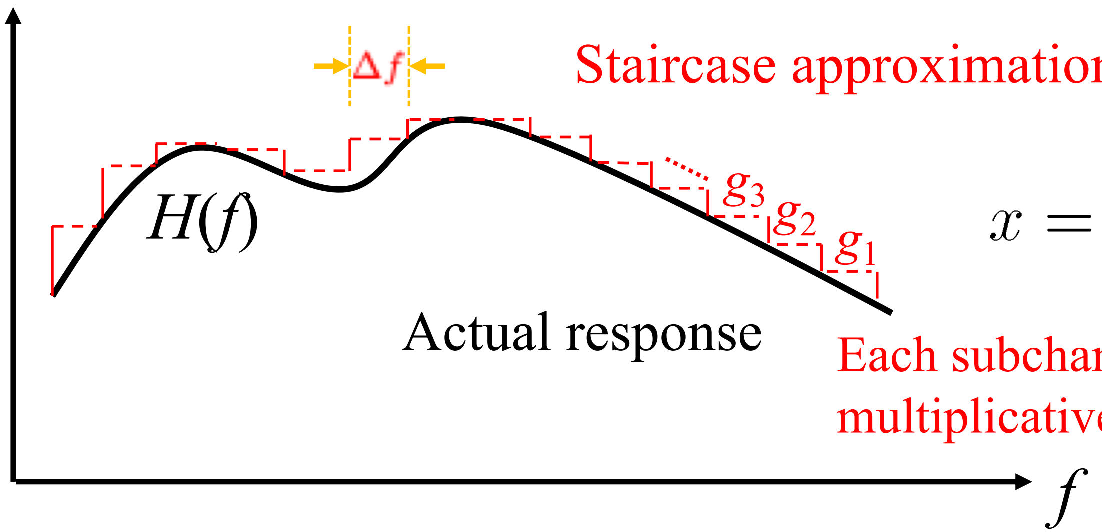
$$\mathrm{SNR} = 2^{2C} - 1$$

- Therefore, we can define the gap between ideal Shannon SNR, and the SNR attainable for a rate R below C as

$$\Gamma = \frac{2^{2C}-1}{2^{2R}-1} = \frac{\mathrm{SNR}}{2^{2R}-1} \quad \text{or} \quad R = \frac{1}{2}\log_2\left(1+\frac{\mathrm{SNR}}{\Gamma}\right)$$

Multichannel Modulation

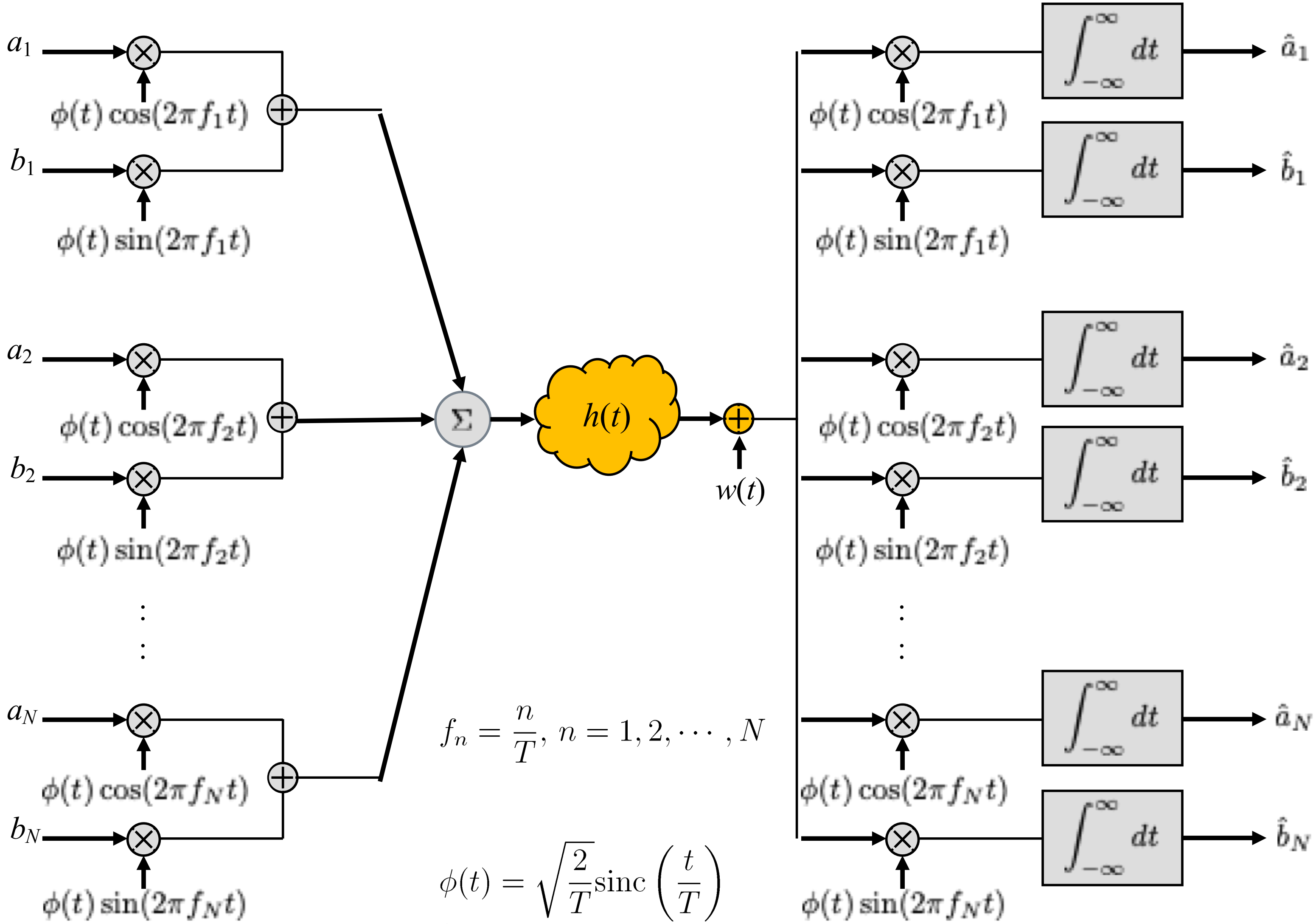
- Multichannel modulation
  - Partition a channel (with squared magnitude response $|H(f)|$ as shown below) into a number of subchannels such that each subchannel becomes approximately AWGN



$$x = g \cdot s + w$$

Each subchannel has different multiplicative gain.

Block diagram of multichannel data transmission system



$$f_n = \frac{n}{T},\ n = 1, 2, \cdots, N$$

$$\phi(t) = \sqrt{\frac{2}{T}} \operatorname{sinc}\left(\frac{t}{T}\right)$$

for $-\infty < t < \infty$

Multichannel Modulation

- Properties of multichannel modulation
  - Property 1: Orthogonality of the two quadrature-modulated sinc functions (in the sense of integration over the entire real line)

