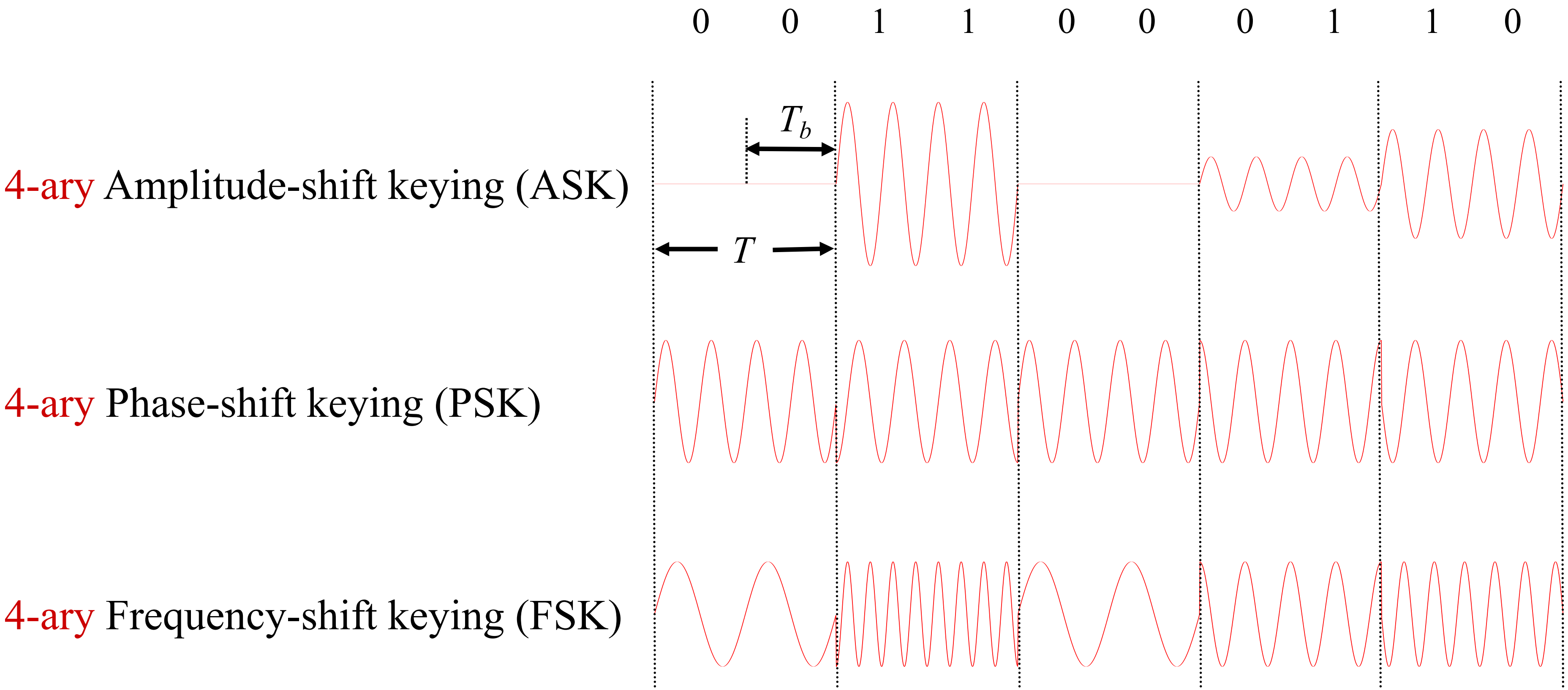
Part 1 Passband Data Transmission Model, PSK and CAP

Passband Data Transmission deals with the Transmission of the Digital Data over the real-valued Passband channel.

Categories of Digital Communications (ASK/PSK/FSK)

