Part 2 Coherent/Non-Coherent PSK/FSK

Coherent Frequency-Shift Keying

- (M-ary) ASK, (M-ary) PSK and (M-ary) FSK are three major categories of digital modulations, in which QAM can be viewed/analyzed similarly to (M-ary) PSK.
- In the sequel, (M-ary) FSK will be introduced and discussed.

Coherent Frequency-Shift Keying

- Binary FSK

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos\left(2\pi f_i t\right), & 0 \le t < T_b \\ 0, & \text{elsewhere} \end{cases}$$

where $i = 1, 2,$ f_i is a multiple of $1/T_b$,

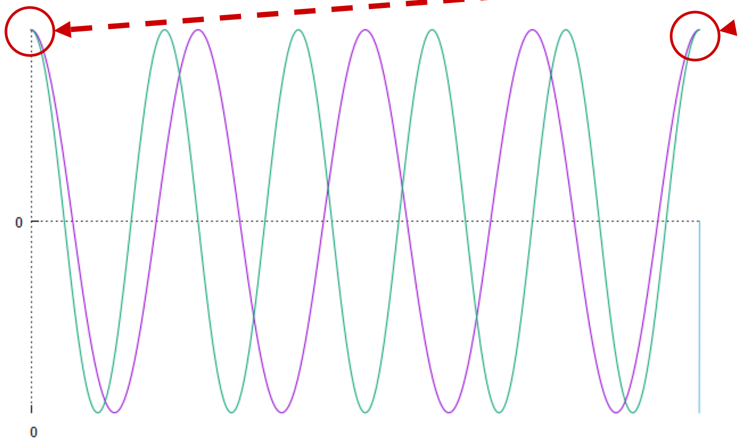
E_b is the transmitted energy per bit, and

T_b is the bit duration.

- Vector space analysis of binary FSK

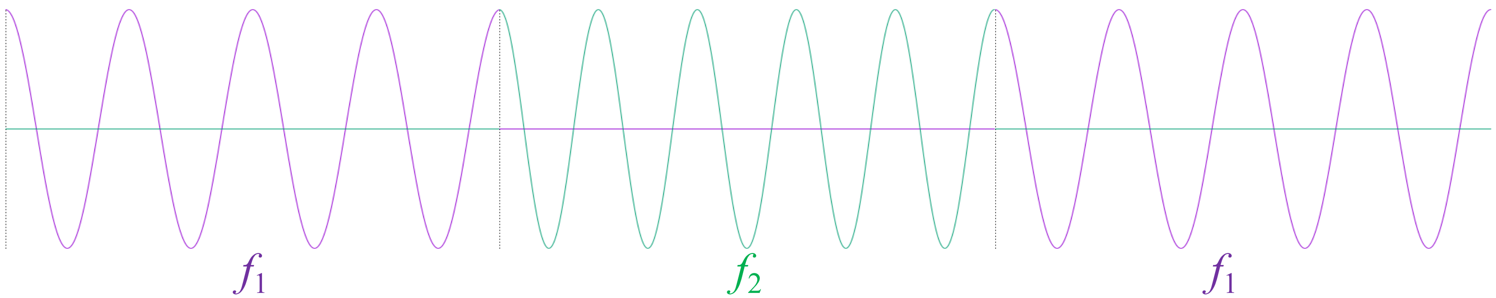
$$\boldsymbol{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\dfrac{2}{T_b}} \cos(2\pi f_1 t) \\ \phi_2(t) = \sqrt{\dfrac{2}{T_b}} \cos(2\pi f_2 t) \end{cases}$$

Since f_i is a multiple of $1/T_b$,
the wave always starts from and ends at the same point.



E. D. Sunde, "Ideal binary pulse transmission by AM and FM," *Bell Labs Technical Journal*, 38, pp. 1357-1426, November 1959.

With this multiple-($1/T_b$) restriction, it becomes "continuous-phase" in every inter-bit transition. Such kind of forced "continuous-phase" signals, known as Sunde's FSK, surely belongs to the general continuous-phase FSK (CPFSK) family.



Coherent Frequency-Shift Keying – Error Probability of BFSK

- Error probability of Binary FSK

$$x(t) = s(t) + w(t)$$

$$\Rightarrow \quad \langle x(t), \phi_i(t) \rangle = \langle s(t), \phi_i(t) \rangle + \langle w(t), \phi_i(t) \rangle$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \text{either } \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\Rightarrow \quad \hat{m} = \arg\max \left\{ P\left(\boldsymbol{x} \middle| \left[\sqrt{E_b}\ 0 \right]^T \right), P\left(\boldsymbol{x} \middle| \left[0\ \sqrt{E_b} \right]^T \right) \right\}$$

$$\Rightarrow \quad x_1 - x_2 \underset{\hat{m}=m_1}{\overset{\hat{m}=m_2}{\lessgtr}} 0$$

$$\boxed{(x_1 - x_2) = \pm\sqrt{E_b} + (w_1 - w_2)}$$

Coherent Frequency-Shift Keying – Error Probability of BFSK

- Error probability of binary FSK
  - Based on the decision rule $y = x_1 - x_2 \underset{s_1 - s_2 = \sqrt{E_b} \text{ is transmitted}}{\overset{s_1 - s_2 = -\sqrt{E_b} \text{ is trasmitted}}{\lessgtr}} 0$

$$\begin{aligned}
P(\text{Error}) &= P\left(-\sqrt{E_b} \text{ transmitted}\right) P\left(y > 0 \,\middle|\, -\sqrt{E_b} \text{ transmitted}\right) \\
&\quad + P\left(+\sqrt{E_b} \text{ transmitted}\right) P\left(y < 0 \,\middle|\, +\sqrt{E_b} \text{ transmitted}\right) \\
&= \frac{1}{2}\int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+\sqrt{E_b})^2/2\sigma^2} dx + \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2} dx \\
&= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2} dx = \Phi\left(\frac{0-\sqrt{E_b}}{\sigma}\right) = \Phi\left(-\sqrt{\frac{E_b}{N_0}}\right)
\end{aligned}$$

$$\boxed{\sigma^2 = N_0 \text{ is the variance of } (w_1 - w_2)}$$

Coherent Frequency-Shift Keying – Error Probability of BFSK

- Comparison between BPSK and BFSK

$$P(\text{BPSK Error}) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)$$

3 dB difference

$$P(\text{BFSK Error}) = \Phi\left(-\sqrt{\frac{E_b}{N_0}}\right)$$

Coherent Frequency-Shift Keying – PSD of BFSK

- Power spectra of binary FSK
  - Assumption: f_1 and f_2 differ by $1/T_b$.
  - Under such assumption,

$$\begin{cases} f_1 = f_c + \dfrac{1}{2T_b} \\ f_2 = f_c - \dfrac{1}{2T_b} \end{cases}$$

is a multiple of $1/T_b$.

(See Slide IDC2-3.)

$$s(t) = \sum_{k=-\infty}^{\infty} g(t - kT_b)\cos\left(2\pi f_c t + I_k \frac{\pi t}{T_b}\right)$$

where $I_k = \pm 1$ with equal probability, and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

and $g(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$

Coherent Frequency-Shift Keying – PSD of BFSK

- General time-averaged power spectra
  - The text derives the (time-averaged) power spectra of the baseband signal as the sum of the in-phase power spectra and the quadrature power spectra.
  - This may not be “correct” in general (See the next Slide).

Coherent Frequency-Shift Keying – PSD of BFSK

$\tilde{s}(t) = g_I(t) + jg_Q(t)$ with real-valued $g_I(t)$ and $g_Q(t)$

$$
\begin{aligned}
\boxed{R_{\tilde{s}\tilde{s}}(t+\tau, t)} &= E[(g_I(t+\tau) + jg_Q(t+\tau))(g_I(t) + jg_Q(t))^*] \\
&= E[g_I(t+\tau)g_I(t)] + E[g_Q(t+\tau)g_Q(t)] \\
&\quad + jE[g_I(t+\tau)g_Q(t)] + jE[g_Q(t+\tau)g_I(t)] \\
&= \boxed{\begin{array}{l} E[g_I(t+\tau)g_I(t)] + E[g_Q(t+\tau)g_Q(t)] \\ \textbf{if, and only if,} \\ E[g_I(t+\tau)g_Q(t)] + E[g_Q(t+\tau)g_I(t)] = 0. \end{array}}
\end{aligned}
$$

Coherent Frequency-Shift Keying – PSD of BFSK

The cross-correlation also affects the resultant power spectra.

$\tilde{s}(t) = \tilde{s}_1(t) + \tilde{s}_2(t)$ with complexed-valued $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$

$$
\begin{aligned}
\boxed{R_{\tilde{s}\tilde{s}}(t+\tau, t)} &= E[(\tilde{s}_1(t+\tau) + \tilde{s}_2(t+\tau))(\tilde{s}_1(t) + \tilde{s}_2(t))^*] \\
&= E[\tilde{s}_1(t+\tau)\tilde{s}_1^*(t)] + E[\tilde{s}_2(t+\tau)\tilde{s}_2^*(t)] \\
&\quad + E[\tilde{s}_1(t+\tau)\tilde{s}_2^*(t)] + E[\tilde{s}_2(t+\tau)\tilde{s}_1^*(t)] \\
&= \boxed{\begin{array}{l} E[\tilde{s}_1(t+\tau)\tilde{s}_1^*(t)] + E[\tilde{s}_2(t+\tau)\tilde{s}_2^*(t)] \\ \textbf{if, and only if,} \\ E[\tilde{s}_1(t+\tau)\tilde{s}_2^*(t)] + E[\tilde{s}_2(t+\tau)\tilde{s}_1^*(t)] = 0 \end{array}}
\end{aligned}
$$

Coherent Frequency-Shift Keying – PSD of BFSK

- Power spectra of binary FSK
  - Since in-phase and quadrature components are independent, and since one of them is zero-mean (See the next few slides), the technique used in text is applicable to binary FSK.

Equivalent baseband signal

$$
\begin{aligned}
s(t) &= \sum_{k=-\infty}^{\infty} g(t-kT_b)\cdot \mathrm{Re}\left\{e^{j(2\pi f_c t + I_k \pi t/T_b)}\right\} \\
&= \mathrm{Re}\left\{\left(\sum_{k=-\infty}^{\infty} g(t-kT_b)e^{jI_k\pi t/T_b}\right)e^{j2\pi f_c t}\right\} \\
\left(\right. &= \left. \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \right) \\
\Rightarrow \tilde{s}(t) &= \sum_{k=-\infty}^{\infty} g(t-kT_b)e^{jI_k\pi t/T_b} \\
&= \sum_{k=-\infty}^{\infty} g(t-kT_b)\cos\left(I_k\pi t/T_b\right) + j\sum_{k=-\infty}^{\infty} g(t-kT_b)\sin\left(I_k\pi t/T_b\right) \\
&= \underbrace{\sum_{k=-\infty}^{\infty} g(t-kT_b)\cos\left(\pi t/T_b\right)}_{g_I(t)} + j\underbrace{\sum_{k=-\infty}^{\infty} I_k g(t-kT_b)\sin\left(\pi t/T_b\right)}_{g_Q(t)}
\end{aligned}
$$

$$g_I(t) = \sum_{k=-\infty}^{\infty} g(t - kT_b)\cos(\pi t/T_b) = \sqrt{\frac{2E_b}{T_b}}\cos(\pi t/T_b)$$

$$\boxed{\bar{R}_{g_I g_I}(\tau)} = \frac{1}{T_b}\int_0^{T_b} \frac{2E_b}{T_b}\cos(\pi(t+\tau)/T_b)\cos(\pi t/T_b)\,dt = \boxed{\frac{E_b}{T_b}\cos(\pi\tau/T_b)}$$

$$\boxed{\bar{S}_{B,g_I}(f) = \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right)\right]}$$

$$\begin{aligned} g_Q(t) &= \sum_{k=-\infty}^{\infty} I_k g(t - kT_b)\sin(\pi t/T_b) \\ &= \sum_{k=-\infty}^{\infty} \underbrace{(-1)^k I_k}_{\tilde{I}_k} \cdot \underbrace{g(t - kT_b)\sin(\pi(t - kT_b)/T_b)}_{\tilde{g}(t-kT_b)} \\ &= \sum_{k=-\infty}^{\infty} \tilde{I}_k \cdot \tilde{g}(t - kT_b) \end{aligned}$$

(See Slides IDC1-33 and IDC1-35.)

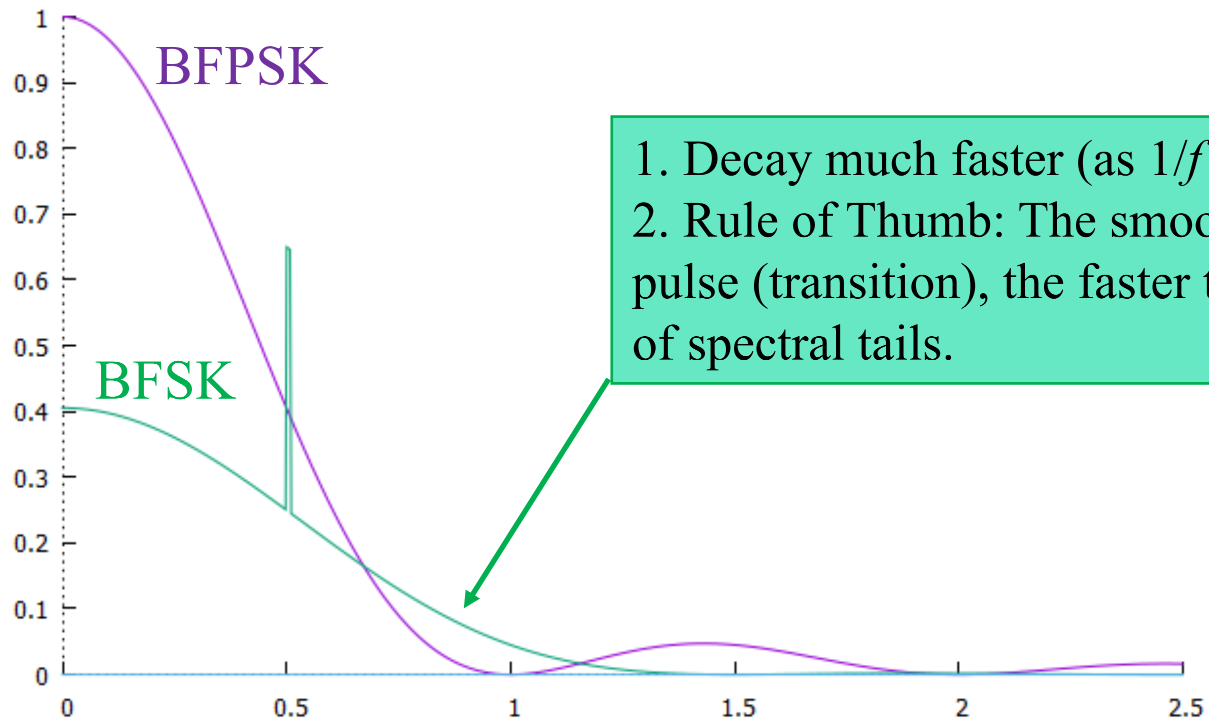
$$
\begin{aligned}
\tilde{G}(f) &= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \sin(\pi t/T_b) e^{-j2\pi f t} dt \\
&= \frac{2\sqrt{2E_b T_b}}{\pi} \left(\frac{\cos(\pi T_b f)}{1 - 4T_b^2 f^2} \right) e^{-j\pi T_b f}
\end{aligned}
$$

$$
\boxed{\bar{S}_{B,g_Q}(f) = \frac{1}{T_b} |\tilde{G}(f)|^2 = \frac{8E_b \cos^2(\pi T_b f)}{\pi^2 (1 - 4T_b^2 f^2)^2}}
$$

$$
\boxed{
\begin{aligned}
\bar{S}_B(f) &= \bar{S}_{B,g_I}(f) + \bar{S}_{B,g_Q}(f) \\
&= \frac{E_b}{2T_b} \left[\delta\left(f - \frac{1}{2T_b} \right) + \delta\left(f + \frac{1}{2T_b} \right) \right] + \frac{8E_b \cos^2(\pi T_b f)}{\pi^2 (1 - 4_b^2 f^2)^2}
\end{aligned}
}
$$

$$\bar{S}_{B,BPSK}(f) = 2E_b\,\mathrm{sinc}^2(T_b f)$$ (From Slide IDC1-36.)

$$\bar{S}_{B,B(CP)FSK}(f) = \frac{E_b}{2T_b}\left[\delta\left(f-\frac{1}{2T_b}\right)+\delta\left(f+\frac{1}{2T_b}\right)\right]+\frac{8E_b\cos^2(\pi T_b f)}{\pi^2(4T_b^2 f^2-1)^2}$$



This should be $\frac{4}{\pi^2} \approx 0.405$

1. Decay much faster (as $1/f^4$)
2. Rule of Thumb: The smoother the pulse (transition), the faster the drop of spectral tails.

$$\frac{1}{4T_b}\delta\left(f-\frac{1}{2T_b}\right)+\frac{1}{4}$$

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Coherent Frequency-Shift Keying – PSD of BFSK

- Final note
  - If BFSK is not continuous phase (due to f_1 and f_2 are not multiple of $1/T_b$), then the "rule of thumb" indicates that the spectral decay will be slower.
  - In fact, the rate of spectral decay for non-continuous-phase FSK will become $1/f^2$.