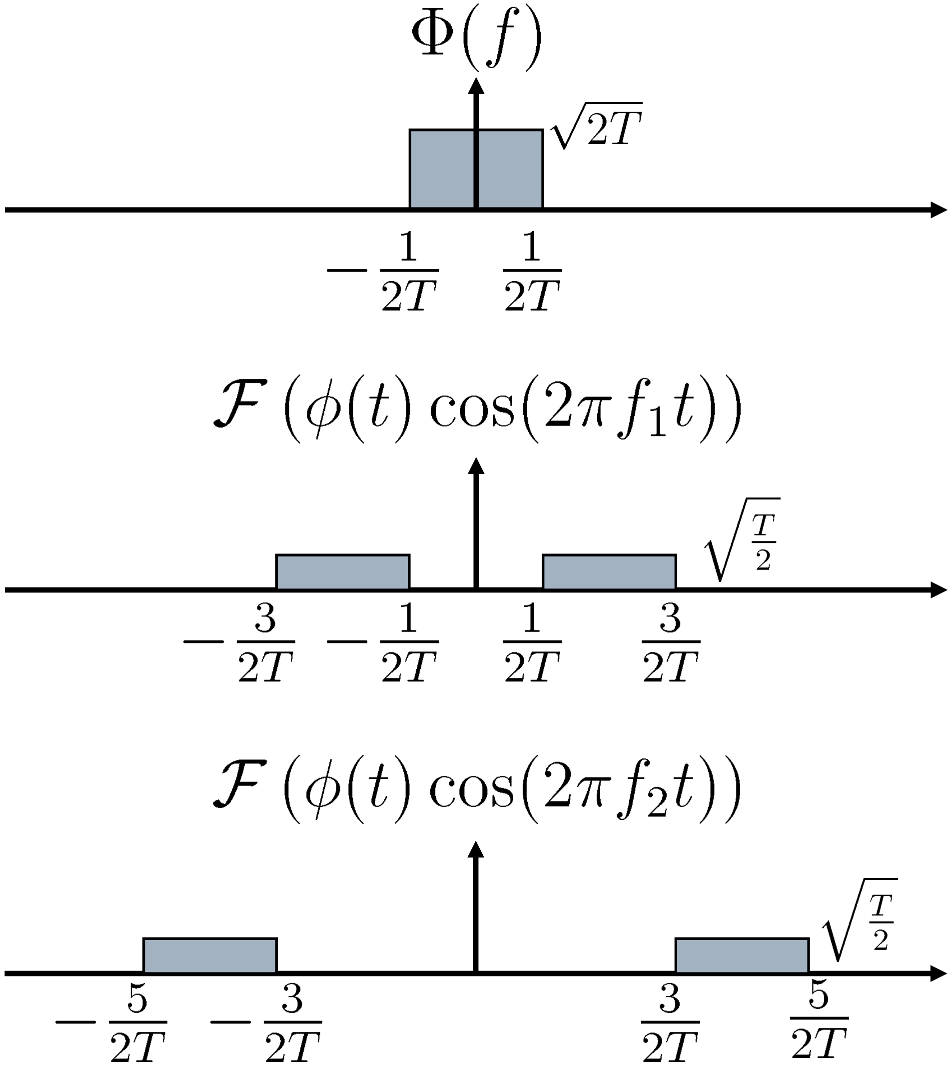
$$
\begin{aligned}
&\langle \phi(t)\cos(2\pi f_n t), \phi(t)\sin(2\pi f_n t)\rangle \\
= &\int_{-\infty}^{\infty} \phi(t)\cos(2\pi f_n t)\phi(t)\sin(2\pi f_n t)dt = 0
\end{aligned}
$$

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$$

where $F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$

and $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt.$

- Property 2: We also have orthogonality among subchannel signals



Introduction of $\phi(t)$ makes no "band overlap" between adjacent subchannels.

Multichannel Modulation

- Property 3: Orthogonality among subchannel signals remains after passing subchannel signals through **linear** channel with arbitrary response *h*.

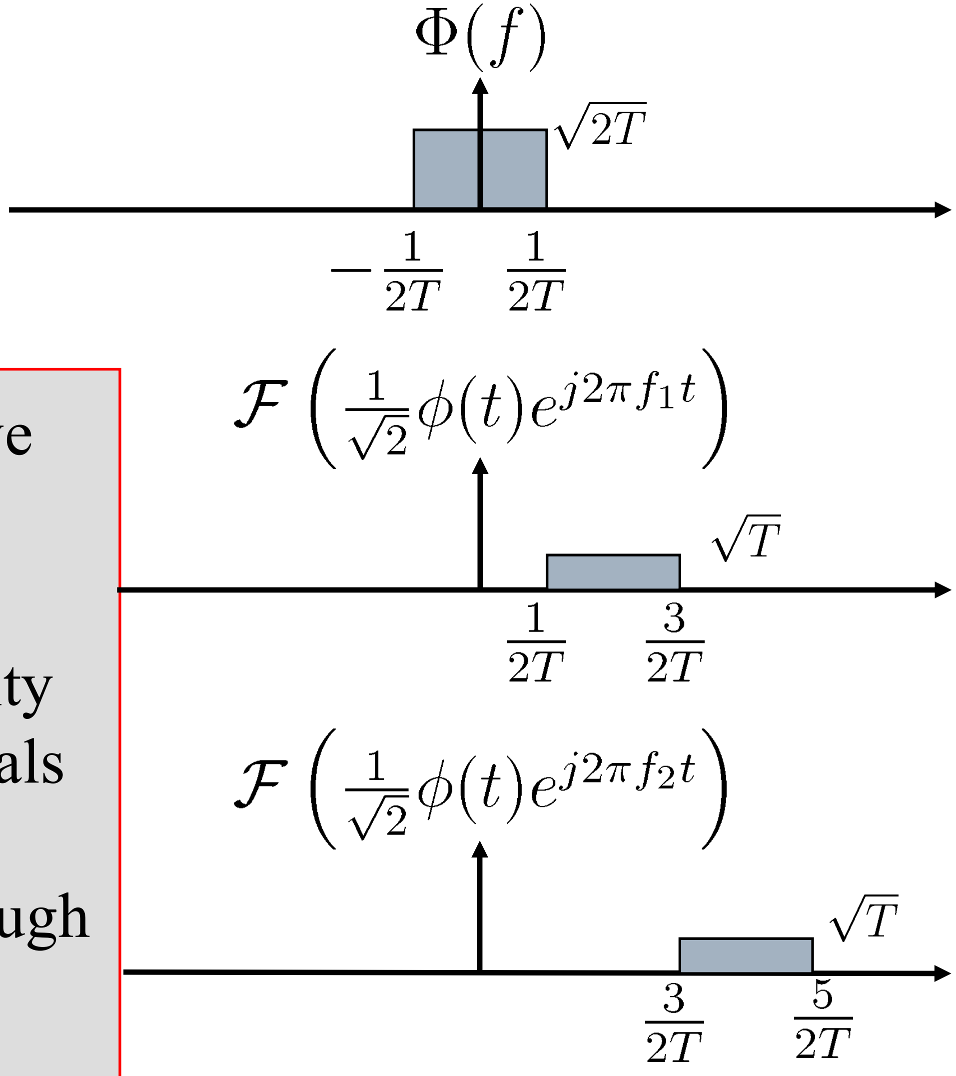
$$\left\{\frac{1}{\sqrt{2}}\phi(t)\exp(j2\pi f_n t)\right\}_{n=1}^{N} \rightarrow \boxed{h(t)} \rightarrow \left\{\frac{1}{\sqrt{2}}\phi(t)\exp(j2\pi f_n t) * h(t)\right\}_{n=1}^{N}$$

Still, no "band overlap" between "convolved" adjacent subchannels.

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$$
where $F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$

and $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt.$

- Property 2: We also have orthogonality among subchannel signals
- Property 3: Orthogonality among subchannel signals remains after passing subchannel signals through **linear** channel with arbitrary response h.



Introduction of $\phi(t)$ makes no "band overlap" between adjacent subchannels.

Multichannel Modulation

- Geometric SNR for multichannel modulation
  - The average rate (in bits per transmission per subchannel)

$$
\begin{aligned}
R &= \frac{1}{N}\sum_{n=1}^{N} R_n \\
&= \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right) \\
&= \frac{1}{2N}\log_2\prod_{n=1}^{N}\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right) \\
&= \frac{1}{2}\log_2\prod_{n=1}^{N}\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right)^{1/N}
\end{aligned}
\qquad
R = \frac{1}{2}\log_2\left(1+\frac{\mathrm{SNR}_{\mathrm{overall}}}{\Gamma}\right)
$$

Multichannel Modulation

We then obtain:

$$\mathrm{SNR}_{\mathrm{overall}} = \Gamma \left(\prod_{n=1}^{N} \left(1 + \frac{\mathrm{SNR}_n}{\Gamma}\right)^{1/N} - 1 \right)$$