- Three basic signaling schemes in *M*-ary digital communications



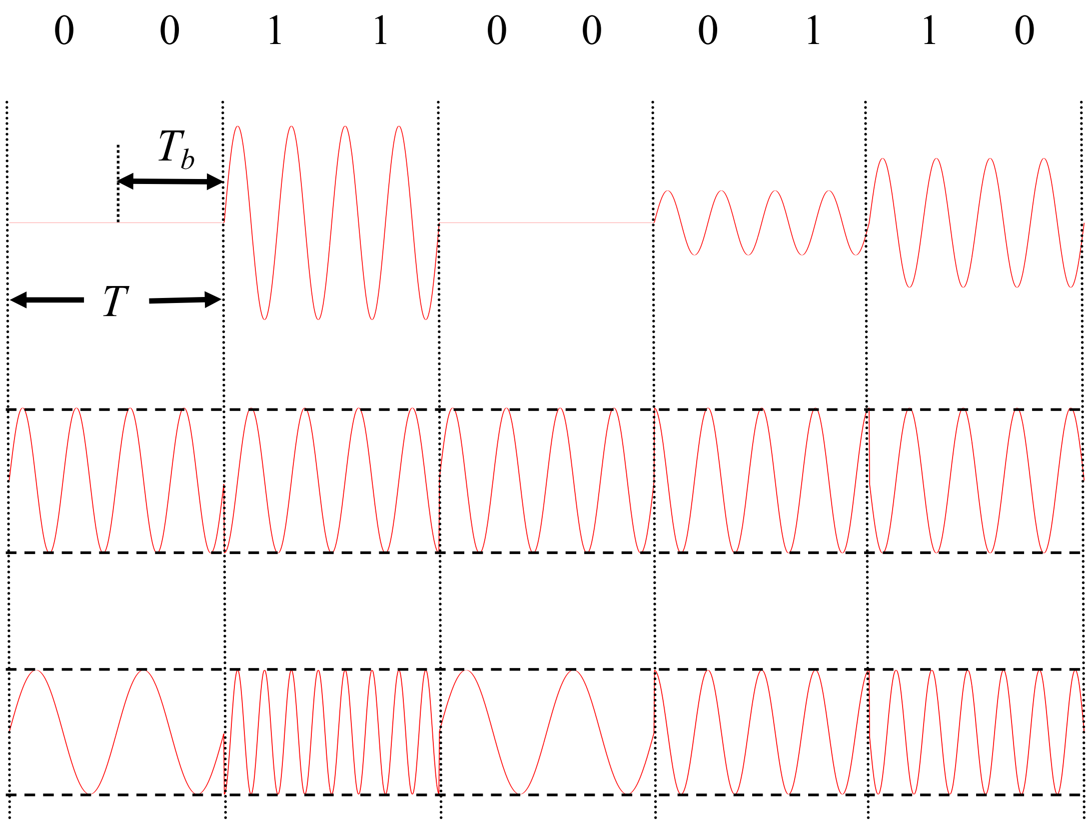
Categories of *M*-ary Digital Communications (Constant Envelope versus Non-Constant Envelope)

- Constant envelope: A necessity for non-linear channels

Non-constant envelope (ASK)

Constant envelope (PSK)

Constant evnelope (FSK)



Categories of Digital Communications (Coherent versus Non-Coherent)

- Coherent technique
  - The transmitter and receiver are required to be synchronized in both carrier phase and bit timing.
- Non-Coherent technique
  - The transmitter and receiver are **not** required to be synchronized in both carrier phase and bit timing.

Roadmap

- We will focus on three factors:
  - Power : A resource in communication
    - Power Spectra
      - The relation between passband signal and baseband signal is easier to identify in spectra view
  - Bandwidth: Another resource in communication
    - Bandwidth efficiency : The ratio of data rate in bits per second to the effectively utilized bandwidth. (Bits/Second/Hz)
  - Probability of M-ary symbol error (Union bound)

Relation between Passband and Baseband Signals

- Math relation between passband and baseband signals (spectrum view)