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

- Memoryless versus continuous-phase

$$
\begin{aligned}
s(t) &= \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_i t) \text{ for } \boxed{0 \le t < T_b} \\
&= \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_c t + I_0 \frac{\pi h}{T_b} t\right)
\end{aligned}
$$

$$
\boxed{f_i = f_c + I_0 \frac{h}{2T_b}}
$$

$$
\text{where } \begin{cases} I_0 = \pm 1 \\ f_c \triangleq \frac{1}{2}(f_1 + f_2) \quad \text{multiple of } 1/T_b \\ h \triangleq \frac{f_1 - f_2}{1/T_b} \quad \text{deviation ratio} \end{cases}.
$$

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

From the previous slide,

$$s(T_b) = \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_c T_b + I_0\pi h\right) = \sqrt{\frac{2E_b}{T_b}}\cos\left(I_0\pi h\right)$$

In order to maintain phase-continuity, $s(t)$ can be of the form:

$$\begin{aligned} s(t) &= \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_i(t-T_b) + I_0\pi h\right) \text{ for } \boxed{T_b \le t < 2T_b} \\ &= \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_c t + I_0\pi h + I_1\pi h\left(\frac{t-T_b}{T_b}\right)\right) \end{aligned}$$

$$\boxed{f_i = f_c + I_1\frac{h}{2T_b}}$$

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

From the previous slide,

$$s(2T_b) = \sqrt{\frac{2E_b}{T_b}} \cos\left(I_0 \pi h + I_1 \pi h\right)$$

In order to maintain phase-continuity, $s(t)$ can be of the form:

$$\begin{aligned} s(t) &= \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_i (t - 2T_b) + I_0 \pi h + I_1 \pi h\right) \text{ for } \boxed{2T_b \le t < 3T_b} \\ &= \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t + I_0 \pi h + I_1 \pi h + I_2 \pi h \left(\frac{t - 2T_b}{T_b}\right)\right) \end{aligned}$$

$$\boxed{f_i = f_c + I_2 \frac{h}{2T_b}}$$

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

- So, in order to maintain phase-continuity subject to that f_c is a multiple of $1/T_b$, we obtain:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t - nT_b}{T_b}\right)\right)$$

for $nT_b \le t < (n+1)T_b$.

Require "memory" of all histories.

- Memoryless, hence, requires:

$$\left(\sum_{k=-\infty}^{\infty} I_k \pi h\right) \bmod 2\pi = \text{constant for all } \{I_k\}_{k=-\infty}^{n-1} \Rightarrow \boxed{h = \text{even integer}}$$

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

- This kind of continuous-phase and memoryless FSK (with h integer) is called Sunde's FSK.

- Sunde's FSK is a special case of the continuous-phase FSK (CPFSK) family.

- For general CPFSK, the system requires to memorize

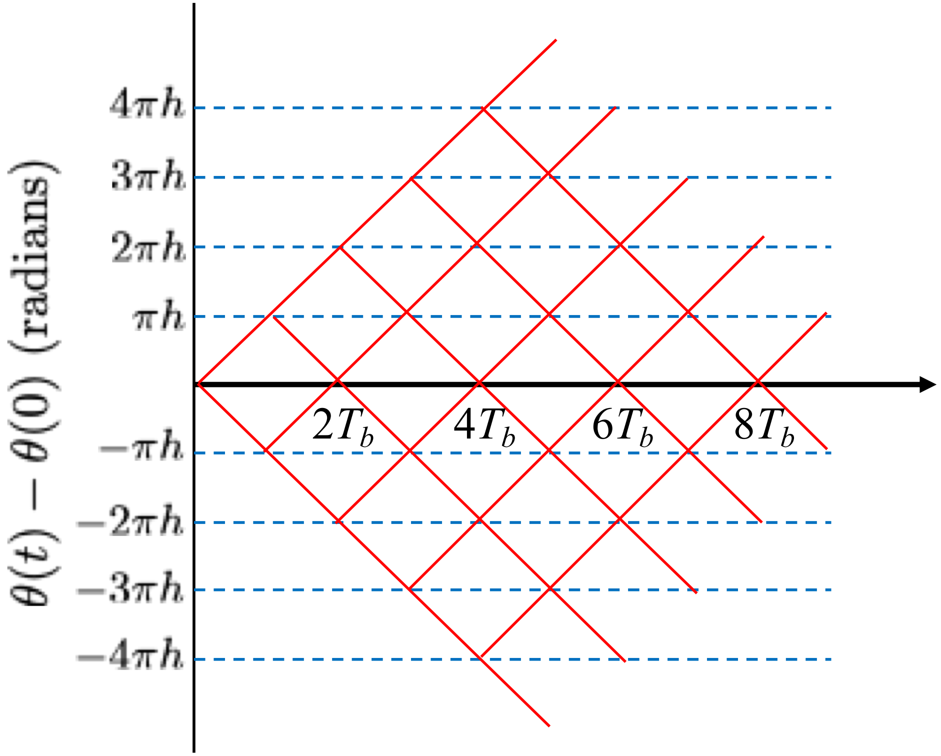
$$\sum_{k=-\infty}^{n-1} I_k \pi h \text{ for every } n$$

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

$$\tilde{s}(t) = \sqrt{\frac{2E_b}{T_b}} e^{j\left(\sum_{k=-\infty}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t-nT_b}{T_b}\right)\right)} \text{ for } nT_b \le t < (n+1)T_b$$

$$\tilde{s}(0) = \sqrt{\frac{2E_b}{T_b}} e^{j\left(\sum_{k=-\infty}^{-1} I_k \pi h\right)}$$

$$\Rightarrow \theta(t) - \theta(0) = \sum_{k=0}^{n-1} I_k \pi h + I_n \pi h \left(\frac{t-nT_b}{T_b}\right) \text{ for } nT_b \le t < (n+1)T_b$$



Phase tree of CPFSK

Coherent Frequency-Shift Keying – Memoryless versus Continuous-Phase

- Minimum shift keying
  - The passband signals respectively for $I_n = -1$ and $I_n = +1$ are better to be coherent orthogonal, i.e., we require

$$\int_{nT_b}^{(n+1)T_b} \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h - \pi h\left(\frac{t-nT_b}{T_b}\right)\right)$$

$$\times \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t + \sum_{k=-\infty}^{n-1} I_k \pi h + \pi h\left(\frac{t-nT_b}{T_b}\right)\right) dt$$

$$= \cancel{\frac{E_b}{T_b}\int_{nT_b}^{(n+1)T_b} \cos\left(4\pi f_c t + 2\sum_{k=-\infty}^{n-1} I_k \pi h\right) dt} + \frac{E_b}{T_b}\int_{nT_b}^{(n+1)T_b} \cos\left(2\pi h\left(\frac{t-nT_b}{T_b}\right)\right) dt$$

$$= 0 + \frac{E_b}{2\pi h}\sin(2\pi h) = 0 \quad \Rightarrow \quad \boxed{(2\pi h) \bmod \pi = 0}$$

Coherent Frequency-Shift Keying – Minimum Shift Keying

- Minimum shift keying
  - $h = ½$ is the minimum h that satisfies "coherent orthogonality" condition; hence, it is named minimum shift keying.

$$h = \frac{f_1 - f_2}{1/T_b} = \frac{1}{2} \Rightarrow (f_1 - f_2) = \frac{1}{2T_b}$$

$$\begin{aligned}\tilde{s}(t) &= \sqrt{\frac{2E_b}{T_b}} e^{j(\pi/2)\left(\sum_{k=-\infty}^{n-1} I_k + I_n\left(\frac{t-nT_b}{T_b}\right)\right)} \text{ for } nT_b \le t < (n+1)T_b \\ &= \sqrt{\frac{2E_b}{T_b}} e^{j\theta(t)}\end{aligned}$$

Coherent Frequency-Shift Keying – Minimum Shift Keying

- Minimum shift keying

$$
\begin{aligned}
e^{j[\theta(t)-\theta(0)]} &= e^{j(\pi/2)\left(\sum_{k=0}^{n-1} I_k + I_n\left(\frac{t-nT_b}{T_b}\right)\right)} \text{ for } nT_b \le t < (n+1)T_b \\
&= \left(\prod_{k=0}^{n-1} e^{j(\pi/2)I_k}\right) \times e^{j(\pi/2)I_n\left(\frac{t-nT_b}{T_b}\right)} \\
&= \left(\prod_{k=0}^{n-1} I_k e^{j(\pi/2)}\right) \times e^{j(\pi/2)I_n\left(\frac{t-nT_b}{T_b}\right)} \\
&= \left(\prod_{k=0}^{n-1} I_k\right) \times e^{j(\pi/2)\left[I_n\left(\frac{t-nT_b}{T_b}\right)+n\right]}
\end{aligned}
$$

Because $I_k = \pm 1$

Coherent Frequency-Shift Keying – Minimum Shift Keying

That $t = T_b$ (hence, $n = 1$) gives that $e^{j[\theta(T_b)-\theta(0)]} = I_0 e^{j(\pi/2)}$.

$\theta(T_b) = \pi/2$	$\theta(0) = 0$	$e^{j(\pi/2-0)} = I_0 e^{j(\pi/2)} = e^{j(\pi/2)}$	$I_0 = 1$
$\theta(T_b) = \pi/2$	$\theta(0) = \pi$	$e^{j(\pi/2-\pi)} = I_0 e^{j(\pi/2)} = -e^{j(\pi/2)}$	$I_0 = -1$
$\theta(T_b) = -\pi/2$	$\theta(0) = \pi$	$e^{j(-\pi/2-\pi)} = I_0 e^{j(\pi/2)} = e^{j(\pi/2)}$	$I_0 = 1$
$\theta(T_b) = -\pi/2$	$\theta(0) = 0$	$e^{j(-\pi/2-0)} = I_0 e^{j(\pi/2)} = -e^{j(\pi/2)}$	$I_0 = -1$

$$\begin{cases} \tilde{s}(T_b) = \sqrt{\dfrac{2E_b}{T_b}} e^{j(\pi/2)\left(\sum_{k=-\infty}^{0} I_k\right)} \\ \tilde{s}(0) = \sqrt{\dfrac{2E_b}{T_b}} e^{j(\pi/2)\left(\sum_{k=-\infty}^{-1} I_k\right)} \end{cases}$$

For $n = 2\ell - 1$, we have

$$
\begin{aligned}
\cos[\theta(t) - \theta(0)] &= \left(\prod_{k=0}^{n} I_k\right) I_n \cos\left(\frac{\pi}{2}\left[I_n\left(\frac{t - nT_b}{T_b}\right) + n\right]\right) \\
&= \left(\prod_{k=0}^{n} I_k\right) I_n \cos\left(\frac{\pi}{2}\left[I_n\left(\frac{t - (2\ell - 1)T_b}{T_b}\right) + 2\ell - 1\right]\right) \\
&= \left(\prod_{k=0}^{n} I_k\right) I_n \cos\left(\frac{\pi}{2}\left[I_n\left(\frac{t - 2\ell T_b}{T_b}\right) + I_n + 2\ell - 1\right]\right) \\
&= \begin{cases} \left(\prod_{k=0}^{n} I_k\right) \cos\left(\frac{\pi}{2}\left[\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell\right]\right), & I_n = 1 \\ -\left(\prod_{k=0}^{n} I_k\right) \cos\left(\frac{\pi}{2}\left[-\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell - 2\right]\right), & I_n = -1 \end{cases} \\
&= \begin{cases} \left(\prod_{k=0}^{2\ell-1} I_k\right) \cos\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right), & \ell \text{ even} \\ -\left(\prod_{k=0}^{2\ell-1} I_k\right) \cos\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right), & \ell \text{ odd} \end{cases}
\end{aligned}
$$

For $n = 2\ell$, we have

$$
\begin{aligned}
\cos[\theta(t)-\theta(0)] &= \left(\prod_{k=0}^{n} I_k\right) I_n \cos\left(\frac{\pi}{2}\left[I_n\left(\frac{t-nT_b}{T_b}\right)+n\right]\right) \\
&= \left(\prod_{k=0}^{n-1} I_k\right) \cos\left(\frac{\pi}{2}\left[I_n\left(\frac{t-2\ell T_b}{T_b}\right)+2\ell\right]\right) \\
&= \begin{cases} \left(\prod_{k=0}^{2\ell-1} I_k\right)\cos\left(\dfrac{\pi(t-2\ell T_b)}{2T_b}\right), & \ell \text{ even} \\ -\left(\prod_{k=0}^{2\ell-1} I_k\right)\cos\left(\dfrac{\pi(t-2\ell T_b)}{2T_b}\right), & \ell \text{ odd} \end{cases}
\end{aligned}
$$

For $(2\ell-1)T_b \le t < (2\ell+1)T_b$,

$$
\cos[\theta(t)-\theta(0)] = (-1)^{\ell} J_{2\ell-1} \cos\left(\frac{\pi(t-2\ell T_b)}{2T_b}\right),
$$

where $J_n \triangleq \prod_{k=0}^{n} I_k$.

For $n = 2\ell$, we have

$$
\begin{aligned}
&\sin[\theta(t) - \theta(0)] \\
&= \left(\prod_{k=0}^{n} I_k\right) I_n \sin\left(\frac{\pi}{2}\left[I_n\left(\frac{t - nT_b}{T_b}\right) + n\right]\right) \\
&= J_{2\ell} I_{2\ell} \sin\left(\frac{\pi}{2}\left[I_{2\ell}\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell\right]\right) \\
&= \begin{cases} J_{2\ell} \sin\left(\frac{\pi}{2}\left[\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell\right]\right), & I_{2\ell} = 1 \\ -J_{2\ell} \sin\left(\frac{\pi}{2}\left[-\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell\right]\right), & I_{2\ell} = -1 \end{cases} \\
&= \begin{cases} J_{2\ell} \sin\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right), & \ell \text{ even} \\ -J_{2\ell} \sin\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right), & \ell \text{ odd} \end{cases}
\end{aligned}
$$

For $n = 2\ell + 1$, we have

$$
\begin{aligned}
\sin[\theta(t) - \theta(0)] &= \left(\prod_{k=0}^{n} I_k\right) I_n \sin\left(\frac{\pi}{2}\left[I_n\left(\frac{t - nT_b}{T_b}\right) + n\right]\right) \\
&= \left(\prod_{k=0}^{n-1} I_k\right) \sin\left(\frac{\pi}{2}\left[I_n\left(\frac{t - (2\ell+1)T_b}{T_b}\right) + 2\ell + 1\right]\right) \\
&= J_{2\ell} \sin\left(\frac{\pi}{2}\left[I_n\left(\frac{t - 2\ell T_b}{T_b}\right) - I_n + 2\ell + 1\right]\right) \\
&= \begin{cases} J_{2\ell} \sin\left(\frac{\pi}{2}\left[\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell\right]\right), & I_n = 1 \\ J_{2\ell} \sin\left(\frac{\pi}{2}\left[-\left(\frac{t - 2\ell T_b}{T_b}\right) + 2\ell + 2\right]\right), & I_n = -1 \end{cases} \\
&= \begin{cases} J_{2\ell} \sin\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right), & \ell \text{ even} \\ -J_{2\ell} \sin\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right), & \ell \text{ odd} \end{cases}
\end{aligned}
$$

Coherent Frequency-Shift Keying – Minimum Shift Keying

For simplicity, assume $\begin{cases} \theta(0) = 0 \\ \tilde{I}_{-1} = 1 \end{cases}$

For $(2\ell - 1)T_b \le t < (2\ell + 1)T_b$,

$$\cos[\theta(t) - \theta(0)] = \cos[\theta(t)] = \tilde{I}_{2\ell-1} \cos\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right),$$

$$\boxed{= \tilde{I}_{2\ell-1} \sin\left(\frac{\pi(t - (2\ell - 1)T_b)}{2T_b}\right)}$$

For $2\ell T_b \le t < (2\ell + 2)T_b$,

$$\sin[\theta(t) - \theta(0)] = \sin[\theta(t)] = \tilde{I}_{2\ell} \sin\left(\frac{\pi(t - 2\ell T_b)}{2T_b}\right),$$

where $\tilde{I}_n \triangleq (-1)^{\lceil n/2 \rceil} \left(\prod_{k=0}^{n} I_k\right)$.