Assume that

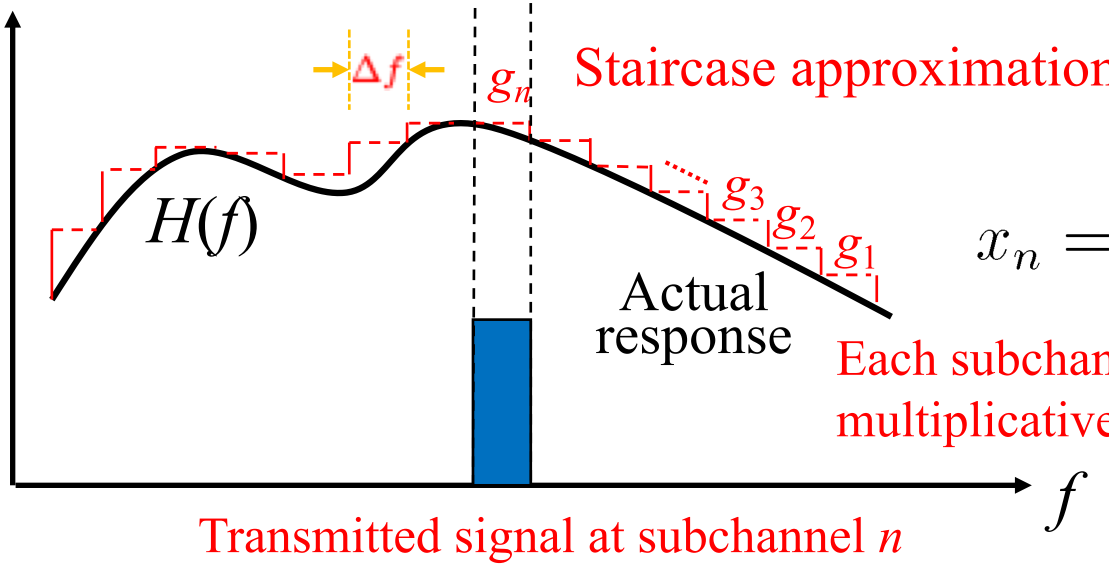
$$\frac{\mathrm{SNR}_n}{\Gamma} \gg 1.$$

$$\mathrm{SNR}_{\mathrm{overall}} \approx \Gamma \left(\prod_{n=1}^{N} \left(\frac{\mathrm{SNR}_n}{\Gamma}\right)^{1/N} \right) = \left(\prod_{n=1}^{N} \mathrm{SNR}_n \right)^{1/N}$$

Geometric mean of individual **SNR**

Multichannel Modulation

- Solution of loading problem – water filling
  - The process of allocating the transmit power P to the individual subchannel so as to maximize the system bit rate is called *loading*.



$$\boxed{\mathrm{SNR}_n = \frac{g_n^2 P_n}{\sigma_n^2}}$$

$$x_n = g_n \cdot s_n + w_n$$

Multichannel Modulation

$$
\max_{\{(P_1,P_2,\cdots,P_N):\sum_{n=1}^{N} P_n \le P\}} \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right)
$$

$$
= \max_{\{(P_1,P_2,\cdots,P_N):\sum_{n=1}^{N} P_n = P\}} \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right)
$$

$$
= \max_{\{(P_1,P_2,\cdots,P_N):\sum_{n=1}^{N} P_n = P\}} \left[\frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right) + \lambda\left(P-\sum_{n=1}^{N} P_n\right)\right]
$$

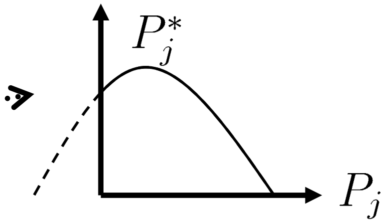
Lagrange multiplier

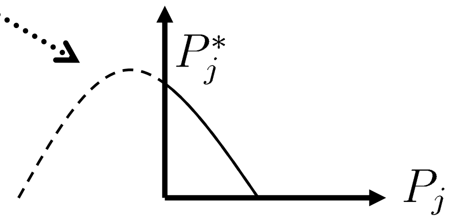
Multichannel Modulation

$$f(P_1, P_2, \cdots, P_n|\lambda) = \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right) + \lambda\left(P - \sum_{n=1}^{N} P_n\right)$$ concave with respect to P_j

Hence,

$$\left.\frac{\partial f(P_1, P_2, \cdots, P_n|\lambda)}{\partial P_j}\right|_{P_j=P_j^*} \begin{cases} = 0, & \text{if } P_j^* > 0 \\ \le 0, & \text{if } P_j^* = 0 \end{cases}$$





As a result,

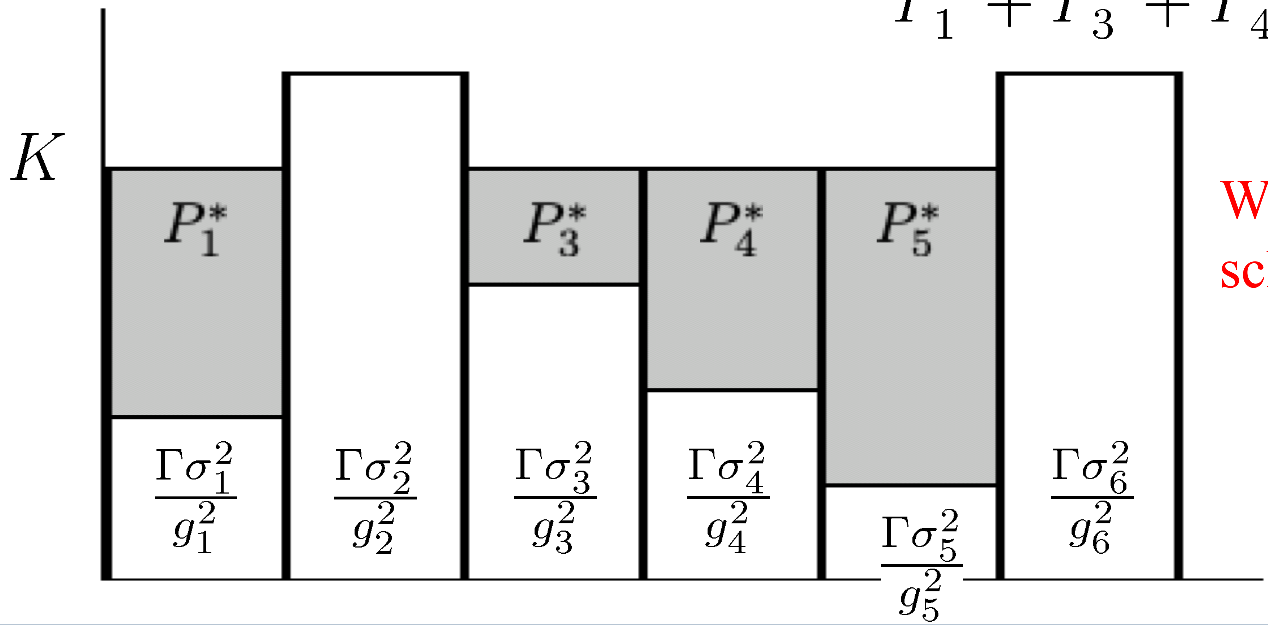
$$\frac{\log_2(e)}{2N}\frac{g_j^2/(\Gamma\sigma_j^2)}{1+g_j^2 P_j^*/(\Gamma\sigma_j)} - \lambda \begin{cases} = 0, & \text{if } P_j^* > 0 \\ \le 0, & \text{if } P_j^* = 0 \end{cases}$$

$$\frac{1}{\Gamma\sigma_j^2/g_j^2 + P_j^*}\begin{cases} = \lambda\frac{2N}{\log_2(e)} = \frac{1}{K}, & \text{if } P_j^* > 0 \\ \le \lambda\frac{2N}{\log_2(e)} = \frac{1}{K}, & \text{if } P_j^* = 0 \end{cases} \quad \text{and} \quad \sum_{j=1}^{N} P_j^* = P$$

$$\Downarrow$$

$$\Gamma\sigma_j^2/g_j^2 + P_j^*\begin{cases} = K, & \text{if } P_j^* > 0 \\ = \Gamma\sigma_j^2/g_j^2 \ge K, & \text{if } P_j^* = 0 \end{cases} \quad \text{and} \quad \sum_{j=1}^{N} P_j^* = P$$

$$P_1^* + P_3^* + P_4^* + P_5^* = P$$



Water filling scheme

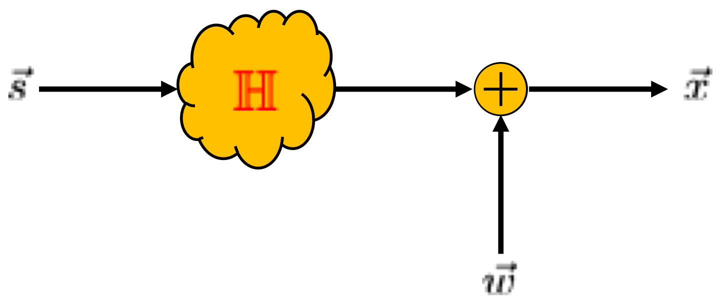
Discrete Multitone

- Impractical in (analog) multichannel modulation
  - Sinc function is time-unlimited (as it is band-limited).
  - The inner product (defined in time-domain) requires to perform integration over the entire real line.
    - Performing integration over a practically finite range will make the (analog) multichannel modulation suboptimal as our text has put it.