$$\tilde{s}(t) = x(t) + jy(t) \quad \text{(Complex) Baseband signal}$$

$$s(t) \quad \text{(Real) Passband signal}$$

$$\Rightarrow s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

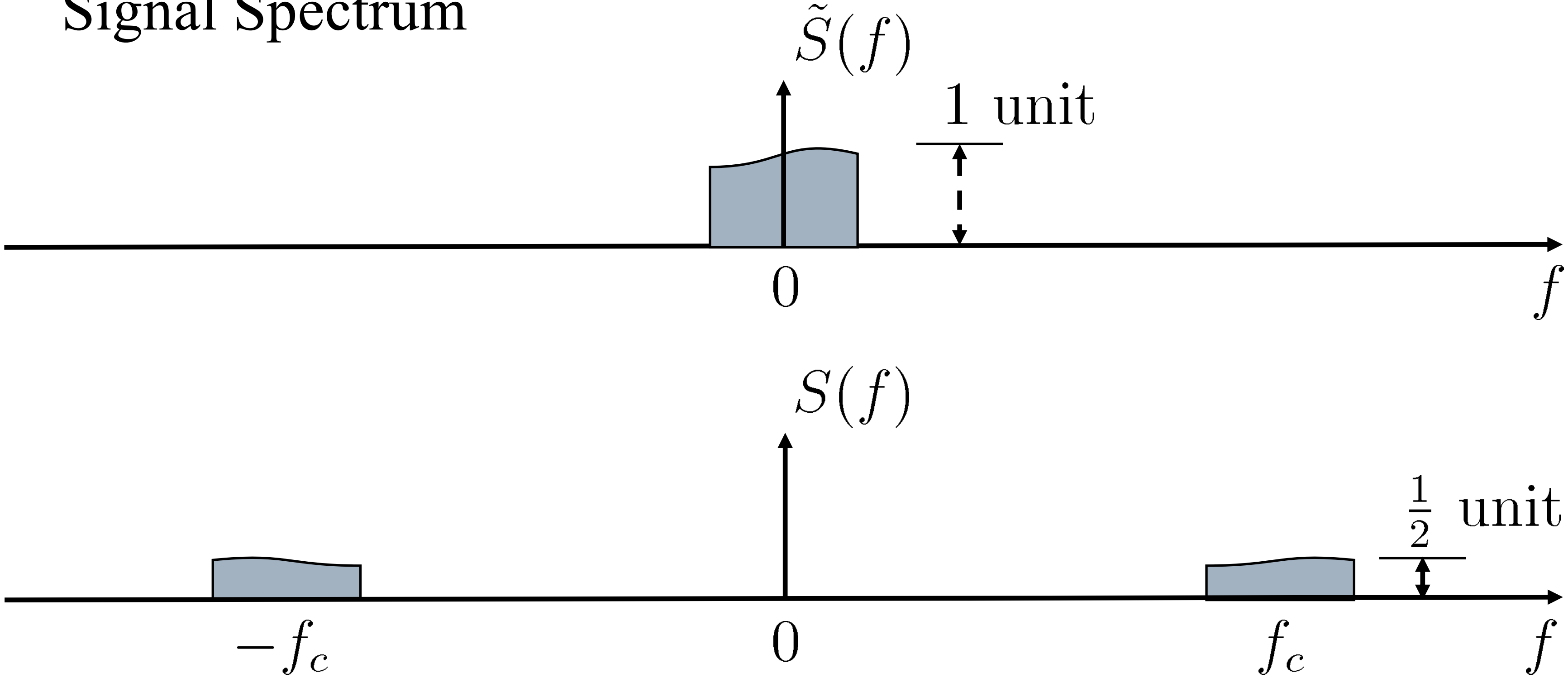
$$= \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\}$$

Relation between Passband and Baseband Signals

$$\begin{aligned}
S(f) &= \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \left\{\mathrm{Re}\left[\tilde{s}(t)e^{j2\pi f_c t}\right]\right\} e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \left\{\frac{1}{2}\left[\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}^*(t)e^{-j2\pi f_c t}\right]\right\} e^{-j2\pi ft}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \tilde{s}(t)e^{-j2\pi(f-f_c)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} \tilde{s}^*(t)e^{-j2\pi(f+f_c)t}dt \\
&= \frac{1}{2}\left[\tilde{S}(f-f_c) + \tilde{S}^*(-f-f_c)\right]
\end{aligned}$$

Relation between Passband and Baseband Signals

Signal Spectrum



Relation between Passband and Baseband Signals

- Math relation between passband and baseband signals (power spectrum view subject to wise-sense stationarity)

$\tilde{s}(t) = x(t) + jy(t)$ (Complex) WSS Baseband signal

$s(t)$ (Real) WSS Passband signal

$$\begin{aligned} \Rightarrow s(t) &= x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t) \\ &= \text{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \end{aligned}$$

Relation between Passband and Baseband Signals

- Let

$$\begin{cases} R_{xx}(\tau) = E[x(t+\tau)x(t)] \\ R_{xy}(\tau) = E[x(t+\tau)y(t)] \\ R_{yx}(\tau) = E[y(t+\tau)x(t)] \\ R_{yy}(\tau) = E[y(t+\tau)y(t)] \end{cases}$$

(Here, we assume they are all WSS.)