Coherent Frequency-Shift Keying – Minimum Shift Keying

$$
\begin{aligned}
s(t) &= \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\}\\
&= \sqrt{\frac{2E_b}{T_b}}\cos[\theta(t)]\cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}}\sin[\theta(t)]\sin(2\pi f_c t)\\
&= \sqrt{\frac{2E_b}{T_b}}\sum_{\ell=0}^{\infty}\Big[\tilde{I}_{2\ell-1}\cdot g(t-(2\ell-1)T_b)\cdot\cos(2\pi f_c t)\\
&\qquad\qquad -\tilde{I}_{2\ell}\cdot g(t-2\ell T_b)\cdot\sin(2\pi f_c t)\Big]
\end{aligned}
$$

where $g(t) = \begin{cases} \sin\left(\dfrac{\pi t}{2T_b}\right), & 0 \le t < 2T_b \\ 0, & \text{otherwise} \end{cases}.$

$$\text{Let } \phi_1(t) = \begin{cases} \sqrt{\dfrac{2}{T_b}} g(t+T_b)\cos(2\pi f_c t), & -T_b \le t < T_b \\ 0, & \text{otherwise} \end{cases}.$$

$$\text{Let } \phi_2(t) = \begin{cases} \sqrt{\dfrac{2}{T_b}} g(t)\sin(2\pi f_c t), & 0 \le t < 2T_b \\ 0, & \text{otherwise} \end{cases}.$$

$$\Rightarrow s(t) = \sqrt{E_b}\sum_{\ell=0}^{\infty}\left[\tilde{I}_{2\ell-1}\cdot\phi_1(t-2\ell T_b) - \tilde{I}_{2\ell}\cdot\phi_2(t-2\ell T_b)\right]$$

$$\begin{aligned} \langle s(t), \phi_1(t-2kT_b)\rangle &= \int_{-\infty}^{\infty} s(t)\phi_1(t-2kT_b)dt \\ &= \int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\phi_1(t-2kT_b)dt \\ &= \sqrt{E_b}\sum_{\ell=0}^{\infty}\left[\tilde{I}_{2\ell-1}\int_{(2k-1)T_b}^{(2k+1)T_b}\phi_1(t-2\ell T_b)\phi_1(t-2kT_b)dt\right. \\ &\qquad \left. -\tilde{I}_{2\ell}\int_{(2k-1)T_b}^{(2k+1)T_b}\phi_2(t-2\ell T_b)\phi_1(t-2kT_b)dt\right] \end{aligned}$$

$k \ge 1$ because $\tilde{I}_{-1}$ is known

$$
\begin{aligned}
\langle s(t), \phi_1(t-2kT_b)\rangle &= \sqrt{E_b}\left[\tilde{I}_{2k-1}\int_{(2k-1)T_b}^{(2k+1)T_b}\phi_1^2(t-2kT_b)dt\right.\\
&\quad -\tilde{I}_{2(k-1)}\int_{(2k-1)T_b}^{2kT_b}\phi_2(t-2(k-1)T_b)\phi_1(t-2kT_b)dt\\
&\quad \left.-\tilde{I}_{2k}\int_{2kT_b}^{(2k+1)T_b}\phi_2(t-2kT_b)\phi_1(t-2kT_b)dt\right]\\
&= \sqrt{E_b}\left[\tilde{I}_{2k-1}\int_{-T_b}^{T_b}\phi_1^2(s)ds - \tilde{I}_{2(k-1)}\int_{-T_b}^{0}\phi_2(s+2T_b)\phi_1(s)ds\right.\\
&\quad \left.-\tilde{I}_{2k}\int_{0}^{T_b}\phi_2(s)\phi_1(s)ds\right]
\end{aligned}
$$

$$\boxed{s = t - 2kT_b}$$

$$
\begin{aligned}
&\int_{-T_b}^{0}\phi_2(s+2T_b)\phi_1(s)ds\\
&= \frac{2}{T_b}\int_{-T_b}^{0}\sin\left(\frac{\pi s}{2T_b}+\pi\right)\sin(2\pi f_c s)\sin\left(\frac{\pi s}{2T_b}+\frac{\pi}{2}\right)\cos(2\pi f_c s)ds\\
&= -\frac{1}{2T_b}\int_{-T_b}^{0}\sin\left(\frac{\pi s}{T_b}\right)\sin(4\pi f_c s)ds = 0.
\end{aligned}
$$

Similarly, $\int_0^{T_b} \phi_2(s)\phi_1(s)ds = 0.$

$$\begin{aligned}
\int_{-T_b}^{T_b} \phi_1^2(s)ds &= \frac{2}{T_b}\int_{-T_b}^{T_b} \sin^2\left(\frac{\pi s}{2T_b}+\frac{\pi}{2}\right)\cos^2(2\pi f_c s)ds \\
&= \frac{2}{T_b}\int_{-T_b}^{T_b} \left(\frac{1+\cos\left(\pi s/T_b\right)}{2}\right)\left(\frac{1+\cos(4\pi f_c s)}{2}\right)ds \\
&= 1.
\end{aligned}$$

Therefore,

$$\boxed{\langle s(t), \phi_1(t-2kT_b)\rangle = \sqrt{E_b}\cdot \tilde{I}_{2k-1}.} \quad \text{for } k \geq 1$$

By following similar procedure,

$$\boxed{\langle s(t), \phi_2(t-2kT_b)\rangle = -\sqrt{E_b}\cdot \tilde{I}_{2k}.} \quad \text{for } k \geq 0$$

$I_n = (-1)^{\lceil n/2\rceil + \lceil (n-1)/2\rceil}\tilde{I}_n\tilde{I}_{n-1} = (-1)^n\tilde{I}_n\tilde{I}_{n-1}$
with initial value $\tilde{I}_{-1} = 1$

Coherent Frequency-Shift Keying – Error Probability of MSK

- Error probability of MSK (Decision rule)

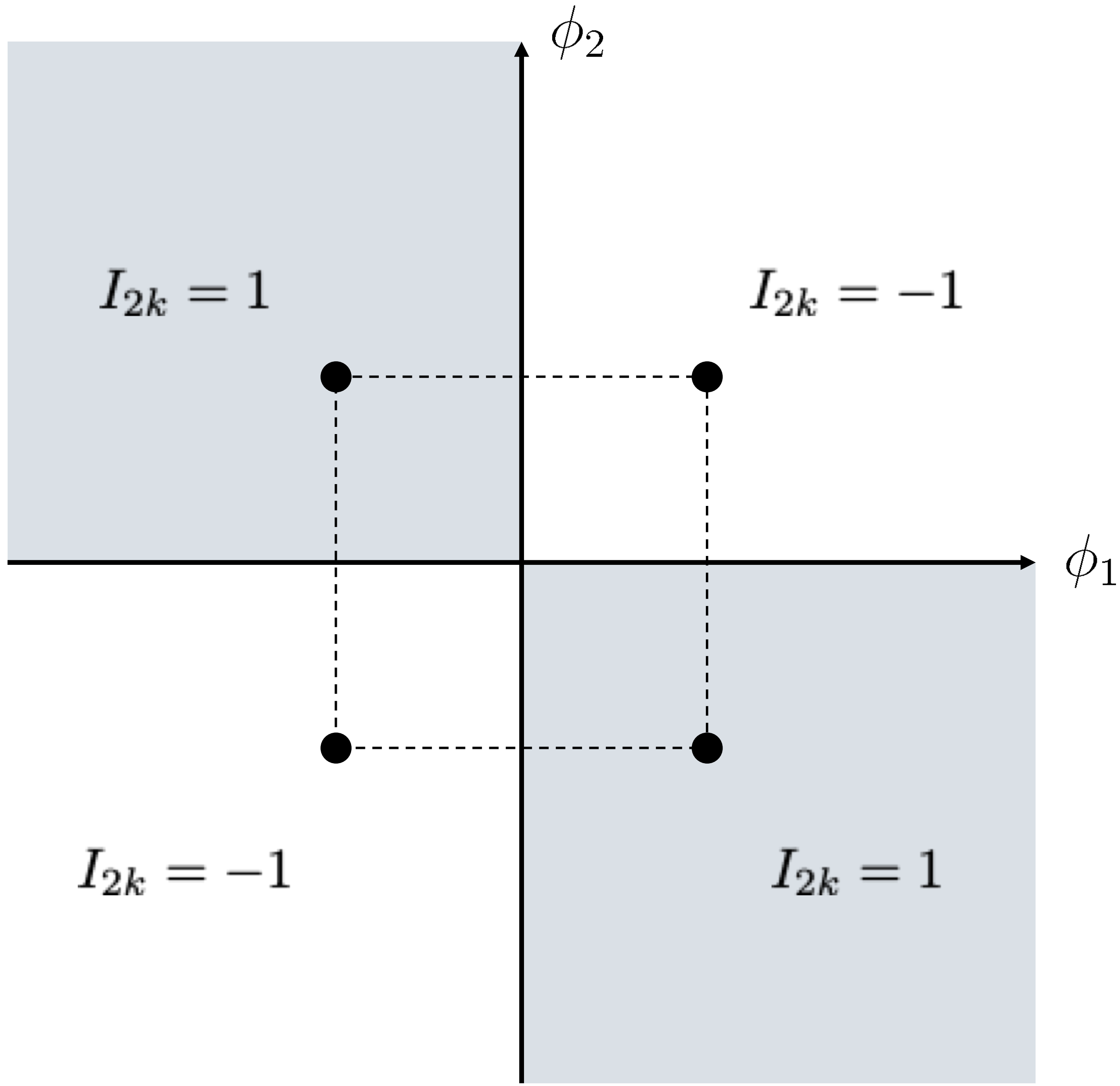
$$x(t) = s(t) + w(t)$$

$$\Rightarrow \begin{cases} \langle x(t), \phi_1(t-2kT_b)\rangle = \langle s(t), \phi_1(t-2kT_b)\rangle + \langle w(t), \phi_1(t-2kT_b)\rangle \\ \langle x(t), \phi_2(t-2kT_b)\rangle = \langle s(t), \phi_2(t-2kT_b)\rangle + \langle w(t), \phi_2(t-2kT_b)\rangle \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \sqrt{E_b}\cdot \tilde{I}_{2k-1} + w_1 \\ x_2 = -\sqrt{E_b}\cdot \tilde{I}_{2k} + w_2 \end{cases}$$

$$\Rightarrow \tilde{I}_{2k-1}\tilde{I}_{2k} = (-1)^{\lceil (2k-1)/2\rceil}\left(\prod_{u=0}^{2k-1} I_u\right)(-1)^{\lceil 2k/2\rceil}\left(\prod_{u=0}^{2k} I_u\right) = I_{2k}$$

implies $\boxed{x_1x_2 \underset{I_{2k}=-1}{\overset{I_{2k}=1}{\lessgtr}} 0}$


ϕ_2
ϕ_1
$I_{2k} = 1$
$I_{2k} = -1$
$I_{2k} = -1$
$I_{2k} = 1$

Error probability of MSK (Quick derivation)

$$
\begin{aligned}
P(\tilde{I}_{2k} \text{ error}) &= \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right) \\
P(\tilde{I}_{2k-1} \text{ error}) &= \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
P(I_{2k} \text{ error}) &= P(\tilde{I}_{2k} \text{ correct})P(\tilde{I}_{2k-1} \text{ error}) \\
&\quad + P(\tilde{I}_{2k} \text{ error})P(\tilde{I}_{2k-1} \text{ correct}) \\
&= 2\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\left[1 - \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\right]
\end{aligned}
}
$$

$$
\boxed{\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)}
$$

- Error probability of MSK (Direct derivation)
  - Based on the decision rule

$$x_1 x_2 \underset{I_{2k}=-1}{\overset{I_{2k}=1}{\lessgtr}} 0$$

$$\begin{aligned}
P(I_{2k}\text{ Error}) &= \frac{1}{4}\int_{[x_1x_2>0]}\frac{1}{2\pi\sigma^2}e^{-(x_1-\sqrt{E_b})^2/2\sigma^2}e^{-(x_2+\sqrt{E_b})^2/2\sigma^2}dx_1dx_2 \quad (\bar{I}_{2k-1},\bar{I}_{2k})=(1,1)\\
&+\frac{1}{4}\int_{[x_1x_2>0]}\frac{1}{2\pi\sigma^2}e^{-(x_1+\sqrt{E_b})^2/2\sigma^2}e^{-(x_2-\sqrt{E_b})^2/2\sigma^2}dx_1dx_2 \quad (\bar{I}_{2k-1},\bar{I}_{2k})=(-1,-1)\\
&+\frac{1}{4}\int_{[x_1x_2\le 0]}\frac{1}{2\pi\sigma^2}e^{-(x_1-\sqrt{E_b})^2/2\sigma^2}e^{-(x_2-\sqrt{E_b})^2/2\sigma^2}dx_1dx_2 \quad (\bar{I}_{2k-1},\bar{I}_{2k})=(1,-1)\\
&+\frac{1}{4}\int_{[x_1x_2\le 0]}\frac{1}{2\pi\sigma^2}e^{-(x_1+\sqrt{E_b})^2/2\sigma^2}e^{-(x_2+\sqrt{E_b})^2/2\sigma^2}dx_1dx_2 \quad (\bar{I}_{2k-1},\bar{I}_{2k})=(-1,1)\\
&= \int_{[x_1x_2>0]}\frac{1}{2\pi\sigma^2}e^{-(x_1-\sqrt{E_b})^2/2\sigma^2}e^{-(x_2+\sqrt{E_b})^2/2\sigma^2}dx_1dx_2\\
&= 2\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\left[1-\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\right]
\end{aligned}$$

Coherent Frequency-Shift Keying – Error Probability of MSK

- Final note on error probability of MSK
  - Since $\{\tilde{I}_n\}$ and $\{I_n\}$ are one-to-one correspondence, we can make $\{\tilde{I}_n\}$ be the true information bits.

$$\{\tilde{I}_n\} \to \{I_n\} \to \text{MSK transmitter} \to \text{Channel} \to \text{MSK receiver} \to \{\tilde{I}_n\}$$

  - In such case, the error rate of MSK becomes the one indicating in Eq. (6.127) as:

$$P(\tilde{I}_n \text{ Error}) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)$$

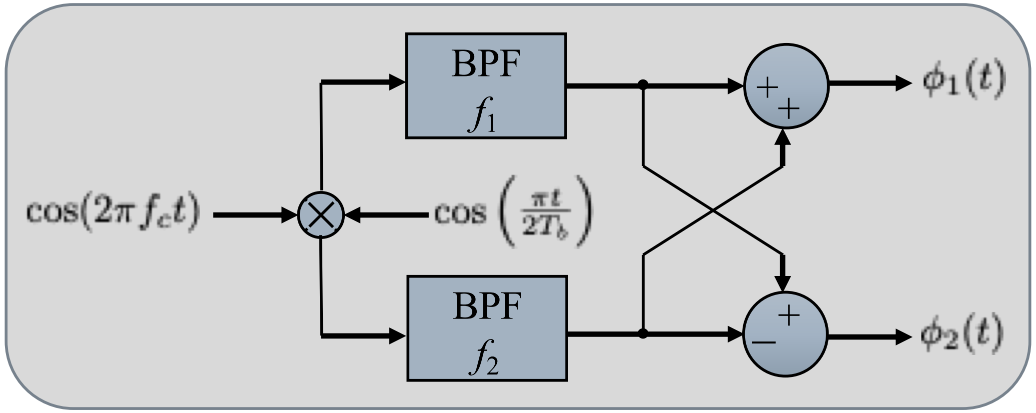
Coherent Frequency-Shift Keying – Block Diagrams