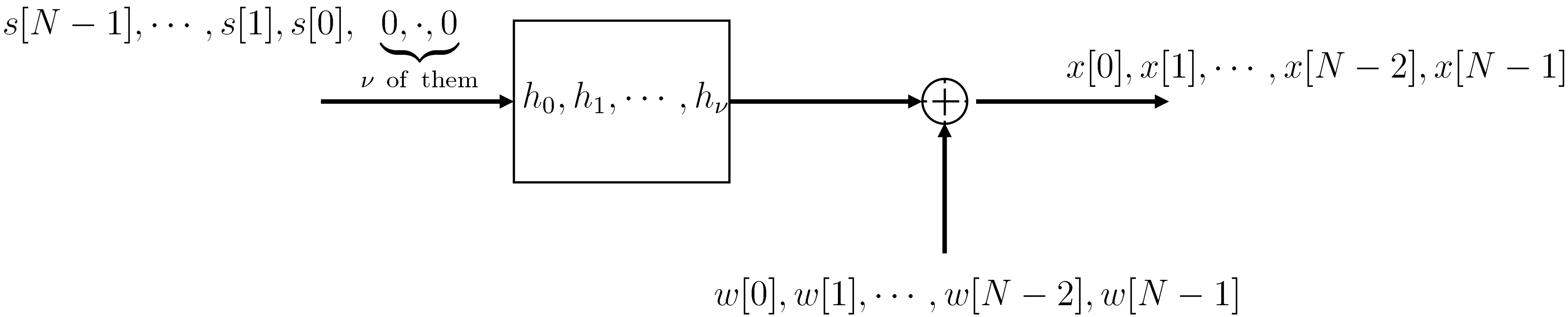
$$\begin{aligned} &\langle \phi(t)\cos(2\pi f_n t), \phi(t)\sin(2\pi f_n t)\rangle \\ = &\int_{-\infty}^{\infty} \phi(t)\cos(2\pi f_n t)\phi(t)\sin(2\pi f_n t)dt = 0 \end{aligned}$$

Discrete Multitone

- A solution
  - Discrete multitone (DMT)
  - Transform *linear discrete convolution* to *circular discrete convolution* by adding *cyclic prefix*.
- Procedure of DMT
  - Sampling the analog signals with sufficiently large sampling rate $1/T_s$.



Linear convolution



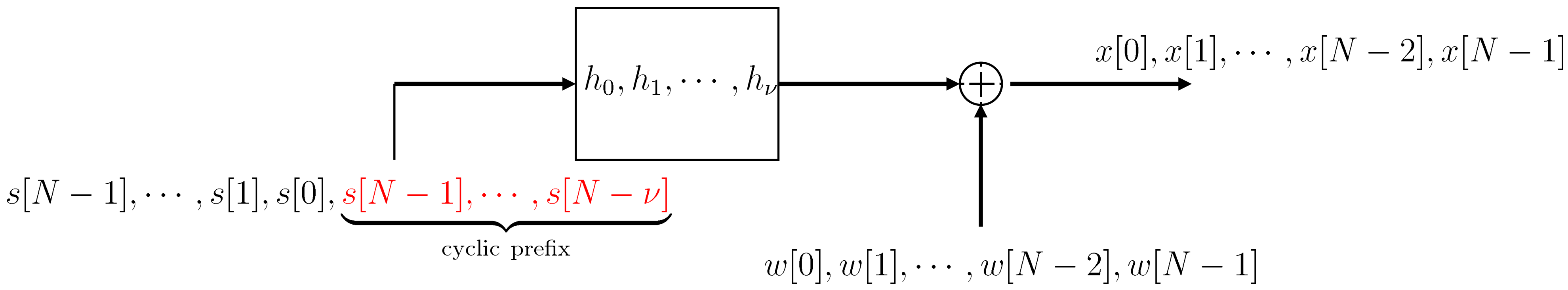
$$\begin{bmatrix} x[N-1] \\ x[N-2] \\ \vdots \\ x[N-\nu-1] \\ x[N-\nu-2] \\ \vdots \\ x[0] \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{\nu-1} & h_\nu & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_{\nu-2} & h_{\nu-1} & h_\nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_\nu \\ 0 & 0 & 0 & \cdots & 0 & 0 & h_0 & \cdots & h_{\nu-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & h_0 \end{bmatrix} \begin{bmatrix} s[N-1] \\ s[N-2] \\ \vdots \\ s[N-\nu-1] \\ s[N-\nu-2] \\ \vdots \\ s[0] \end{bmatrix} + \begin{bmatrix} w[N-1] \\ w[N-2] \\ \vdots \\ w[N-\nu-1] \\ w[N-\nu-2] \\ \vdots \\ w[0] \end{bmatrix}$$

- The above formula is valid under the assumption that

$$s[-\nu] = s[-\nu+1] = \cdots = s[-1] = 0$$

- Without the guard period, ISI occurs.

Circular convolution



$$
\begin{bmatrix} x[N-1] \\ x[N-2] \\ \vdots \\ x[N-\nu-1] \\ x[N-\nu-2] \\ \vdots \\ x[0] \end{bmatrix}
=
\begin{bmatrix}
h_0 & h_1 & h_2 & \cdots & h_{\nu-1} & h_\nu & 0 & \cdots & 0 \\
0 & h_0 & h_1 & \cdots & h_{\nu-2} & h_{\nu-1} & h_\nu & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_\nu \\
h_\nu & 0 & 0 & \cdots & 0 & 0 & h_0 & \cdots & h_{\nu-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
h_1 & h_2 & h_3 & \cdots & h_\nu & 0 & 0 & \cdots & h_0
\end{bmatrix}
\begin{bmatrix} s[N-1] \\ s[N-2] \\ \vdots \\ s[N-\nu-1] \\ s[N-\nu-2] \\ \vdots \\ s[0] \end{bmatrix}
+
\begin{bmatrix} w[N-1] \\ w[N-2] \\ \vdots \\ w[N-\nu-1] \\ w[N-\nu-2] \\ \vdots \\ w[0] \end{bmatrix}
$$

- Instead of zeroing the guard period, how about letting

$$s[-k] = s[N-k] \text{ for } k = 1, 2, \cdots, \nu$$

Circular convolution to Discrete Fourier Transform

$$\boldsymbol{x} = \mathbb{H}_{\text{circulant}}\boldsymbol{s} + \boldsymbol{w}$$

- **Spectral decomposition of a circulant matrix**

$$\mathbb{H}_{\text{circulant}} = \mathbb{Q}^{\dagger}\boldsymbol{\Lambda}\mathbb{Q}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix, and

$$\mathbb{Q} = \frac{1}{\sqrt{N}}\begin{bmatrix} e^{-j\frac{2\pi}{N}(N-1)(N-1)} & \cdots & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}(N-1)} & 1 \\ e^{-j\frac{2\pi}{N}(N-1)(N-2)} & \cdots & e^{-j\frac{2\pi}{N}2(N-2)} & e^{-j\frac{2\pi}{N}(N-2)} & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ e^{-j\frac{2\pi}{N}(N-1)} & \cdots & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}} & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}$$

Circular convolution to Discrete Fourier Transform

$$\boldsymbol{x} = \mathbb{Q}^{\dagger}\boldsymbol{\Lambda}\mathbb{Q}\boldsymbol{s} + \boldsymbol{w}$$

$$\Rightarrow \mathbb{Q}\boldsymbol{x} = \mathbb{Q}\mathbb{Q}^{\dagger}\boldsymbol{\Lambda}\mathbb{Q}\boldsymbol{s} + \mathbb{Q}\boldsymbol{w} \qquad \boxed{\mathbb{Q}\mathbb{Q}^{\dagger} = \mathbb{I}}$$

$$\Rightarrow \boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{S} + \boldsymbol{W}$$

$$\Rightarrow X_k = \lambda_k S_k + W_k \text{ for } k = 0, 1, \cdots, N-1$$

(Here, $\{\lambda_k\}_{k=0}^{N-1}$ are assumed "known" or "can-be-accurately estimated".)