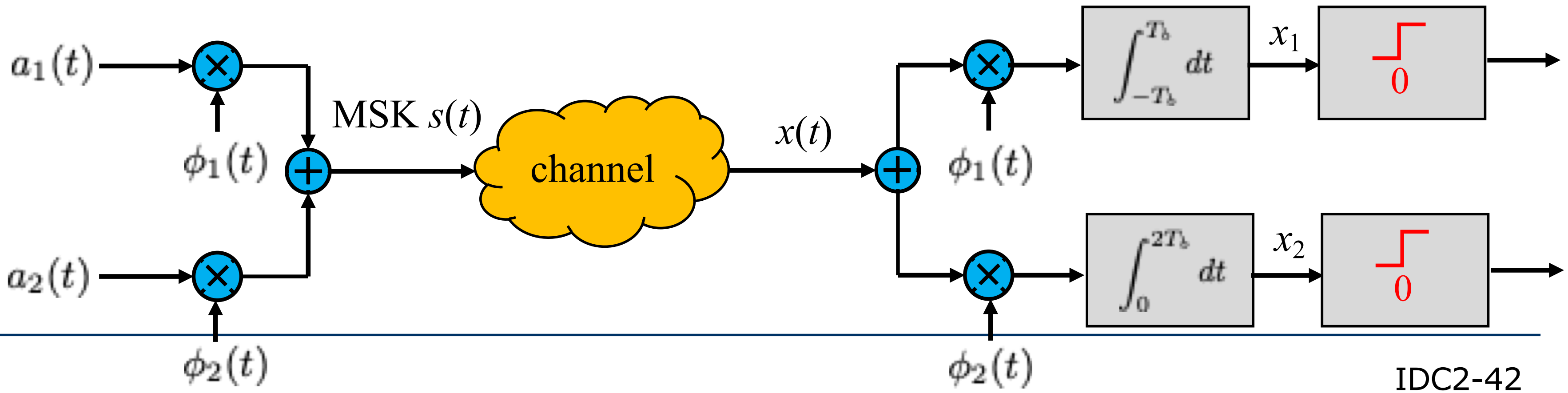
$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right) \cos(2\pi f_c t), & -T_b \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sqrt{\frac{1}{2T_b}} \left[\cos(2\pi f_2 t) + \cos(2\pi f_1 t)\right], & -T_b \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right) \sin(2\pi f_c t), & 0 \le t < 2T_b \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sqrt{\frac{1}{2T_b}} \left[\cos(2\pi f_2 t) - \cos(2\pi f_1 t)\right], & 0 \le t < 2T_b \\ 0, & \text{otherwise} \end{cases}$$





Coherent Frequency-Shift Keying – PSD of MSK

- Power spectra of MSK

$$\tilde{s}(t) = \sum_{\ell=-\infty}^{\infty} \left[\tilde{I}_{2\ell-1} \cdot g(t-(2\ell-1)T_b) \cdot + j\tilde{I}_{2\ell} \cdot g(t-2\ell T_b) \right]$$

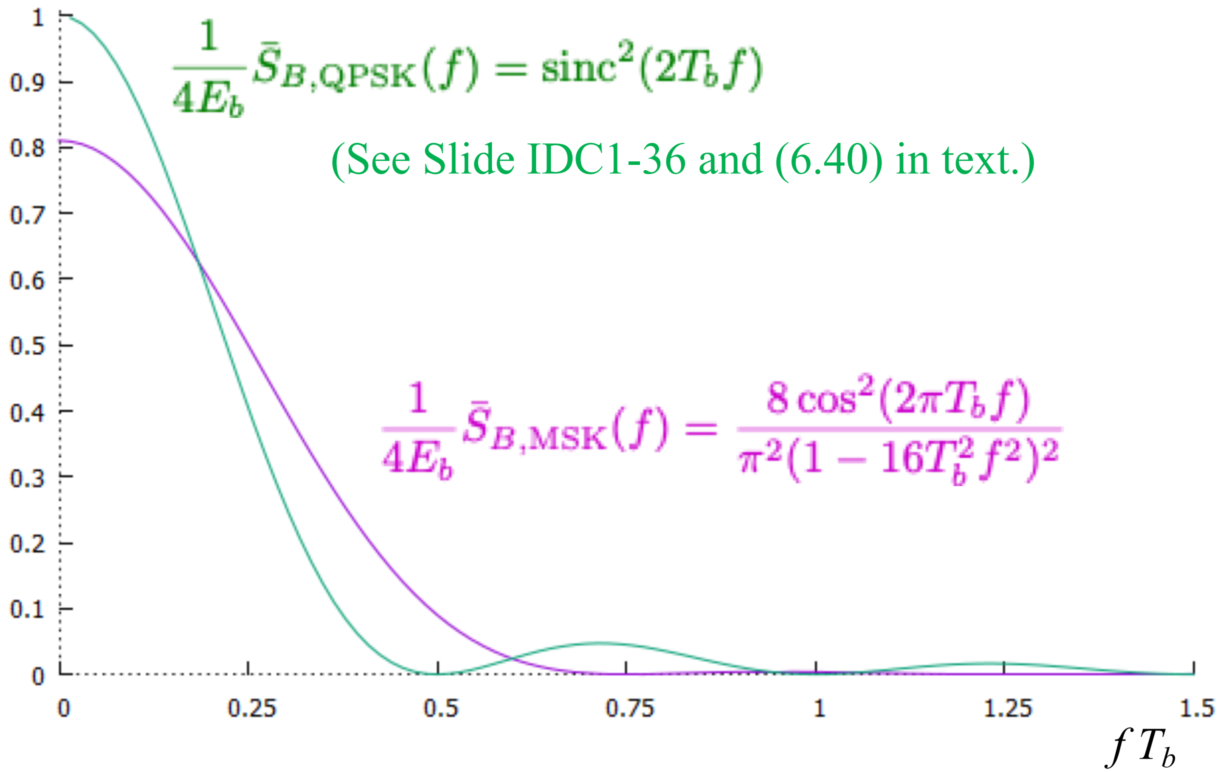
$$\text{where } g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right), & 0 \le t < 2T_b \\ 0, & \text{otherwise} \end{cases}.$$

  - No cross-correlation between g_I and g_Q (cf. Slide IDC2-10)

$$\bar{S}_{B,\mathrm{MSK}}(f) = 2\left(\frac{1}{2T_b}\right)|G(f)|^2 = \frac{32E_b\cos^2(2\pi T_b f)}{\pi^2(1-16T_b^2 f^2)^2}$$

Coherent Frequency-Shift Keying – PSD of MSK

- MSK decays as the inverse fourth power of frequency.
- QPSK decays as the inverse second power of frequency.



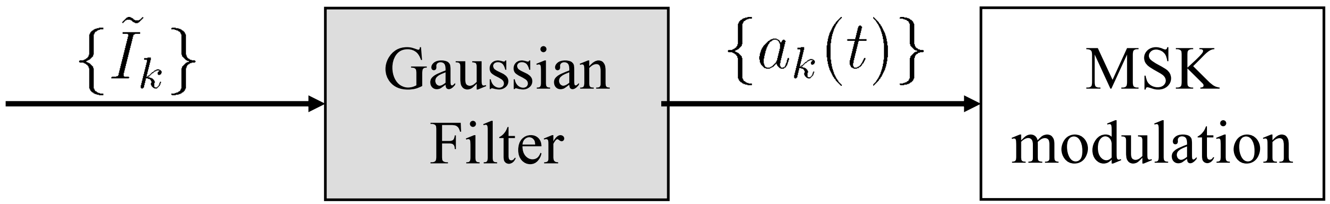
Coherent Frequency-Shift Keying – GMSK

- Gaussian-filtered MSK (GMSK)
  - MSK has the merits of
    - Constant envelope
    - Relatively narrow bandwidth (compared with QPSK)
    - Same coherent detection performance as QPSK
  - Can we further improve out-of-band characteristics of MSK (to fulfill the stringent requirements of certain applications such as GSM)?
    - Answer: GMSK

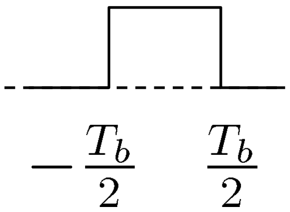
With $g(t) = \begin{cases} \sin\left(\dfrac{\pi t}{2T_b}\right), & 0 \le t < 2T_b \\ 0, & \text{otherwise} \end{cases}$.

$$\begin{aligned} s_{\text{MSK}}(t) &= \sqrt{\frac{2E_b}{T_b}} \sum_{\ell=0}^{\infty} \Big[\tilde{I}_{2\ell-1} \cdot g(t-(2\ell-1)T_b) \cdot \cos(2\pi f_c t) \\ &\qquad - \tilde{I}_{2\ell} \cdot g(t-2\ell T_b) \cdot \sin(2\pi f_c t) \Big] \end{aligned}$$

$$\begin{aligned} s_{\text{GMSK}}(t) &= \sqrt{\frac{2E_b}{T_b}} \sum_{\ell=0}^{\infty} \Big[a_{2\ell-1}(t) \star g(t-(2\ell-1)T_b) \cdot \cos(2\pi f_c t) \\ &\qquad - a_{2\ell}(t) \star g(t-2\ell T_b) \cdot \sin(2\pi f_c t) \Big] \end{aligned}$$



Coherent Frequency-Shift Keying – GMSK



$$H(f) = \exp\left(-\frac{\log(2)}{2}\left(\frac{f}{W}\right)^2\right) = 2^{-\frac{(f/W)^2}{2}}$$

$$h(t) = \sqrt{\frac{2\pi}{\log(2)}}W\exp\left(-\frac{2\pi^2}{\log(2)}W^2t^2\right)$$

$$W = 3 \text{ dB baseband bandwidth}$$

$a(t)$

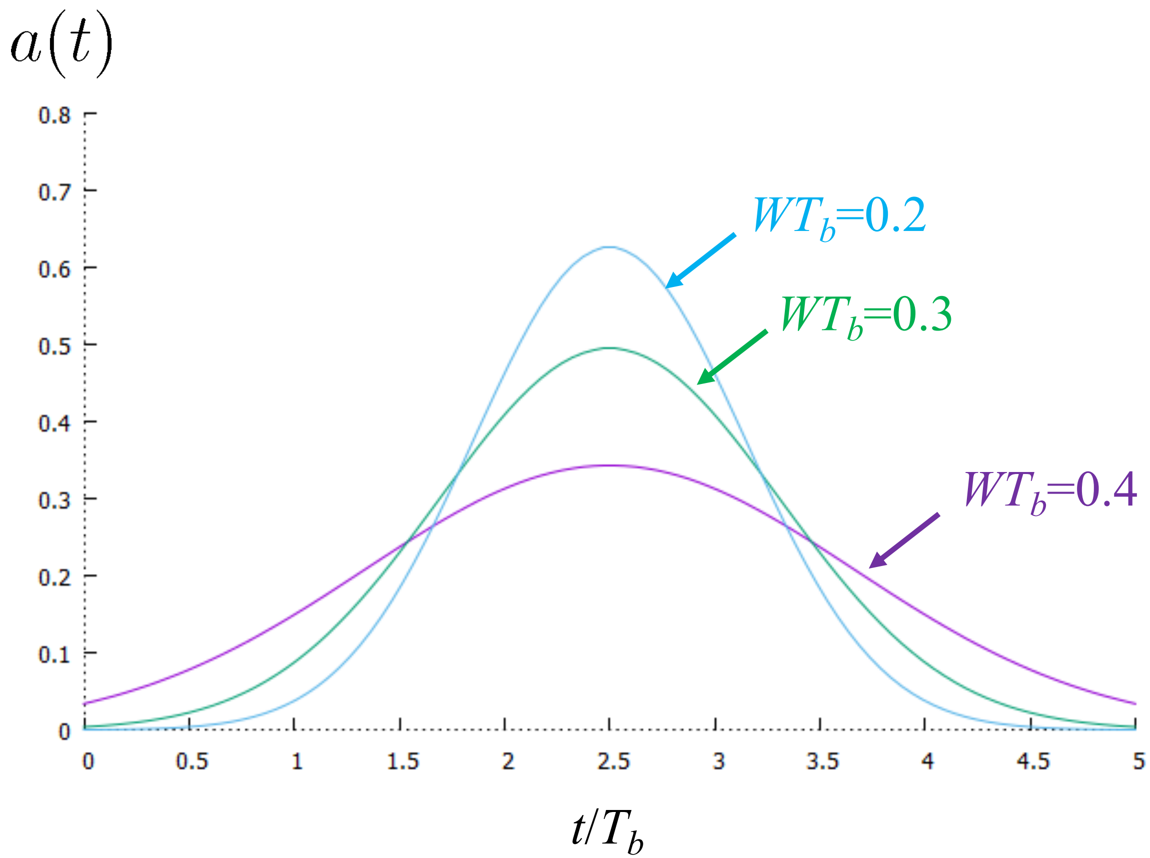
Gaussian filter

$$a(t) = \Phi\left(-\frac{2\pi W T_b}{\sqrt{\log(2)}}\left(\frac{t}{T_b}-\frac{1}{2}\right)\right) - \Phi\left(-\frac{2\pi W T_b}{\sqrt{\log(2)}}\left(\frac{t}{T_b}+\frac{1}{2}\right)\right)$$

$$\boxed{\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)}$$

Coherent Frequency-Shift Keying – GMSK

- Approximate (truncate and time-shift) the noncausal Gaussian filter by a causal filter
  - Shift in time by $2.5T_b$ and truncate at $\pm 2.5T_b$



Coherent Frequency-Shift Keying – GMSK

- In the limiting case, GMSK corresponds to the case of ordinary MSK.

$\boxed{\text{Let } v = WT_b.}$

$$
\begin{aligned}
\lim_{v\uparrow\infty} a(t) &= \lim_{v\uparrow\infty}\left[\Phi\left(-\frac{2\pi v}{\sqrt{\log(2)}}\left(\frac{t}{T_b}-\frac{1}{2}\right)\right)-\Phi\left(-\frac{2\pi v}{\sqrt{\log(2)}}\left(\frac{t}{T_b}+\frac{1}{2}\right)\right)\right]\\
&= \begin{cases} 1, & -\dfrac{T_b}{2} < t < \dfrac{T_b}{2}\\ \dfrac{1}{2}, & t = -\dfrac{T_b}{2} \text{ or } t = \dfrac{T_b}{2}\\ 0, & \text{otherwise}\end{cases}\\
&= \text{Input waveform}
\end{aligned}
$$

Appendix (Recall the PSD of Line Coded Signals)

- A usual general PSD formula is (See my Slides 2-30 and 6-64 for Introduction to Communication Systems):

$$\overline{\text{PSD}} = \lim_{T\to\infty} \frac{1}{2T} E[S(f)S_{2T}^*(f)], \text{ where } s_{2T}(t) = s(t)\cdot \mathbf{1}\{|t| \le T\}.$$

For a line coded signal, $s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b)$, where $g(t) = 0$ outside $[0, T_\text{b})$.