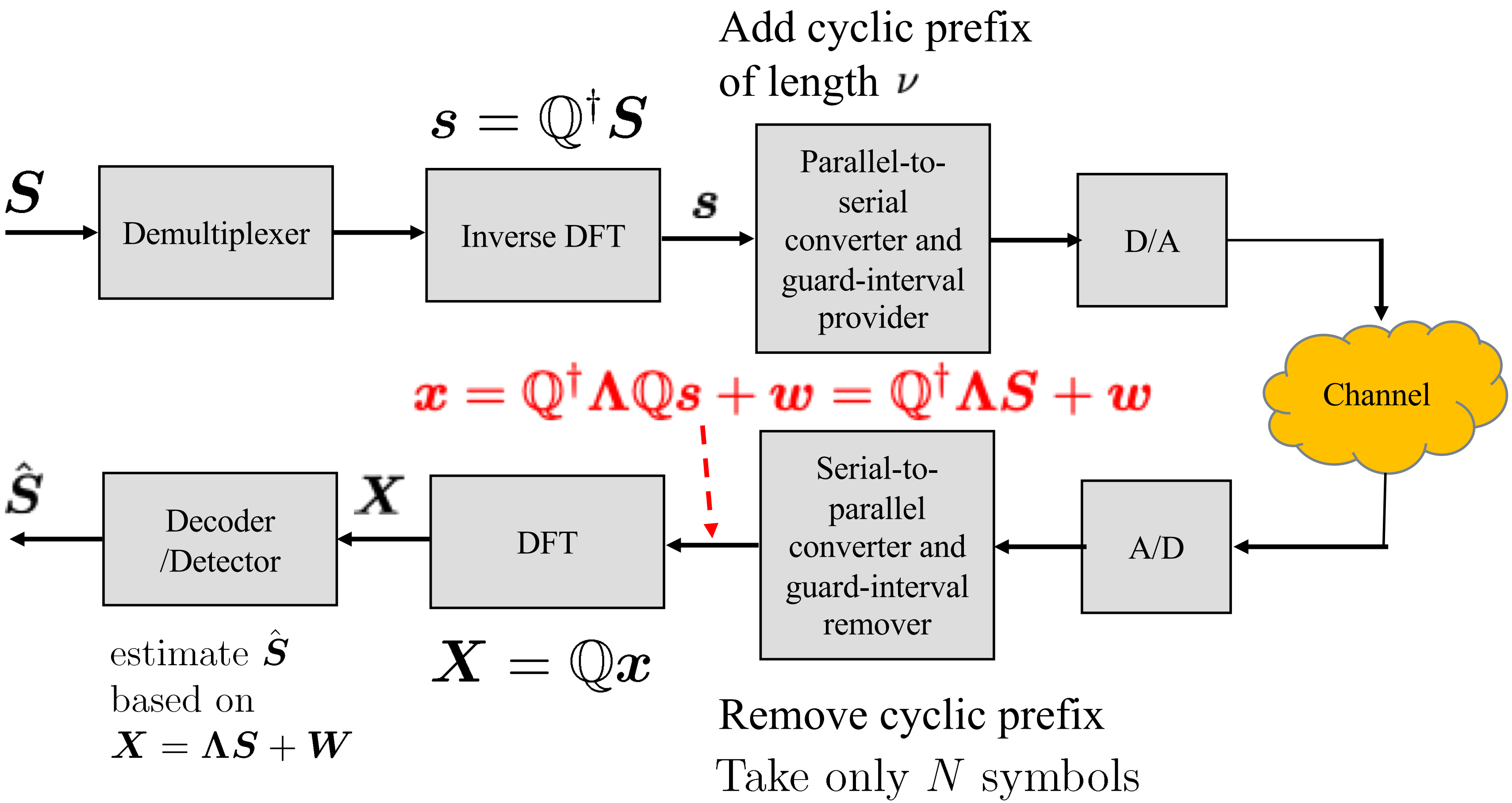
Discrete Multitone

- With cyclic prefix, the discrete Fourier transform (DTF) technique can be used straightforwardly.
  - DFT transform pair
    - Analysis equation versus synthesis equation

$$\begin{cases} X[k] = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi}{N}kn\right) & \text{for } k = 0, 1, \cdots, N-1 \\ x[n] = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{k=0}^{N-1} X[k] \exp\left(j\frac{2\pi}{N}kn\right) & \text{for } n = 0, 1, \cdots, N-1 \end{cases}$$

$$\boxed{\begin{cases} \boldsymbol{X} = \mathbb{Q} \, \boldsymbol{x} \\ \boldsymbol{x} = \mathbb{Q}^{\dagger} \, \boldsymbol{X} \end{cases}}$$

Discrete Multitone



Synchronization

- Modes of synchronization
  - Carrier synchronization (carrier recovery)
    - Including the estimate of carrier phase and frequency
  - Symbol synchronization (clock recovery)
    - So as to know the timing for sampling and product-integrator

Synchronization

- Example: Decision-directed synchronization for M-ary PSK

$$s_k(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \alpha_k) \text{ for } 0 \le t < T$$

where E symbol energy, T symbol period, and
$\alpha_k \in \{0, \frac{2\pi}{M}, \ldots, (M-1)\frac{2\pi}{M}\}$.

  - Due to channel effect, we receive

$$x_k(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \alpha_k + \theta) + w(t) \text{ for } \tau \le t \le T + \tau$$

Note that for $-T+\tau \le t < \tau$, the received signal will be $x_{k-1}(t)$, a function of α_{k-1}. Hence, knowing τ is essential.

Synchronization

- Given that τ is accurately estimated, we shall estimate θ through likelihood ratio function.

$$\begin{cases} \phi_1(t) = \sqrt{\dfrac{2}{T}}\cos(2\pi f_c t), & \tau \le t < T+\tau \\ \phi_2(t) = \sqrt{\dfrac{2}{T}}\sin(2\pi f_c t), & \tau \le t < T+\tau \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \text{ where } \begin{cases} x_i = \displaystyle\int_\tau^{T+\tau} x_k(t)\phi_i(t)dt \\ s_i = \displaystyle\int_\tau^{T+\tau} \sqrt{\dfrac{2E}{T}}\cos(2\pi f_c t + \alpha_k + \theta)\phi_i(t)dt \\ w_i = \displaystyle\int_\tau^{T+\tau} w(t)\phi_i(t)dt \end{cases}$$

Synchronization

- For an observation window of size L_0, i.e., $k = 0, 1, \ldots, L_0-1$,

$$\begin{aligned}
\hat{\theta} &= \arg\max_{\theta} f(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{L_0-1}|\theta) \\
&= \arg\max_{\theta} \frac{1}{(\pi N_0)^{L_0}} \exp\left(-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k - \boldsymbol{s}_k(\theta)\|^2\right) \\
&= \arg\min_{\theta} \sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k - \boldsymbol{s}_k(\theta)\|^2 \\
&= \arg\max_{\theta} \sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T \boldsymbol{s}_k(\theta)
\end{aligned}$$

$$\boxed{\boldsymbol{s}_k(\theta) = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix} = \begin{bmatrix} \sqrt{E}\cos(\alpha_k + \theta) \\ -\sqrt{E}\sin(\alpha_k + \theta) \end{bmatrix}}$$

$$\hat{\theta} = \arg\max_{\theta} \sum_{k=0}^{L_0-1} [x_{1,k}\cos(\alpha_k+\theta) - x_{2,k}\sin(\alpha_k+\theta)] = \arg\max_{\theta} \ell(\theta)$$