Hence, $S(f) = G(f) \sum_{n=-\infty}^{\infty} a_n e^{-j2\pi f n T_\text{b}}$ and $S_{2NT_\text{b}}(f) = G(f) \sum_{n=-N}^{N-1} a_n e^{-j2\pi f n T_\text{b}}$.

$$\Rightarrow \overline{\text{PSD}} = \lim_{N\to\infty} \tfrac{1}{2NT_\text{b}} |G(f)|^2 \left(\sum_{n=-\infty}^{\infty} \sum_{m=-N}^{N-1} E[a_n a_m^*] e^{-j2\pi f (n-m) T_\text{b}} \right).$$

Appendix (Recall the PSD of Line Coded Signals)

$$\text{PSD} = \lim_{N\to\infty} \frac{1}{2NT_b} |G(f)|^2 \left(\sum_{n=-\infty}^{\infty} \sum_{m=-N}^{N-1} E[a_n a_m^*] e^{-j2\pi f(n-m)T_b} \right)$$

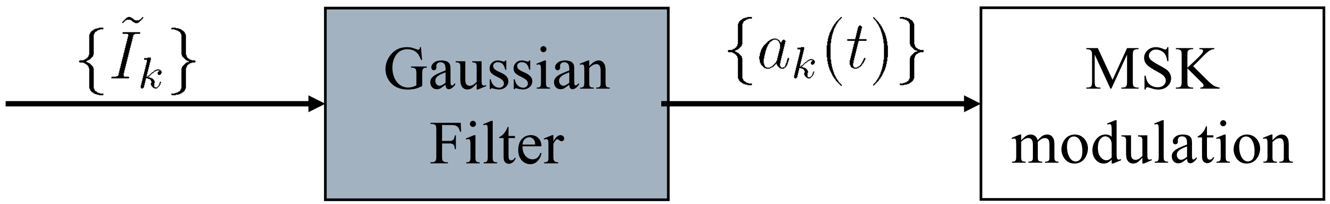
$$= |G(f)|^2 \lim_{N\to\infty} \frac{1}{2NT_b} \left(\sum_{m=-N}^{N-1} \sum_{n=-\infty}^{\infty} \phi_a(n-m) e^{-j2\pi f(n-m)T_b} \right)$$

$$= |G(f)|^2 \lim_{N\to\infty} \frac{1}{2NT_b} \left(\sum_{m=-N}^{N-1} \sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fkT_b} \right)$$

$$= |G(f)|^2 \frac{1}{T_b} \left(\sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fkT_b} \right)$$

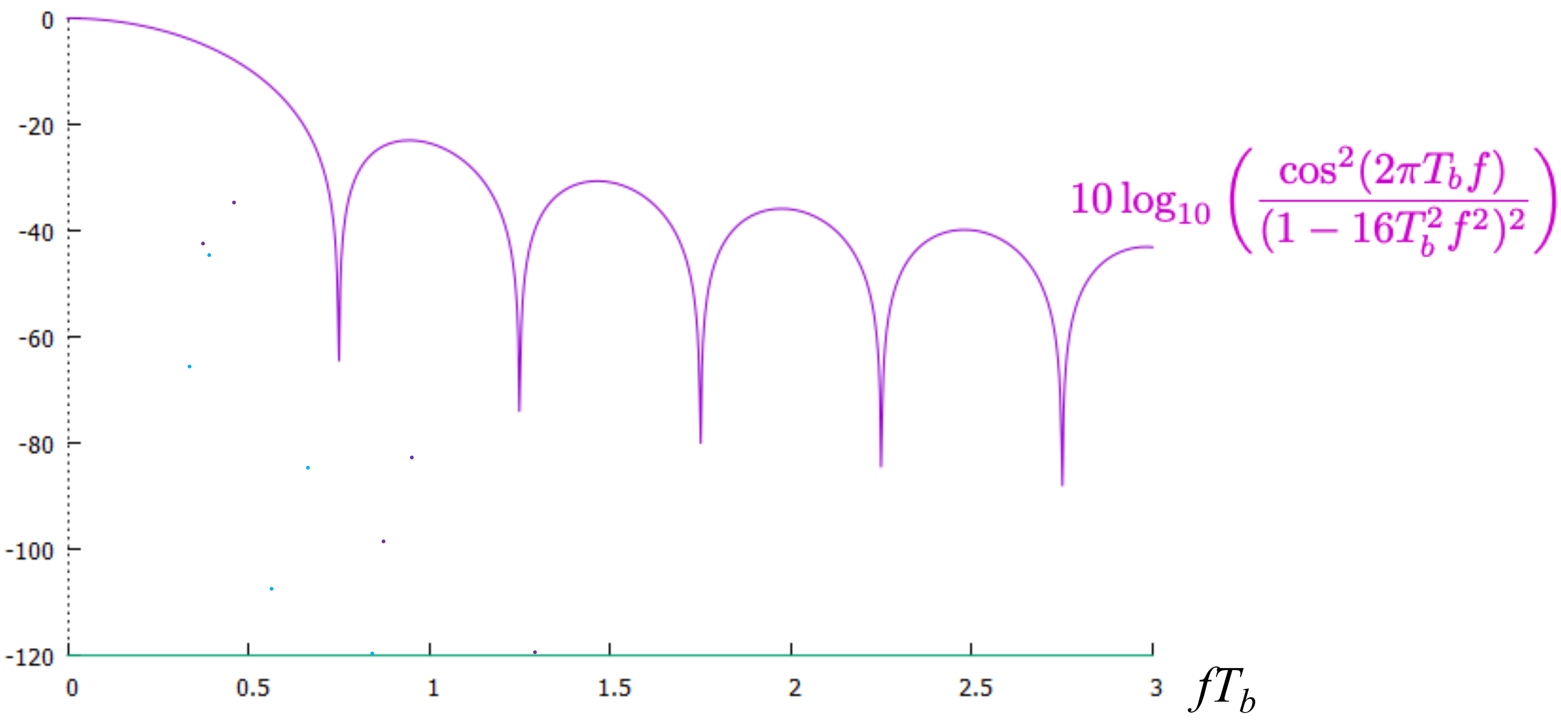
$$
\begin{aligned}
s_{\text{MSK}}(t) &= \sqrt{\frac{2E_b}{T_b}} \sum_{\ell=0}^{\infty} \Big[\tilde{I}_{2\ell-1} \cdot g(t-(2\ell-1)T_b) \cdot \cos(2\pi f_c t) \\
&\qquad\qquad\qquad\qquad - \tilde{I}_{2\ell} \cdot g(t-2\ell T_b) \cdot \sin(2\pi f_c t) \Big] \\
s_{\text{GMSK}}(t) &= \sqrt{\frac{2E_b}{T_b}} \sum_{\ell=0}^{\infty} \Big[a_{2\ell-1}(t) \star g(t-(2\ell-1)T_b) \cdot \cos(2\pi f_c t) \\
&\qquad\qquad\qquad\qquad - a_{2\ell}(t) \star g(t-2\ell T_b) \cdot \sin(2\pi f_c t) \Big]
\end{aligned}
$$

The GMSK signal is not in the "line-coding" form except when WT_b tends to infinity (which results in MSK)". Hence, its PSD is in general difficult to obtain. Figure 6.33 in textbook was obtained using the approximation approach proposed by G. J. Garrison in 1975 ("A power spectral density analysis for digital FM", *IEEE Trans. Commun.*, vol. 23, pp. 1228-1243, Nov. 1975).



Coherent Frequency-Shift Keying – GMSK

- More compact in power spectra for time-bandwidth product WT_b less than unity (See Figure 6.33 in textbook)



Coherent Frequency-Shift Keying – GMSK

- Error probability of GMSK

It is known from Slide IDC2-41 that

$$P_{\text{MSK}}(\tilde{I}_n \text{ Error}) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)$$

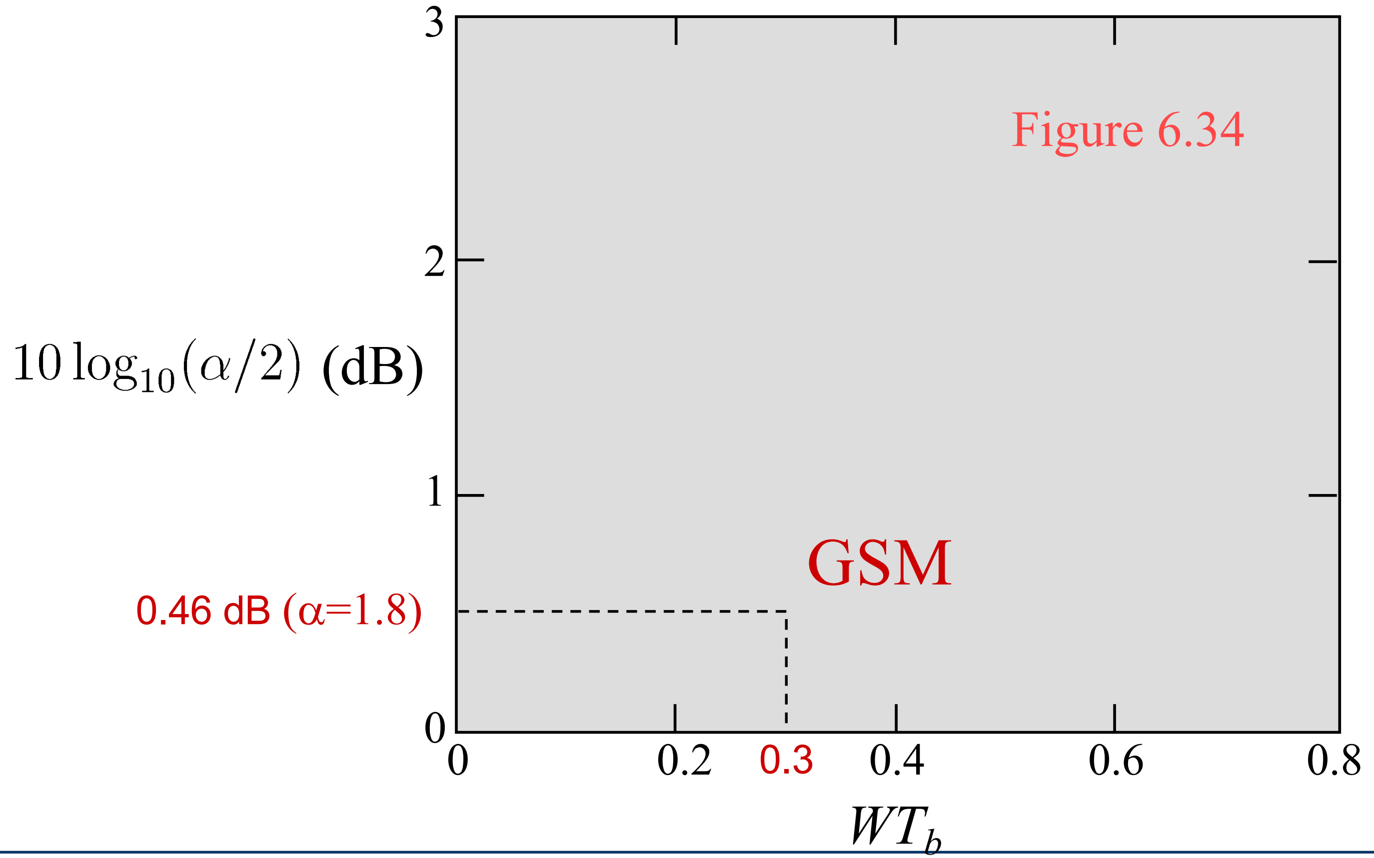
Assume that

$$P_{\text{GMSK}}(\tilde{I}_n \text{ Error}) = \Phi\left(-\sqrt{\alpha\frac{E_b}{N_0}}\right)$$

Find α empirically.

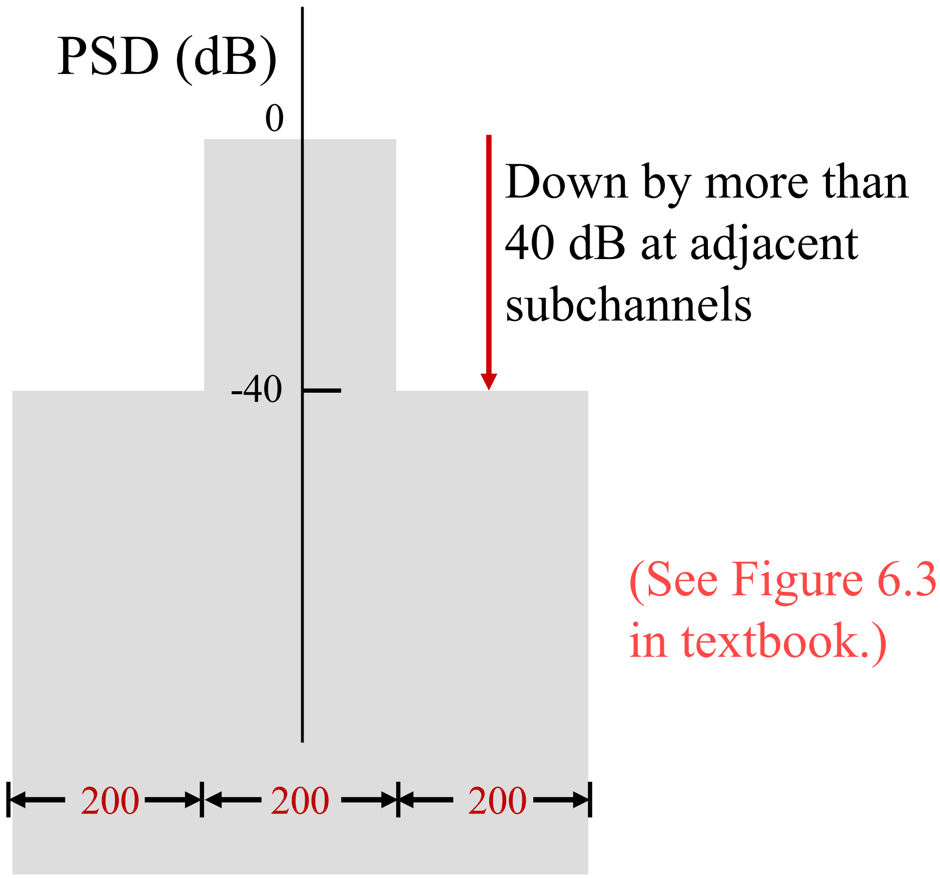
Coherent Frequency-Shift Keying – GMSK

Performance degrades (from MSK) due to intersymbol interference that is introduced by Gaussian filter (See Figure 6.34 in textbook).



Coherent Frequency-Shift Keying – GMSK

- Power spectra of GMSK for GSM
  - $WT_b = 0.3$
  - 99% of the RF power is confined to a bandwidth of 250 KHz.
  - Each subchannel is 200 KHz wide for transmitting data at 271 kbps.



(See Figure 6.35 in textbook.)

Coherent Frequency-Shift Keying – *M*-ary FSK

- *M*-ary FSK

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos\left(2\pi f_c t + i\dfrac{\pi t}{2T}\right), & 0 \le t < T \\ 0, & \text{elsewhere} \end{cases}$$

where $i = \pm 1, \pm 3, \ldots, \pm(M-1)$, f_c is a multiple of $1/(2T)$,

E is the transmitted energy per symbol, and

T is the symbol duration.

- Orthogonality

$$\phi_i(t) = \frac{1}{\sqrt{E}} s_i(t), \text{ where } \{\phi_i(t)\}_{i=1}^{M} \text{ orthonormal.}$$

Coherent Frequency-Shift Keying – *M*-ary FSK

- Error probability bound of *M*-ary FSK

$$\boxed{P_{e,M} \le (M-1)P_{e,2}} = (M-1)\Phi\left(-\sqrt{\frac{E}{N_0}}\right)$$

(See the next slide.)

- Power spectra of *M*-ary FSK (No derivation)

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} g(t-kT)e^{jI_k\pi t/(2T)}$$

where $I_k = \pm 1, \cdots, \pm(M-1)$ with equal prob., and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E}{T}}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
P_{e,M} &= \sum_{i=1}^{M} \Pr(s_i \text{ transmitted}) \Pr(\text{decision} \neq s_i | s_i \text{ transmitted}) \\
&= \sum_{i=1}^{M} \Pr(s_i \text{ transmitted}) \Pr(\text{decision} = s_1 \text{ or } \cdots \\
&\qquad \text{or decision} = s_{i-1} \text{ or decision} = s_{i+1} \text{ or } \cdots \\
&\qquad \text{or decision} = s_1 | s_i \text{ transmitted}) \text{ ------------ } (1) \\
&\leq \sum_{i=1}^{M} \Pr(s_i \text{ transmitted}) \sum_{k=1, k\neq i}^{M} \Pr(\text{decision} = s_k | s_i \text{ transmitted}) \\
&\leq \boxed{\sum_{i=1}^{M} \Pr(s_i \text{ transmitted}) (M-1) \max_{1\leq k\leq M, k\neq i} \Pr(\text{decision} = s_k | s_i \text{ transmitted})} \\
&\leq \boxed{(M-1) \max_{1\leq \ell, k\leq M, \ell\neq k} \Pr(\text{decision} = s_k | s_\ell \text{ transmitted})}
\end{aligned}
$$

Union (upper) bound

$$
\begin{aligned}
P_{e,M} &= (1) \\
&\geq \boxed{\sum_{i=1}^{M} \Pr(s_i \text{ transmitted}) \max_{1\leq k\leq M, k\neq i} \Pr(\text{decision} = s_k | s_i \text{ transmitted})}
\end{aligned}
$$

Lower bound

Upper bound = Lower bound, when $M = 2$

Coherent Frequency-Shift Keying – *M*-ary FSK

- Power spectra of *M*-ary FSK (Not in the line-coding form)

$$\begin{aligned}\tilde{s}(t) &= \sum_{k=-\infty}^{\infty} g(t-kT)e^{jI_k\pi t/(2T)} \\ &= \sum_{k=-\infty}^{\infty} e^{jI_k\pi k/2} g(t-kT)e^{jI_k\pi(t-kT)/(2T)}\end{aligned}$$

where $I_k = \pm 1, \cdots, \pm(M-1)$ with equal prob., and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E}{T}}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

Appendix (Excluded from exam)

This is outside the current scope of the text. Just provide it for those who are interested in the derivation.

$$
\begin{aligned}
R_{\tilde{s}\tilde{s}}(t+\tau, t) &= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} g(t+\tau-kT)g(t-\ell T)E\left[e^{jI_k\pi(t+\tau)/(2T)}e^{-jI_\ell\pi t/(2T)}\right] \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty,\ell\neq k}^{\infty} g(t+\tau-kT)g(t-\ell T)E\left[e^{jI_k\pi(t+\tau)/(2T)}\right]E\left[e^{-jI_\ell\pi t/(2T)}\right] \\
&+ \sum_{k=-\infty}^{\infty} g(t+\tau-kT)g(t-kT)E\left[e^{jI_k\pi\tau/(2T)}\right]
\end{aligned}
$$

where $E\left[e^{jI_k v}\right] = \frac{2}{M}\sum_{u=1}^{M/2}\cos\left((2u-1)v\right)$ for even M.