which implies

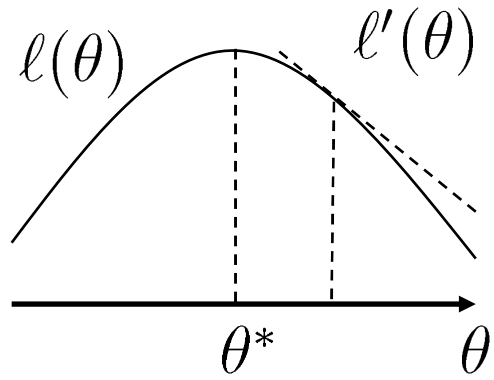
$$\begin{aligned} 0 &= \partial\ell(\theta)/\partial\theta = \partial\left(\sum_{k=0}^{L_0-1}[x_{1,k}\cos(\alpha_k+\theta) - x_{2,k}\sin(\alpha_k+\theta)]\right)\Bigg/\partial\theta \\ &= \sum_{k=0}^{L_0-1}[-x_{1,k}\sin(\alpha_k+\theta) - x_{2,k}\cos(\alpha_k+\theta)] \\ &= \sum_{k=0}^{L_0-1}\operatorname{Im}\left\{(x_{1,k}-jx_{2,k})e^{-j(\alpha_k+\theta)}\right\} \end{aligned}$$

$$\left(= \sum_{k=0}^{L_0-1}\operatorname{Im}\left\{a_k^*\tilde{x}_k e^{-j\theta}\right\}\right) \quad \boxed{\text{As the text puts } \begin{cases} \tilde{x}_k = x_{1,k} - jx_{2,k} \\ a_k = e^{j\alpha_k} \end{cases}}.$$

Synchronization

- Adaptive or recursive algorithm for ML estimation

$$\begin{cases} \text{decrease the current } \hat{\theta}, & \text{if } \ell'(\theta) < 0 \\ \text{increase the current } \hat{\theta}, & \text{if } \ell'(\theta) > 0 \end{cases}$$

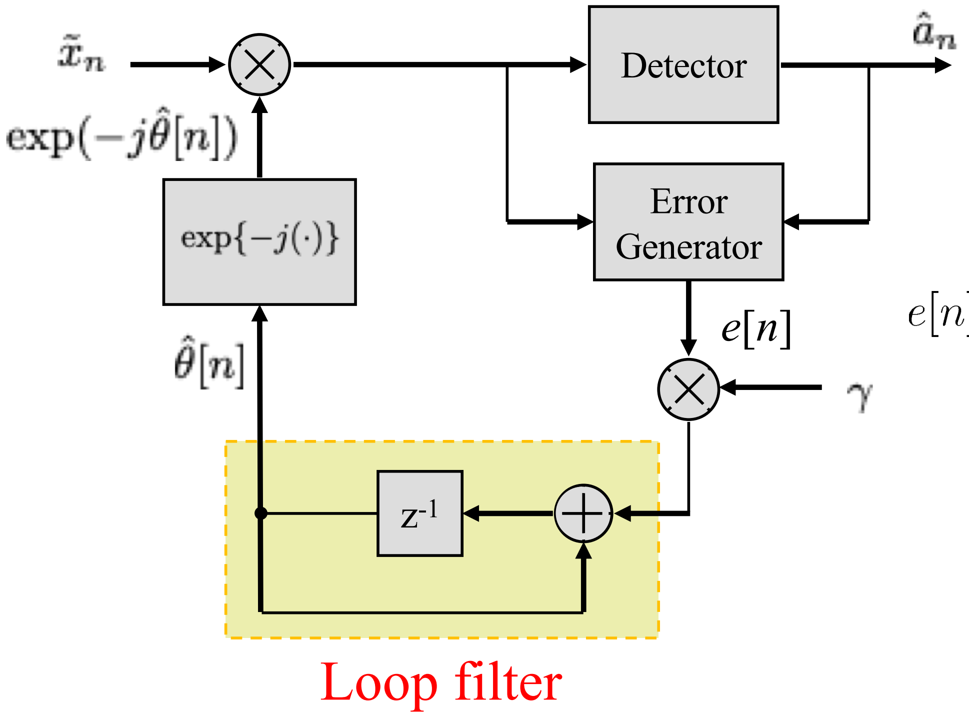


$$\begin{aligned} \hat{\theta}[n+1] &= \hat{\theta}[n] + \gamma \cdot \ell'(\hat{\theta}[n]) \\ &= \hat{\theta}[n] + \gamma \cdot \sum_{k=0}^{L_0-1} \operatorname{Im}\left\{ a_k^* \tilde{x}_k e^{-j\hat{\theta}[n]} \right\} \end{aligned}$$

$\boxed{\gamma > 0 \text{ step size}}$

Synchronization

- First-order digital (loop) filter



$$e[n] = \mathrm{Im}\left\{a_n^* \tilde{x}_n e^{-j\hat{\theta}[n]}\right\}$$

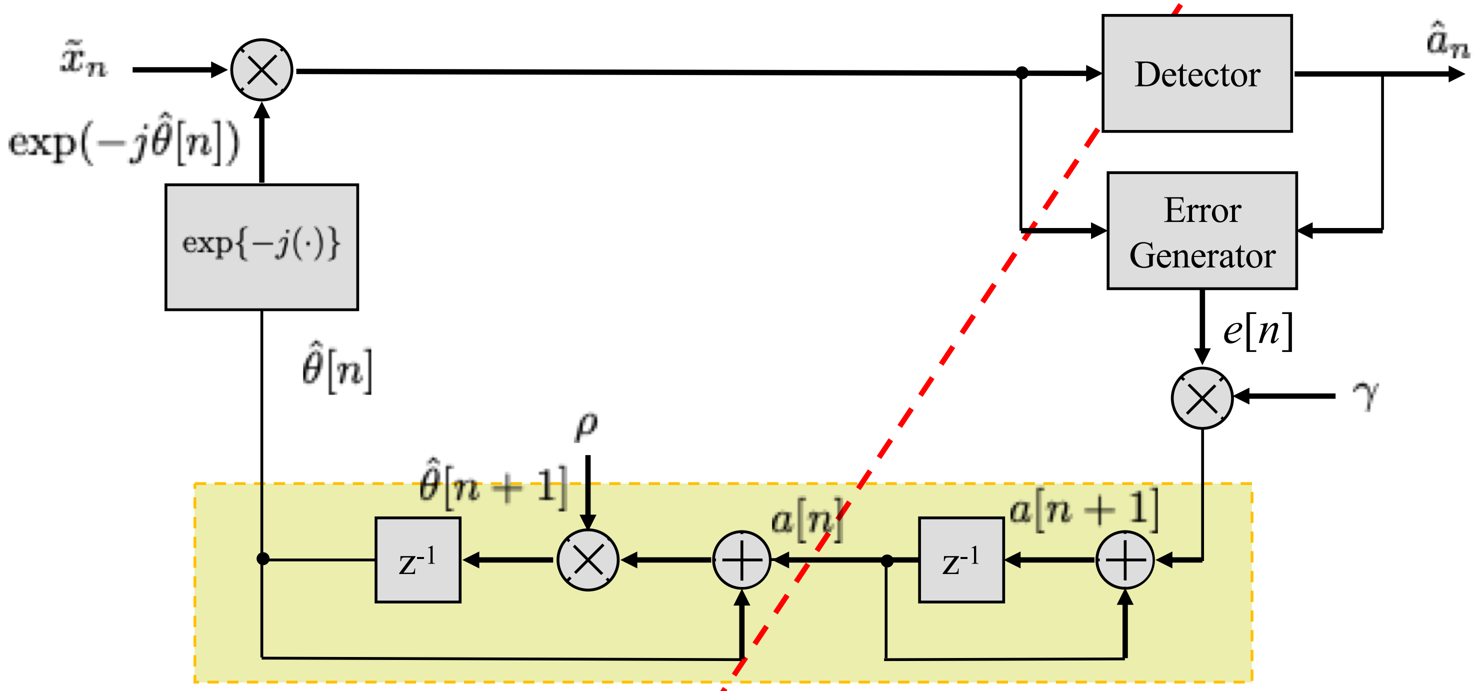
with $L_0 = 1$

$$\begin{cases}\hat\theta[n+1]=\rho\hat\theta[n]+\rho a[n]\\ a[n+1]=a[n]+\gamma e[n]\end{cases}\Rightarrow\begin{cases}\hat\theta[n+2]=\rho\hat\theta[n+1]+\rho a[n+1] & (1)\\ \hat\theta[n+1]=\rho\hat\theta[n]+\rho a[n] & (2)\end{cases}\Rightarrow (1)-(2)$$

(Note that $a[n+1]-a[n]=\gamma e[n]$.)