$$\tilde{S}(f) = \sum_{k=-\infty}^{\infty} e^{jI_k\pi k/2} e^{-j2\pi fkT} \mathcal{F}\left\{g(t)e^{jI_k\pi t/(2T)}\right\}$$

$$= \sum_{k=-\infty}^{\infty} e^{jI_k\pi k/2} e^{-j2\pi fkT} G\left(f - \frac{I_k}{4T}\right)$$

$$\tilde{S}_{2NT_b}(f) = \sum_{m=-N}^{N-1} e^{jI_m\pi m/2} e^{-j2\pi fmT} G\left(f - \frac{I_m}{4T}\right)$$

This is outside the current scope of the text. Just provide it for those who are interested in the derivation.

$$\overline{\mathrm{PSD}}(f) = \lim_{N\to\infty} \frac{1}{2NT_b} E\left[\sum_{k=-\infty}^{\infty} e^{jI_k\pi k/2} e^{-j2\pi fkT} G\left(f - \frac{I_k}{4T}\right)\right.$$

$$\left.\sum_{m=-N}^{N-1} e^{-jI_m\pi m/2} e^{j2\pi fmT} G^*\left(f - \frac{I_m}{4T}\right)\right]$$

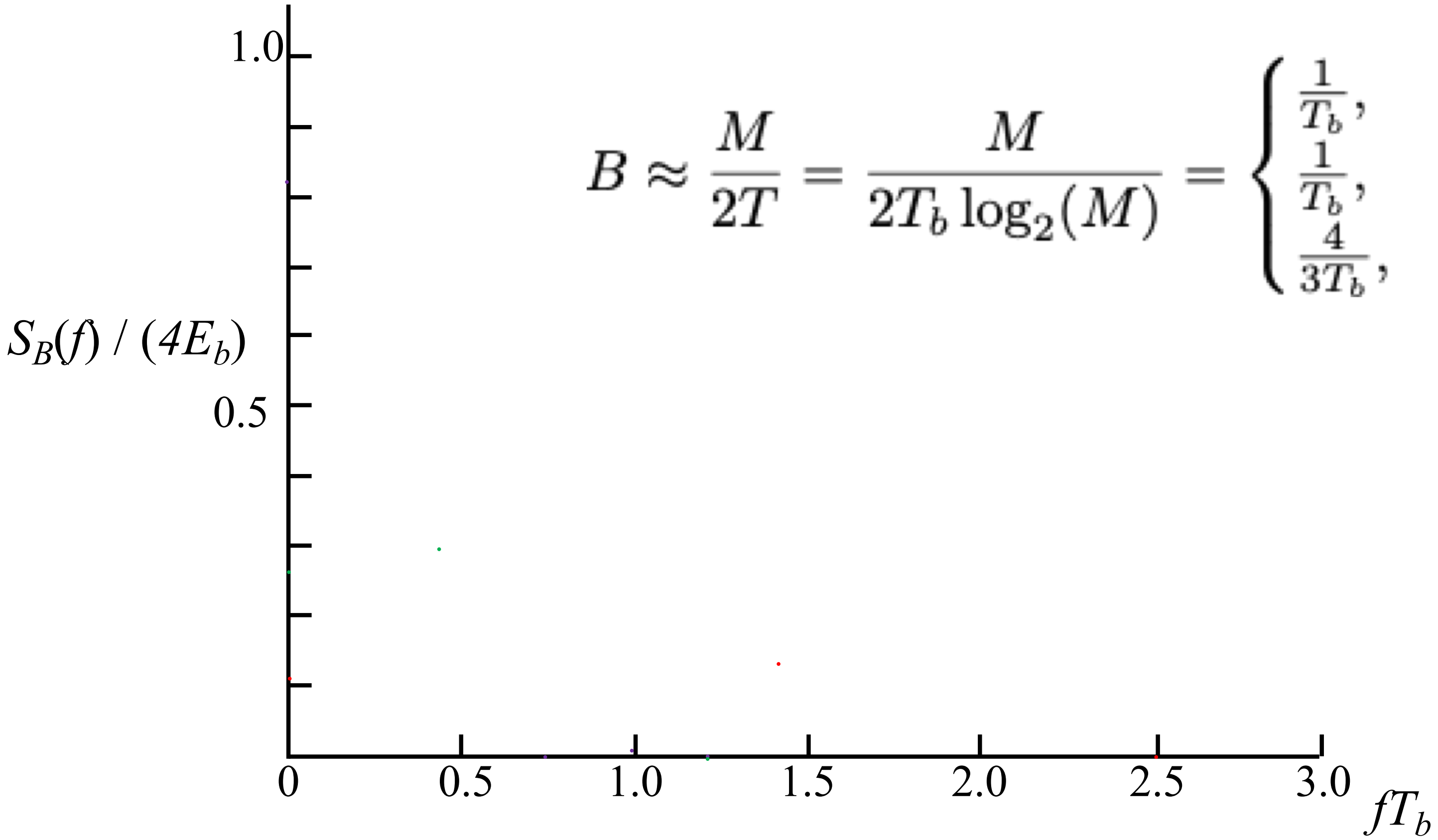
$$= \lim_{N\to\infty} \frac{1}{2NT_b} \sum_{m=-N}^{N-1} \sum_{k=-\infty}^{\infty} e^{-j2\pi f(k-m)T}$$

$$E\left[e^{jI_k\pi k/2} G\left(f - \frac{I_k}{4T}\right) e^{-jI_m\pi m/2} G^*\left(f - \frac{I_m}{4T}\right)\right]$$

$= \ldots\ldots$ (no straightforward analysis)

Coherent Frequency-Shift Keying – *M*-ary FSK

See Figure 6.36 in textbook.



$$B \approx \frac{M}{2T} = \frac{M}{2T_b \log_2(M)} = \begin{cases} \frac{1}{T_b}, & M = 2 \\ \frac{1}{T_b}, & M = 4 \\ \frac{4}{3T_b}, & M = 8 \end{cases}$$

Coherent Frequency-Shift Keying – *M*-ary FSK

- Spectral efficiency of *M*-ary FSK

$$\left.\begin{aligned} B &= M\frac{1}{2T} \\ R_b &= \frac{1}{T_b} \\ T &= T_b \log_2(M) \end{aligned}\right\} \Rightarrow \rho = \frac{R_b}{B} = \frac{2\log_2(M)}{M} \text{ bits/seconds/Hz}$$

M	2	4	8	16	32	64
ρ	1	1	0.75	0.5	0.3125	0.1815

Larger *M implies* worse spectral efficiency.

Detection of Signals with Unknown Phase

- How to deal with unknown phase θ, e.g., in FSK?

$$x(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_i t + \theta) + w(t) = s_i(t) + w(t)$$

  - Answer: Noncoherent receiver
- How to remove the requirement of phase information at noncoherent receiver?
  - Answer: Take the expectation with respect to all possible θ.

Detection of Signals with Unknown Phase

- (Conditional) likelihood ratio test
  - For known θ,

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_i t + \theta)$$
$$\phi_i(t) = \frac{1}{\sqrt{E}} s_i(t)$$

$$x(t) = s_k(t) + w(t)$$
$$\Rightarrow \quad \langle x(t), \phi_i(t)\rangle = \langle s_k(t), \phi_i(t)\rangle + \langle w(t), \phi_i(t)\rangle$$
$$\Rightarrow \quad x_i = (\text{either } \sqrt{E} \text{ or } 0) + w_i = s_{i,k} + w_i$$

The distribution of w_i has nothing to do with θ.

$$\Rightarrow \quad f(x_i|s_{i,k}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i - s_{i,k})^2/(2\sigma^2)}$$

$$\sigma^2 = N_0/2$$

$$\Rightarrow \quad \text{decision} = \arg\max_{1\le k\le M} \prod_{i=1}^{M} f(x_i|s_{i,k}) = \arg\max_{1\le k\le M} \prod_{i=1}^{M} e^{s_{i,k}x_i/\sigma^2}$$

$$s_{i,k} = \begin{cases} \sqrt{E}, & i = k \\ 0, & \text{otherwise} \end{cases}$$

Detection of Signals with Unknown Phase

$$\Rightarrow \quad \text{decision} = \arg \max_{1\le k\le M} \prod_{i=1}^{M} e^{s_{i,k}x_i/\sigma^2} \qquad \boxed{s_{i,k} = \begin{cases} \sqrt{E}, & i = k \\ 0, & \text{otherwise} \end{cases}}$$

$$\Rightarrow \quad \text{decision} = \arg\max \left\{ e^{\sqrt{E}x_1/\sigma^2}, e^{\sqrt{E}x_2/\sigma^2}, \cdots, e^{\sqrt{E}x_M/\sigma^2} \right\}$$

$$\text{where } x_i = \langle x(t), \phi_i(t) \rangle = \int_0^T \sqrt{\frac{2}{T}} x(t) \cos(2\pi f_i t + \theta) dt.$$

- However, θ is unknown! So, let's average it out.

$$\text{decision} = \arg\max \left\{ E_\theta \left[e^{\sqrt{E}x_1/\sigma^2} \right], E_\theta \left[e^{\sqrt{E}x_2/\sigma^2} \right], \cdots, E_\theta \left[e^{\sqrt{E}x_M/\sigma^2} \right] \right\}$$

Hence, $E_\theta\left[e^{\sqrt{E}x_i/\sigma^2}\right] = E_\theta\left[e^{\sqrt{\frac{8E}{TN_0^2}}\int_0^T x(t)\cos(2\pi f_i t+\theta)dt}\right]$

Since

$$
\begin{aligned}
&\int_0^T x(t)\cos(2\pi f_i t+\theta)dt \\
&\quad= \cos(\theta)\int_0^T x(t)\cos(2\pi f_i t)dt - \sin(\theta)\int_0^T x(t)\sin(2\pi f_i t)dt \\
&\quad= \ell_i[\cos(\theta)\cos(\beta_i) - \sin(\theta)\sin(\beta_i)] = \boxed{\ell_i\cos(\theta+\beta_i)}
\end{aligned}
$$

where

$$\ell_i = \left[\left(\int_0^T x(t)\cos(2\pi f_i t)dt\right)^2 + \left(\int_0^T x(t)\sin(2\pi f_i t)dt\right)^2\right]^{1/2}$$

$$\beta_i = \arctan\left(\int_0^T x(t)\sin(2\pi f_i t)dt \Big/ \int_0^T x(t)\cos(2\pi f_i t)dt\right)$$

Assume θ is uniform distributed over $[-\pi, \pi)$.

$$\begin{aligned}
E_\theta\left[e^{\sqrt{E}x_i/\sigma^2}\right] &= E_\theta\left[e^{\sqrt{\frac{8E}{TN_0^2}}\ell_i\cos(\theta+\beta_i)}\right]\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\sqrt{\frac{8E}{TN_0^2}}\ell_i\cos(\theta+\beta_i)}\,d\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\sqrt{\frac{8E}{TN_0^2}}\ell_i\cos(\theta)}\,d\theta\\
&= I_0\left(\sqrt{\frac{8E}{TN_0^2}}\ell_i\right)
\end{aligned}$$

The modified Bessel function of zero kind is a monotonically increasing function.

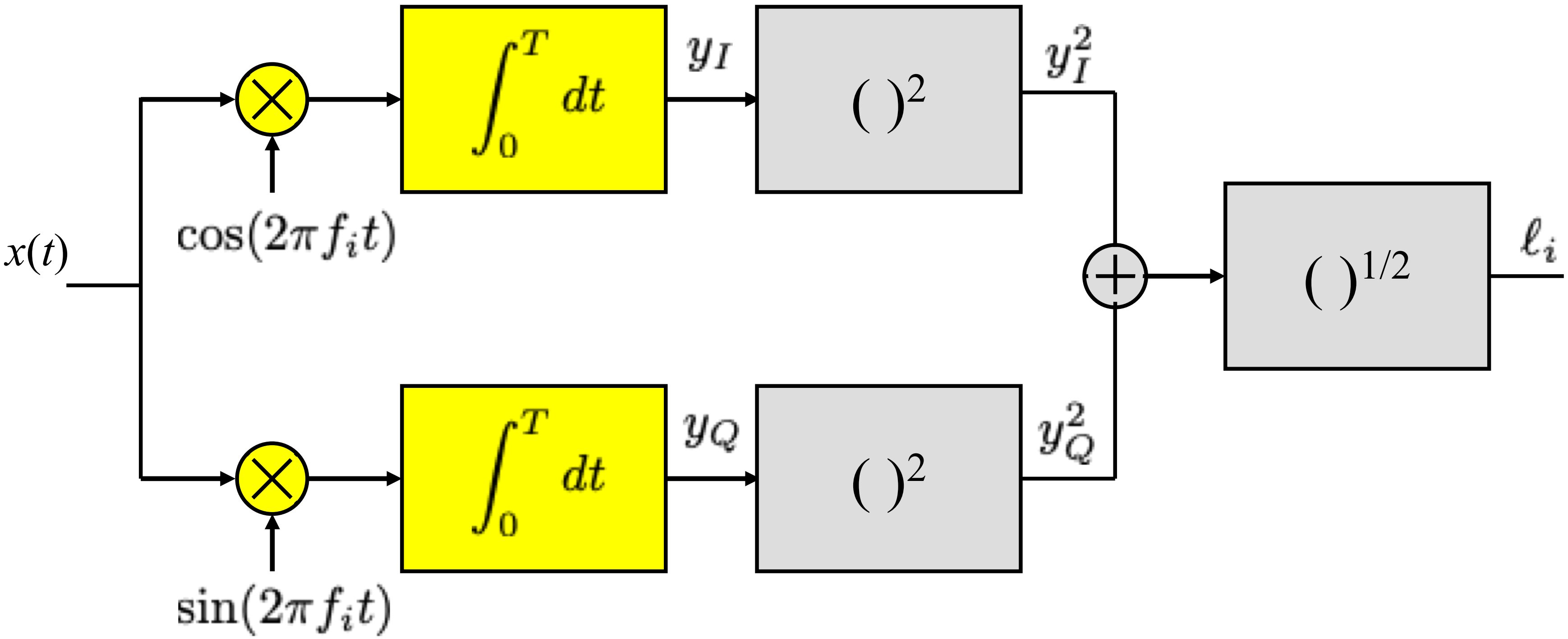
$$
\begin{aligned}
\text{decision} &= \arg\max_{1\le i\le M} E_\theta\left[e^{\sqrt{E}x_i/\sigma^2}\right]\\
&= \arg\max_{1\le i\le M} I_0\left(\sqrt{\frac{8E}{TN_0^2}}\ell_i\right)\\
&= \arg\max_{1\le i\le M} \ell_i\\
&= \arg\max_{1\le i\le M} \ell_i^2, \quad \text{since } \ell_i \ge 0.
\end{aligned}
$$

The modified Bessel function of zero kind is a monotonically increasing function.

The receiver is therefore named as *quadratic receiver*.