Synchronization

- An example of second-order digital (loop) filter



$$\hat\theta[n+2]=(1+\rho)\hat\theta[n+1]-\rho\hat\theta[n]+\rho\gamma e[n]$$

Synchronization

- The previous discussion estimates θ subject to either known or accurately estimated τ.
- We are thus required to estimate τ without the knowledge of θ.

$$\Rightarrow \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} = \begin{bmatrix} s_1(\alpha_k, \theta, \tau|\tau_0) \\ s_2(\alpha_k, \theta, \tau|\tau_0) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \text{ where}$$

$$\begin{cases} x_i(\tau) = \int_\tau^{T+\tau} x_k(t)\phi_i(t)dt \\ s_i(\alpha_k, \theta, \tau|\tau_0) = \int_\tau^{T+\tau} \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \alpha_k + \theta)\mathbf{1}\{\tau_0 \le t < \tau_0 + T\}\phi_i(t)dt \\ w_i = \int_\tau^{T+\tau} w(t)\phi_i(t)dt \end{cases}$$

$\boxed{\tau_0 \text{ is the true timing delay}}$

$\longrightarrow$ Its statistics is independent of τ.

Synchronization

- For an observation window of size L_0, i.e., $k = 0, 1, \ldots, L_0-1$,

$$
\begin{aligned}
\hat{\tau} &= \arg\max_{\tau} f(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{L_0-1} | \{\alpha_k\}, \theta, \tau, \tau_0) \\
&= \arg\max_{\tau} \frac{1}{(\pi N_0)^{L_0}} \exp\left(-\frac{1}{N_0} \sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k(\tau) - \boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0)\|^2 \right) \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \exp\left(\frac{2}{N_0} \boldsymbol{x}_k(\tau)^T \boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0) \right)
\end{aligned}
$$

$$
\boxed{\boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0) = \begin{bmatrix} \sqrt{\tilde{E}} \cos(\alpha_k + \theta) \\ -\sqrt{\tilde{E}} \sin(\alpha_k + \theta) \end{bmatrix}}
$$

(See the next slide.)

As an example, assume $\tau_0 > \tau$ (this condition is required for the first three steps),

$$
\begin{aligned}
s_1(\alpha_k, \theta, \tau | \tau_0) &= \frac{2\sqrt{E}}{T} \int_{\tau_0}^{T+\tau} \cos(2\pi f_c t + \alpha_k + \theta) \cos(2\pi f_c t) dt \\
&= \frac{\sqrt{E}}{T} \int_{\tau_0}^{T+\tau} \left[\cos(4\pi f_c t + \alpha_k + \theta) + \cos(\alpha_k + \theta)\right] dt \\
&= \underbrace{\cancel{\frac{\sqrt{E}}{T} \int_{\tau_0}^{T+\tau} \cos(4\pi f_c t + \alpha_k + \theta) dt}}_{\text{approximately zero}} + \frac{\sqrt{E}}{T} \int_{\tau_0}^{T+\tau} \cos(\alpha_k + \theta) dt \\
&\approx \sqrt{E} \left(\frac{T - |\tau_0 - \tau|}{T} \right) \cos(\alpha_k + \theta) \\
&= \sqrt{\tilde{E}} \cos(\alpha_k + \theta).
\end{aligned}
$$

This is valid for both $\tau_0 > \tau$ and $\tau_0 \leq \tau$.

Synchronization

- One way to solve the problem of unknown θ is to average out all possible θ in the maximization operation.

$$
\begin{aligned}
\hat{\tau} &= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\frac{2}{N_0}\boldsymbol{x}_k(\tau)^T \boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\right) d\theta \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\frac{2\sqrt{\tilde{E}}}{N_0}\left(|a_k\tilde{x}_k(\tau)|\cos(\arg[\tilde{x}_k(\tau)]-\arg[a_k]-\theta)\right)\right\} d\theta \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|\cos(\theta)\right\} d\theta \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} I_0\left(\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|\right)
\end{aligned}
$$

(See the next slide.)

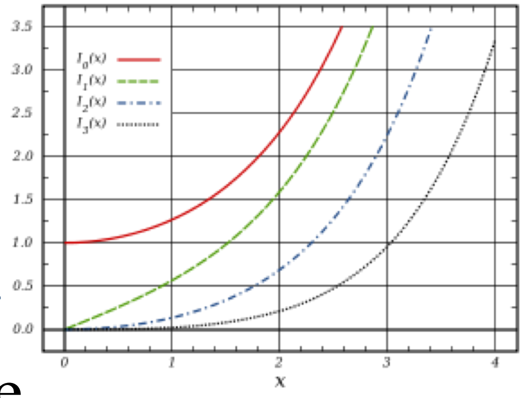
Since $|a_k| = 1$, and the integrand is periodic with period 2π.

Since $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\varphi)}d\varphi$.

$$\boxed{\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)=\begin{bmatrix}\sqrt{\tilde{E}}\cos(\alpha_k+\theta)\\ -\sqrt{\tilde{E}}\sin(\alpha_k+\theta)\end{bmatrix}}$$

$$\begin{aligned}
&\frac{1}{\sqrt{\tilde{E}}}\boldsymbol{x}_k(\tau)^T\boldsymbol{s}_k(\alpha_k,\theta,\tau|\tau_0)\\
&= x_{1,k}(\tau)\cos(\alpha_k+\theta)-x_{2,k}(\tau)\sin(\alpha_k+\theta)\\
&= \mathrm{Re}\left\{[x_{1,k}(\tau)-jx_{2,k}(\tau)]e^{-j(\alpha_k+\theta)}\right\}\\
&= \mathrm{Re}\left\{a_k^*\tilde{x}_k(\tau)e^{-j\theta}\right\}\\
&= \mathrm{Re}\left\{|a_k\tilde{x}_k(\tau)|\exp\left(j\left[\arg(\tilde{x}_k(\tau))-\arg(a_k)-\theta\right]\right)\right\}\\
&= |a_k\tilde{x}_k(\tau)|\cos\left(\arg(\tilde{x}_k(\tau))-\arg(a_k)-\theta\right).
\end{aligned}$$

$$\boxed{\text{As the text puts }\left\{\begin{array}{lcl}\tilde{x}_k&=&x_{1,k}-jx_{2,k}\\ a_k&=&e^{j\alpha_k}\end{array}\right.}$$



Synchronization

- Square approximation of logarithm of the modified Bessel function of zero order

$$\begin{aligned}\log I_0(x) &= \log\left(\sum_{m=0}^{\infty}\frac{(x/2)^{2m}}{(m!)^2}\right)\\ &\approx \log\left(1+\frac{x^2}{4}\right) \quad \text{for } x \text{ small}\\ &\approx \frac{x^2}{4} \quad \text{for } x \text{ small}\end{aligned}$$

(Take the first two terms, i.e., *m*=0 and *m*=1)

$$\hat{\tau} = \arg\max_{\tau}\sum_{k=0}^{L_0-1}\log\left\{I_0\left(\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|\right)\right\} \approx \arg\max_{\tau}\sum_{k=0}^{L_0-1}|\tilde{x}_k(\tau)|^2$$

$$\text{provided that } \frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)| \text{ small}$$

Synchronization

- Realization of square approximation

$$\hat{\tau} = \arg\max_{\tau} \sum_{k=0}^{L_0-1} |\tilde{x}_k(\tau)|^2$$

$$\begin{aligned}
\Rightarrow \sum_{k=0}^{L_0-1} \frac{\partial |\tilde{x}_k(\tau)|^2}{\partial \tau} &= \sum_{k=0}^{L_0-1} \frac{\partial \left(\mathrm{Re}^2\{\tilde{x}_k(\tau)\} + \mathrm{Im}^2\{\tilde{x}_k(\tau)\} \right)}{\partial \tau} \\
&= \sum_{k=0}^{L_0-1} 2 \cdot \mathrm{Re}\{\tilde{x}_k(\tau)\} \cdot \mathrm{Re}'\{\tilde{x}_k(\tau)\} + 2 \cdot \mathrm{Im}\{\tilde{x}_k(\tau)\} \cdot \mathrm{Im}'\{\tilde{x}_k(\tau)\} \\
&= 2 \sum_{k=0}^{L_0-1} \mathrm{Re}\left\{\tilde{x}_k^*(\tau)\tilde{x}_k'(\tau)\right\} = 0
\end{aligned}$$

Synchronization

$$x_i(\tau) = \int_{\tau}^{T+\tau} x_k(t)\phi_i(t)dt$$

- Early-late approximation of the derivative

$$\begin{aligned}
\tilde{x}'_n(\tau) &= \tilde{x}'_n(nT + \hat{\tau}_n) \\
&\approx \frac{\overbrace{\tilde{x}\left(nT + \frac{T}{2} + \hat{\tau}[n+1/2]\right)}^{\text{late}} - \overbrace{\tilde{x}\left(nT - \frac{T}{2} + \hat{\tau}[n-1/2]\right)}^{\text{early}}}{T} \\
&\approx \frac{\tilde{x}\left(nT + \frac{T}{2} + \hat{\tau}[n]\right) - \tilde{x}\left(nT - \frac{T}{2} + \hat{\tau}[n-1]\right)}{T}
\end{aligned}$$

since no estimations for $\hat{\tau}[n+1/2]$ and $\hat{\tau}[n-1/2]$ are performed

Synchronization

$$\hat{\tau}[n+1] = \hat{\tau}[n] + \gamma \cdot e[n]$$

where $e[n] = \mathrm{Re}\left\{\tilde{x}^*(nT + \hat{\tau}[n])\left[\tilde{x}\left(nT + \frac{T}{2} + \hat{\tau}[n]\right) - \tilde{x}\left(nT - \frac{T}{2} + \hat{\tau}[n-1]\right)\right]\right\}.$

Final note: For every $\hat{\tau}[n]$, the realization requires the sample values of $\tilde{x}(nT + \hat{\tau}[n])$ and $\tilde{x}(nT + T/2 + \hat{\tau}[n])$.

Experiment: Carrier Recovery and Symbol Timing

- Experiment: Carrier phase θ recovery subject to known timing information τ
  - QPSK and error-free
  - $L_0 = 1$ and symbol energy $E = 1$ for simplicity

$$s_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k) \text{ for } 0 \le t < T$$

where T symbol period, and $\alpha_k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

$$\Rightarrow x_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k + \theta) \text{ for } \tau \le t \le T + \tau$$

$$\begin{cases} \phi_1(t) = \sqrt{\dfrac{2}{T}}\cos(2\pi f_c t), & \tau \le t < T+\tau \\ \phi_2(t) = \sqrt{\dfrac{2}{T}}\sin(2\pi f_c t), & \tau \le t < T+\tau \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix}, \text{ where } \begin{cases} x_{i,k} = \displaystyle\int_\tau^{T+\tau} x_k(t)\phi_i(t)dt \\ s_{i,k} = \displaystyle\int_\tau^{T+\tau} \sqrt{\dfrac{2}{T}}\cos(2\pi f_c t + \alpha_k + \theta)\phi_i(t)dt \end{cases}$$

$$\boxed{\boldsymbol{s}_k(\theta) = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix} = \begin{bmatrix} \cos(\alpha_k + \theta) \\ -\sin(\alpha_k + \theta) \end{bmatrix}}$$

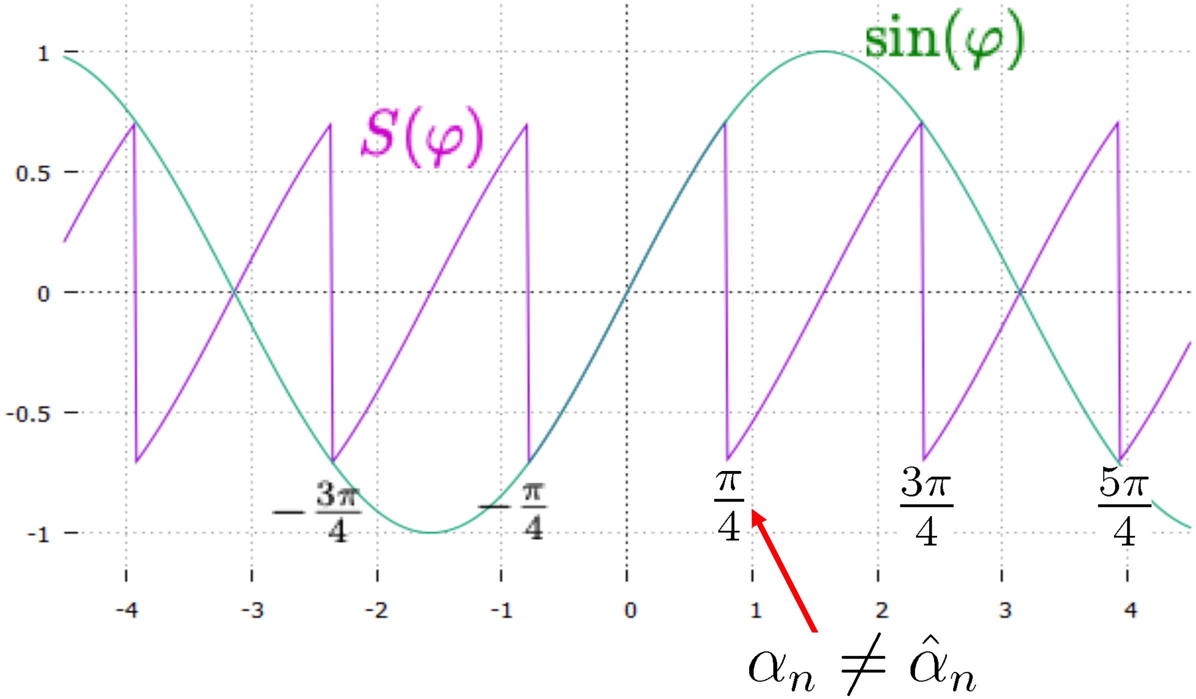
$$\Rightarrow \hat{\theta} = \arg\max_{\tilde{\theta}} \boldsymbol{x}_k^T \boldsymbol{s}_k(\tilde{\theta}) = \arg\max_{\tilde{\theta}} \boldsymbol{s}_k^T(\theta)\boldsymbol{s}_k(\tilde{\theta}) = \arg\max_{\tilde{\theta}} \ell(\tilde{\theta})$$

$$\Rightarrow 0 = \frac{\partial \ell(\tilde{\theta})}{\partial \tilde{\theta}} = \mathrm{Im}\left\{\hat{a}_k^* \tilde{x}_k e^{-j\tilde{\theta}}\right\} = \mathrm{Im}\left\{e^{-j\hat{\alpha}_k} e^{j(\alpha_k+\theta)} e^{-j\tilde{\theta}}\right\} = \mathrm{Im}\left\{e^{j(\alpha_k-\hat{\alpha}_k)} e^{j(\theta-\tilde{\theta})}\right\}$$

$\hat{\alpha}_k$ is the feedback directed decision.
α_k is the actual transmitted signal.

$$\boxed{\begin{cases} \tilde{x}_k = x_{1,k} - j x_{2,k} \\ \hat{a}_k = e^{j\hat{\alpha}_k} \end{cases}}$$

$$\varphi = \theta - \hat{\theta}[n]$$
$$S(\varphi) = E\left[e[n] | \varphi\right] = \sin(\varphi) \quad \text{(S-curve)}$$
$$e[n] = \mathrm{Im}\left\{e^{j(\alpha_n - \hat{\alpha}_n)} e^{j(\theta - \hat{\theta}[n])}\right\} = \sin(\varphi) \text{ if ideally } \alpha_n = \hat{\alpha}_n$$



$$\boxed{\ell(\tilde{\theta}) = \cos(\theta - \tilde{\theta}) \quad \text{and} \quad \frac{\partial^2 \ell(\tilde{\theta})}{\partial^2 \tilde{\theta}} = -\cos(\varphi)}$$