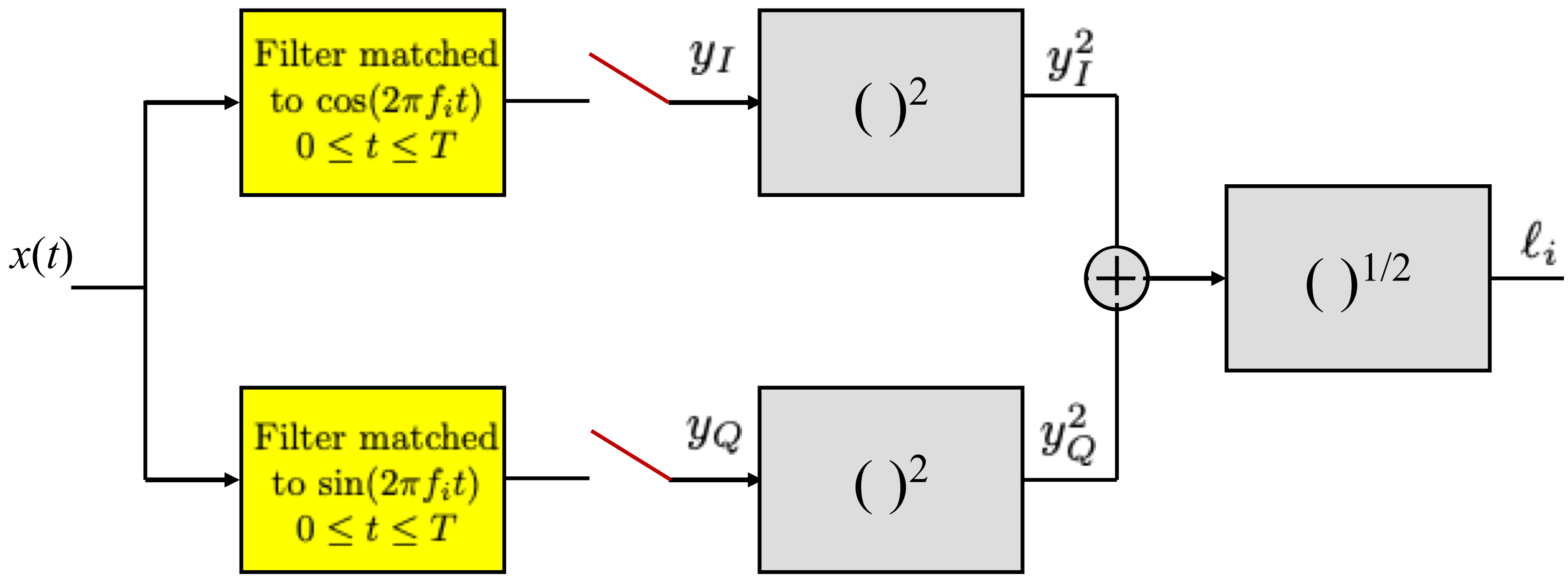
$$
\ell_i = \left[\left(\int_0^T x(t)\cos(2\pi f_i t)dt\right)^2 + \left(\int_0^T x(t)\sin(2\pi f_i t)dt\right)^2\right]^{1/2}
$$

Detection of Signals with Unknown Phase



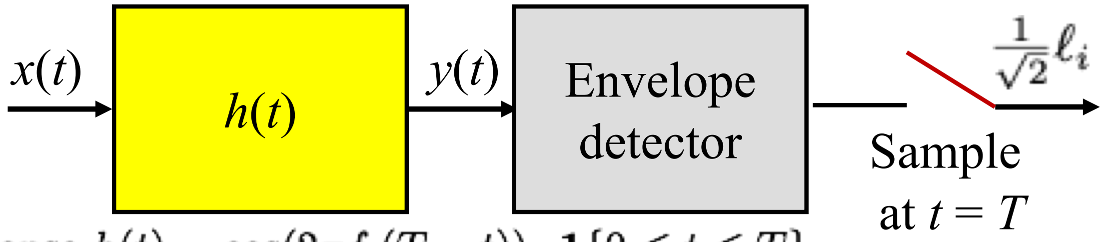
Detection of Signals with Unknown Phase

- Alternative realization of quadratic receiver
  - Quadrature receiver using matched filter



Detection of Signals with Unknown Phase

- Another alternative realization of quadratic receiver
  - Noncoherent matched filter



Impulse response $h(t) = \cos(2\pi f_i(T-t)) \cdot \mathbf{1}\{0 \le t \le T\}$

$$\begin{aligned} y(t) &= \int_{t-T}^{t} x(\tau)\cos(2\pi f_i(T-(t-\tau)))d\tau \\ &= \cos[2\pi f_i(T-t)]\int_{t-T}^{t} x(\tau)\cos(2\pi f_i\tau)d\tau - \sin[2\pi f_i(T-t)]\int_{t-T}^{t} x(\tau)\sin(2\pi f_i\tau)d\tau \\ &= \ell_i(t)\cdot\cos[2\pi f_i(T-t)+\beta_i(t)] \end{aligned}$$

Detection of Signals with Unknown Phase

- Envelope detector = squarer + lowpass filter + square-rooter

$$
\begin{aligned}
y^2(t) &= \ell_i^2(t)\cdot\cos^2[2\pi f_i(T-t)+\beta_i(t)] \\
&= \frac{1}{2}\ell_i^2(t)+\frac{1}{2}\ell_i^2(t)\cdot\cos[4\pi f_i(T-t)+2\beta_i(t)] \\
&\overset{\text{lowpass}}{\rightarrow} \frac{1}{2}\ell_i^2(t) \\
&\overset{\text{rooter}}{\rightarrow} \frac{1}{\sqrt{2}}\ell_i(t)
\end{aligned}
$$

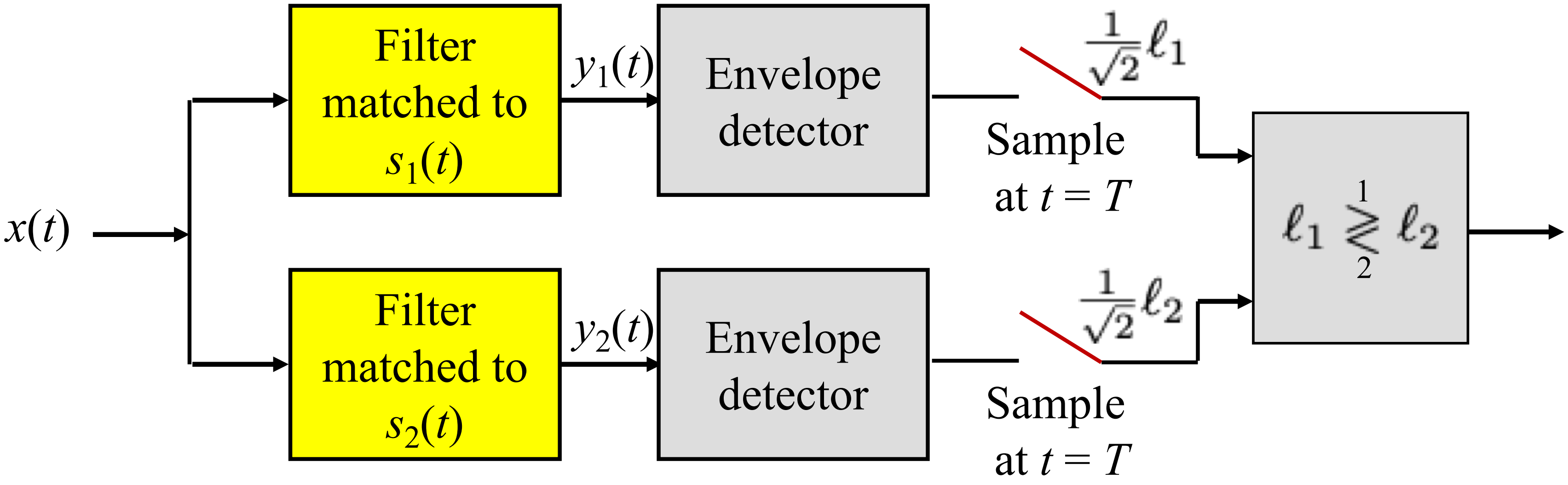
- Final note
  - The merit of noncoherent matched filter over quadratic receiver using matched filter is that the latter actually samples the output before the lowpass filter (i.e., high-frequency signal) while the former samples the output after the lowpass filter (i.e., true envelope signal). Hence, the latter has a much higher demand on the accuracy of sampling time.

Noncoherent Orthogonal Modulation

- Definition of noncoherent orthogonal modulation
  - The signals remain orthogonal and have the same energy regardless of the unknown carrier phase.
  - Example. Binary FSK introduced previously

Noncoherent Orthogonal Modulation

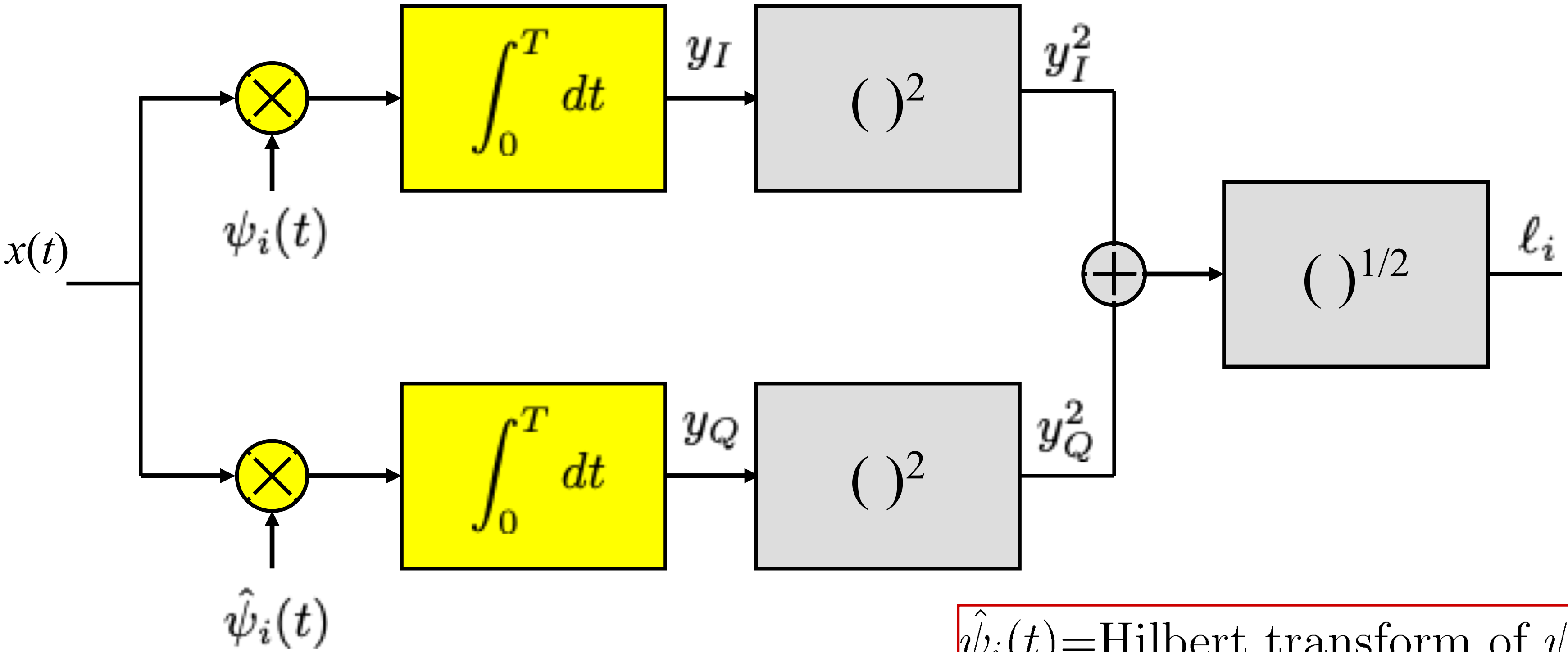
- Noncoherent matched filter



$s_1(t)$ and $s_2(t)$ are orthogonal.

Noncoherent Orthogonal Modulation

- Noncoherent matched filter = Quadrature receiver if $\psi_i(t)$ and $\hat{\psi}_i(t)$ are properly chosen.



$\hat{\psi}_i(t)$=Hilbert transform of $\psi_i(t)$

Error Rate of Noncoherent Receiver

- Error rate of noncoherent receiver for binary orthogonal modulated signals
  - Assume $s_1(t)$ is transmitted and θ is known.

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_i t + \theta)$$

$$\phi_i(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_i t)$$

$$\hat{\phi}_i(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_i t)$$

$$x(t) = s_i(t) + w(t)$$

$$\Rightarrow \begin{cases} x_{I,1} = \langle x(t), \phi_1(t)\rangle = \sqrt{E}\cos(\theta) + w_1 \\ x_{Q,1} = \langle x(t), \hat{\phi}_1(t)\rangle = \sqrt{E}\sin(\theta) + w_2 \\ x_{I,2} = \langle x(t), \phi_2(t)\rangle = w_3 \\ x_{Q,2} = \langle x(t), \hat{\phi}_2(t)\rangle = w_4 \end{cases}$$

$\{w_i\}$ i.i.d. zero-mean Gaussian with variance $N_0/2$

$$\ell_i = \left[\left(\underbrace{\int_0^T x(t)\cos(2\pi f_i t)dt}_{x_I}\right)^2 + \left(\underbrace{\int_0^T x(t)\sin(2\pi f_i t)dt}_{x_Q}\right)^2\right]^{1/2}$$

See Slide IDC2-70

Error Rate of Noncoherent Receiver

- Based on the decision rule $\ell_1^2 \underset{i=1}{\overset{i=2}{\lessgtr}} \ell_2^2$

$$
\begin{aligned}
&P(\text{Error}|s_1(t)\text{ transmitted})\\
&\quad= \Pr\left(\ell_1^2 \le \ell_2^2 \,|s_1(t)\text{ transmitted}\right)\\
&\quad= \Pr\left(x_{I,1}^2 + x_{Q,1}^2 \le x_{I,2}^2 + x_{Q,2}^2\right)\\
&\quad= \Pr\left(x_{I,1}^2 + x_{Q,1}^2 \le \ell_2^2\right)
\end{aligned}
$$

All four are independent.

$$
\begin{aligned}
x_{I,1} &\sim \mathcal{N}(\sqrt{E}\cos(\theta), N_0/2)\\
x_{Q,1} &\sim \mathcal{N}(\sqrt{E}\sin(\theta), N_0/2)\\
x_{I,2} &\sim \mathcal{N}(0, N_0/2)\\
x_{Q,2} &\sim \mathcal{N}(0, N_0/2)
\end{aligned}
$$

$$
\begin{aligned}
x_{I,1} &\sim \mathcal{N}(\sqrt{E}\cos(\theta), N_0/2)\\
x_{Q,1} &\sim \mathcal{N}(\sqrt{E}\sin(\theta), N_0/2)\\
\ell_2 &\sim \text{Rayleigh distributed with } E[\ell_2^2] = N_0
\end{aligned}
$$

(Cf. Section 1.12 in textbook or my slide 3-46)

$$
\begin{aligned}
&P(\text{Error}|s_1(t)\text{ transmitted})\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi N_0}e^{-[x_{I,1}-\sqrt{E}\cos(\theta)]^2/N_0}e^{-[x_{Q,1}-\sqrt{E}\sin(\theta)]^2/N_0}\left(\int_{\sqrt{x_{I,1}^2+x_{Q_1}^2}}^{\infty}\frac{2\ell_2}{N_0}e^{-\ell_2^2/N_0}d\ell_2\right)dx_{I,1}dx_{Q,1}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi N_0}e^{-[x_{I,1}-\sqrt{E}\cos(\theta)]^2/N_0}e^{-[x_{Q,1}-\sqrt{E}\sin(\theta)]^2/N_0}\left(e^{-[x_{I,1}^2+x_{Q_1}^2]/N_0}\right)dx_{I,1}dx_{Q,1}\\
&= \frac{1}{2}e^{-E/(2N_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi(N_0/2)}e^{-[x_{I,1}-\sqrt{E}\cos(\theta)/2]^2/(N_0/2)}e^{-[x_{Q,1}-\sqrt{E}\sin(\theta)/2]^2/(N_0/2)}dx_{I,1}dx_{Q,1}\\
&= \frac{1}{2}e^{-E/(2N_0)}.
\end{aligned}
$$

Similarly,

$$
P(\text{Error}|s_2(t)\text{ transmitted}) = \frac{1}{2}e^{-E/(2N_0)}.
$$

Consequently,

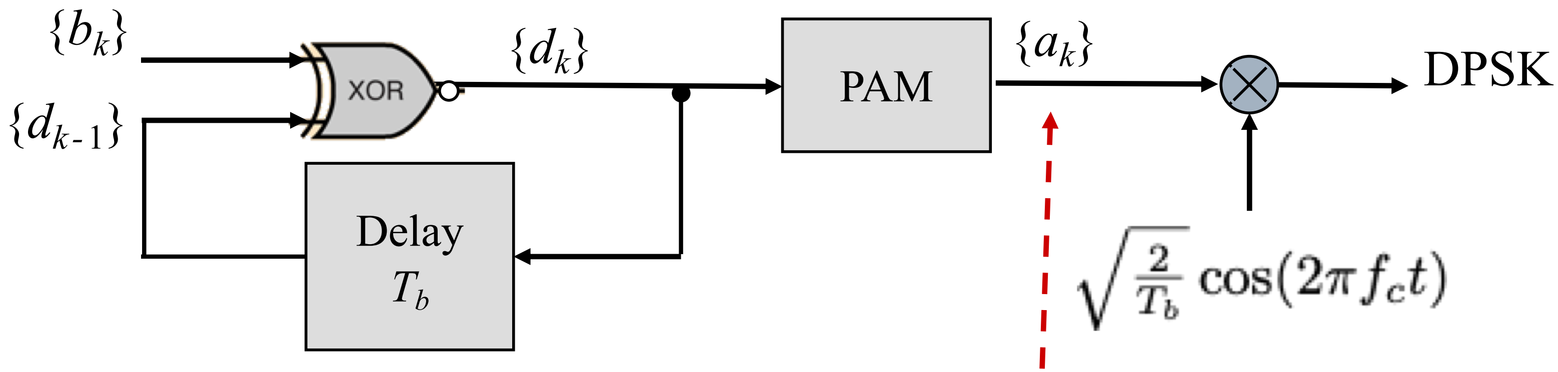
$$
P(\text{Error}) = \frac{1}{2}e^{-E/(2N_0)}.
$$

Note that the resulting error rate has nothing to do with θ.

Differential Phase-Shift Keying

- Transmitter of differential phase-shift keying

$$d_k = \overline{d_{k-1} \oplus b_k} \qquad a_k = 2d_k - 1$$

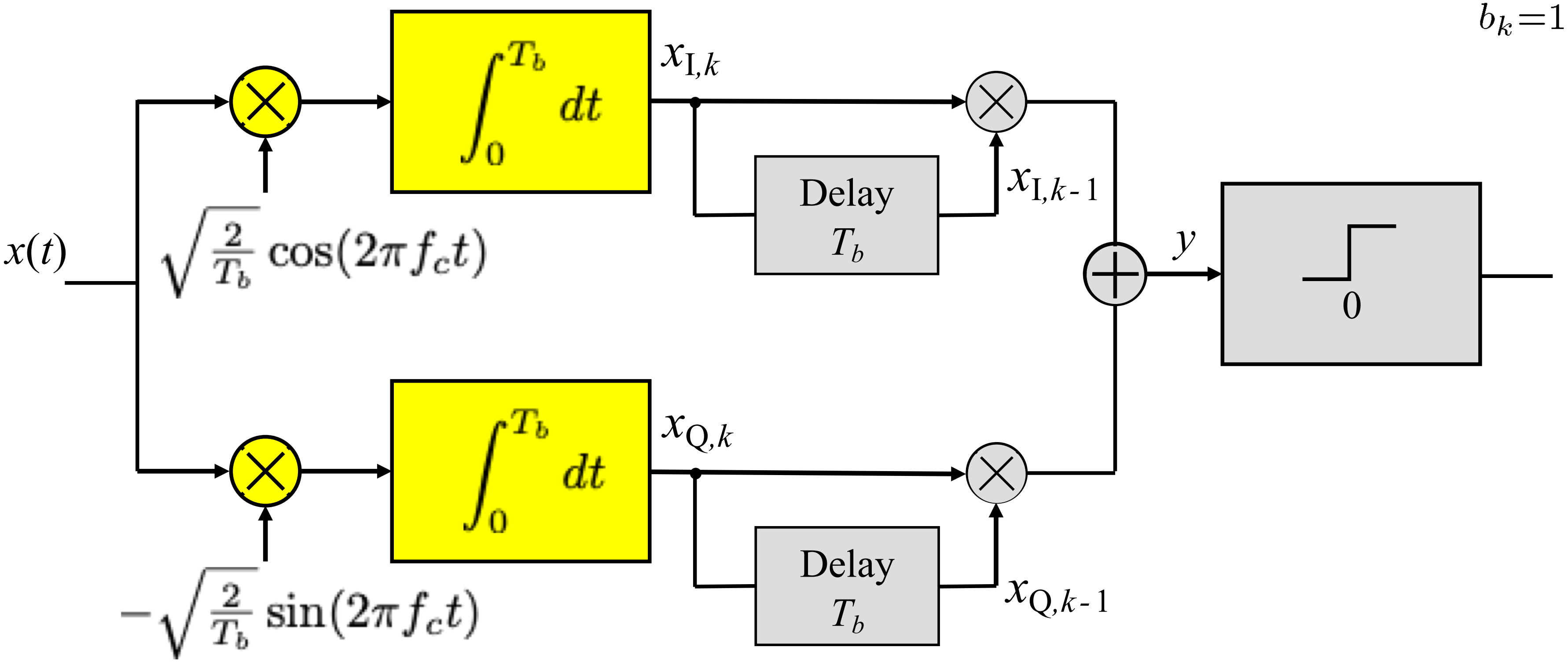


$$\text{Transmitted phase} = \begin{cases} 0, & d_k = 1 \\ \pi, & d_k = 0 \end{cases}$$

Differential Phase-Shift Keying

- Receiver of differential phase-shift keying

$$y = x_{I,k-1}x_{I,k} + x_{Q,k-1}x_{Q_k} \underset{b_k=1}{\overset{b_k=0}{\lessgtr}} 0$$



$$\boxed{\begin{aligned}
s(t) &= \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + (1-d_k)\pi + \theta) \text{ for } kT_b \le t < (k+1)T_b\\
\phi_1(t) &= \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)\\
\phi_2(t) &= -\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)
\end{aligned}}$$

$$x(t) = s(t) + w(t)$$

$$\Rightarrow \left\{\begin{aligned}
x_{I,k-1} &= \langle x(t), \phi_1(t)\rangle = \sqrt{E_b}\cos((1-d_{k-1})\pi + \theta) + w_{I,k-1}\\
&= \boxed{-(-1)^{d_{k-1}}\sqrt{E_b}\cos(\theta) + w_{I,k-1}}\\
x_{Q,k-1} &= \langle x(t), \phi_2(t)\rangle = \sqrt{E_b}\sin((1-d_{k-1})\pi + \theta) + w_{Q,k-1}\\
&= \boxed{-(-1)^{d_{k-1}}\sqrt{E_b}\sin(\theta) + w_{Q,k-1}}\\
x_{I,k} &= \langle x(t), \phi_1(t)\rangle = \sqrt{E_b}\cos((1-d_k)\pi + \theta) + w_{I,k}\\
&= \boxed{-(-1)^{d_k}\sqrt{E_b}\cos(\theta) + w_{I,k}}\\
x_{Q,k} &= \langle x(t), \phi_2(t)\rangle = \sqrt{E_b}\sin((1-d_k)\pi + \theta) + w_{Q,k}\\
&= \boxed{-(-1)^{d_k}\sqrt{E_b}\sin(\theta) + w_{Q,k}}
\end{aligned}\right.$$

$$\begin{cases} -(-1)^{d_{k-1}}\sqrt{E_b}\cos(\theta) = \boxed{s_I} \\ -(-1)^{d_{k-1}}\sqrt{E_b}\sin(\theta) = \boxed{s_Q} \\ -(-1)^{d_k}\sqrt{E_b}\cos(\theta) = \boxed{-(-1)^{b_k} s_I} \\ -(-1)^{d_k}\sqrt{E_b}\sin(\theta) = \boxed{-(-1)^{b_k} s_Q} \end{cases}$$

$$\boxed{s_I^2 + s_Q^2 = E_b}$$

$$d_k = \overline{d_{k-1} \oplus b_k}$$

$$\boxed{-(-1)^{d_k} = (-1)^{d_{k-1}}(-1)^{b_k}}$$

All four are independent.

$$\begin{aligned} P(\text{Error}|b_k = 1) &= \Pr\left(x_{I,k-1}x_{I,k} + x_{Q,k-1}x_{Q,k} \le 0\right) \\ &= \Pr\left(u_I^2 + u_Q^2 \le v_I^2 + v_Q^2\right) \end{aligned}$$

$$\begin{aligned} x_{I,k-1} &\sim \mathcal{N}(s_I, N_0/2) \\ x_{Q,k-1} &\sim \mathcal{N}(s_Q, N_0/2) \\ x_{I,k} &\sim \mathcal{N}(s_I, N_0/2) \\ x_{Q,k} &\sim \mathcal{N}(s_Q, N_0/2) \end{aligned}$$

$$\begin{aligned} u_I - 2s_I &\sim \mathcal{N}(0, N_0) \\ u_Q - 2s_Q &\sim \mathcal{N}(0, N_0) \\ v_I &\sim \mathcal{N}(0, N_0) \\ v_Q &\sim \mathcal{N}(0, N_0) \end{aligned}$$

$$\boxed{\begin{cases} u_I = x_{I,k-1} + x_{I,k} \\ u_Q = x_{Q,k-1} + x_{Q,k} \\ v_I = x_{I,k-1} - x_{I,k} \\ v_Q = x_{Q,k-1} - x_{Q,k} \end{cases}}$$

All four are **also** independent.

$$
\begin{aligned}
& P(\text{Error}|b_k = 1) \\
&= \Pr\left(u_I^2 + u_Q^2 \le \ell_2^2\right) \quad \begin{array}{l} u_I \sim \mathcal{N}(2s_I, N_0) \\ u_Q \sim \mathcal{N}(2s_Q, N_0) \\ \ell_2 \sim \text{Rayleigh distributed with } E[\ell_2^2] = 2N_0 \end{array} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi N_0} e^{-(u_I-2s_I)^2/(2N_0)} e^{-(u_Q-2s_I)^2/(2N_0)} \left(\int_{\sqrt{u_I^2+u_Q^2}}^{\infty} \frac{\ell_2}{N_0} e^{-\ell_2^2/(2N_0)} d\ell_2\right) du_I du_Q \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi N_0} e^{-(u_I-2s_I)^2/(2N_0)} e^{-(u_Q-2s_I)^2/(2N_0)} \left(e^{-(u_I^2+u_Q^2)/(2N_0)}\right) du_I du_Q \\
&= \frac{1}{2} e^{-(s_I^2+s_Q^2)/N_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi N_0} e^{-(u_I-s_I)^2/N_0} e^{-(u_Q-s_Q)^2/N_0} du_I du_Q \\
&= \frac{1}{2} e^{-E_b/N_0}.
\end{aligned}
$$

Similarly,

$$P(\text{Error}|b_k = 0) = \frac{1}{2} e^{-E_b/N_0}.$$

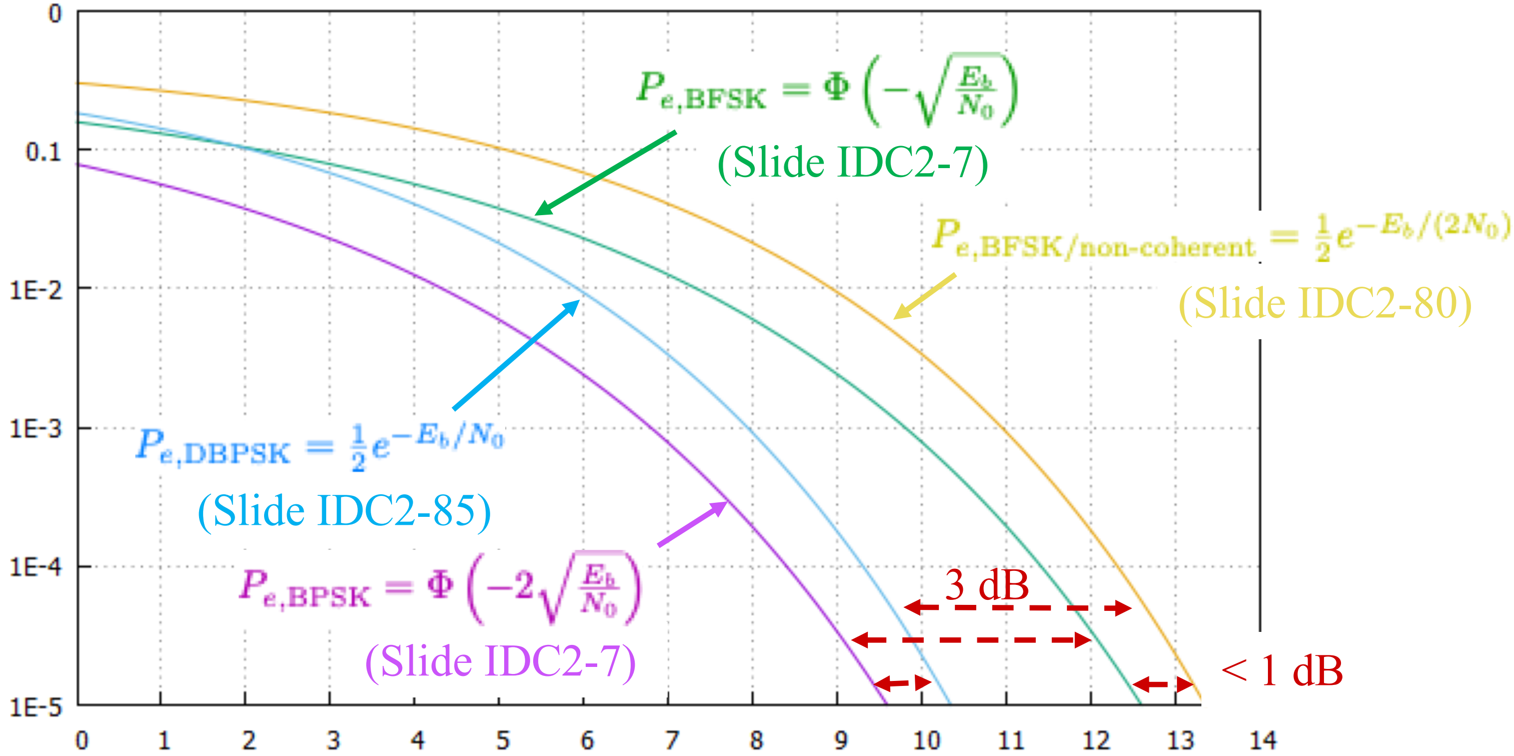
Consequently,

$$P(\text{Error}) = \frac{1}{2} e^{-E_b/N_0}.$$

Note that the resulting error rate has nothing to do with θ.

Comparison of Digital Modulation Schemes

- The performance degradation from coherent to noncoherent counterpart is less than 1 dB ($10^{1/10}=1.259$).



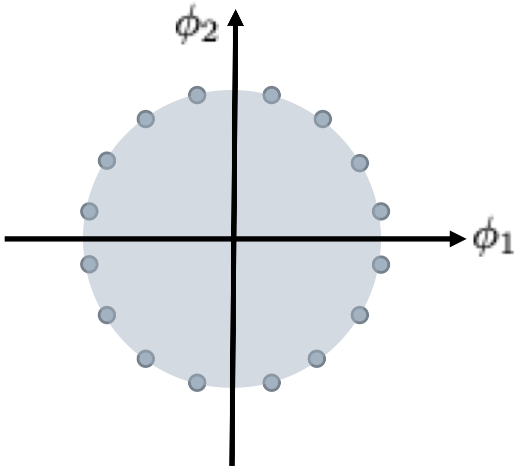
Comparison of Digital Modulation Schemes

- The power-bandwidth requirement of M-ary PSK with respect to binary PSK
  - $M = 4$ offers the best tradeoff between power and bandwidth requirement, which explains why QPSK is widely used in practice.

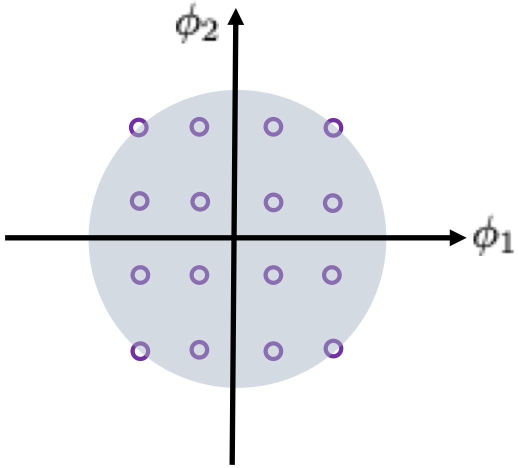
	Value of M	$\dfrac{(\text{Bandwidth})_{M-\text{ary}}}{(\text{Bandwidth})_{\text{Binary}}}$		$\dfrac{(\text{SNR required for SER}=10^{-4})_{M-\text{ary}}}{(\text{SNR required for SER}=10^{-4})_{\text{Binary}}}$	
Under the same T_b	4	0.5	(Slide IDC1-49)	0.34 dB	(Slide IDC1-43)
	8	0.333	(Slide IDC 1-63)	3.91 dB	(Slide IDC1-65)
	16	0.25	(Slide IDC 1-63)	8.52 dB	(Slide IDC1-65)
	32	0.2	(Slide IDC 1-63)	13.52 dB	(Slide IDC1-65)

Comparison of Digital Modulation Schemes

- Comparison between M-ary PSK and M-ary QAM
  - M-ary QAM outperforms M-ary PSK as can be easily seen from the below constellations.
  - However, M-ary QAM requires higher linearity in, e.g., power amplifier design.



16-ary PSK



16-ary QAM