- That $s(t)$ is WSS implies

$$\boxed{R_{ss}(\tau) = E[s(t+\tau)s(t)] \text{ is irrelevant to } t.}$$

Relation between Passband and Baseband Signals

$$
\begin{aligned}
R_{ss}(\tau) &= E\Big[\big(x(t+\tau)\cos(2\pi f_c(t+\tau)) - y(t+\tau)\sin(2\pi f_c(t+\tau)\big) \\
&\qquad \big(x(t)\cos(2\pi f_c t) \qquad\qquad - y(t)\sin(2\pi f_c t)\big)\Big] \\
&= R_{xx}(\tau)\ \cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_{yy}(\tau)\ \sin(2\pi f_c(t+\tau)\sin(2\pi f_c t) \\
&\quad -R_{xy}(\tau)\ \cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) - R_{yx}(\tau)\ \sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&= R_{xx}(\tau)\ \frac{\cos(2\pi f_c\tau)+\cos(2\pi f_c(2t+\tau))}{2} + R_{yy}(\tau)\ \frac{\cos(2\pi f_c\tau)-\cos(2\pi f_c(2t+\tau))}{2} \\
&\quad -R_{xy}(\tau)\ \frac{\sin(2\pi f_c(2t+\tau))-\sin(2\pi f_c\tau)}{2} - R_{yx}(\tau)\ \frac{\sin(2\pi f_c(2t+\tau))+\sin(2\pi f_c\tau)}{2} \\
&= \frac{1}{2}[R_{xx}(\tau)+R_{yy}(\tau)]\cos(2\pi f_c\tau) + \frac{1}{2}[R_{xx}(\tau)-R_{yy}(\tau)]\cos(2\pi f_c(2t+\tau)) \\
&\quad +\frac{1}{2}[R_{xy}(\tau)-R_{yx}(\tau)]\sin(2\pi f_c\tau) - \frac{1}{2}[R_{xy}(\tau)+R_{yx}(\tau)]\sin(2\pi f_c(2t+\tau))
\end{aligned}
$$

Then, $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$.

Relation between Passband and Baseband Signals

$$
\begin{aligned}
R_{\tilde{s}\tilde{s}}(\tau) &= E[(x(t+\tau)+jy(t+\tau))(x(t)+jy(t))^*] \\
&= R_{xx}(\tau)+R_{yy}(\tau)+jR_{yx}(\tau)-jR_{xy}(\tau) \\
&= 2[R_{xx}(\tau)+jR_{yx}(\tau)]
\end{aligned}
$$

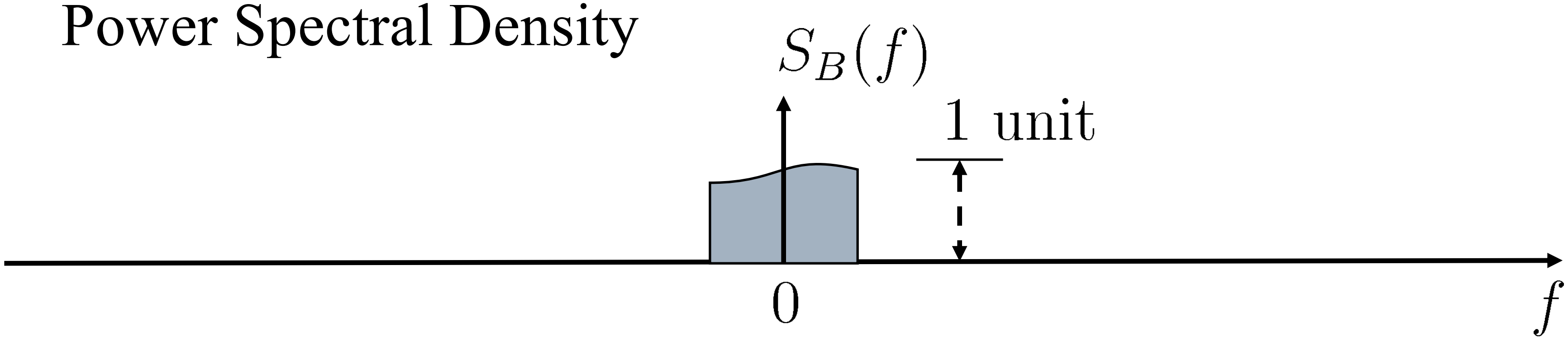
$$
\begin{aligned}
R_{ss}(\tau) &= R_{xx}(\tau)\cos(2\pi f_c\tau)-R_{yx}(\tau)\sin(2\pi f_c\tau) \\
&= \mathrm{Re}\left\{[R_{xx}(\tau)+jR_{yx}(\tau)]\,e^{j2\pi f_c\tau}\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}\right\}
\end{aligned}
$$

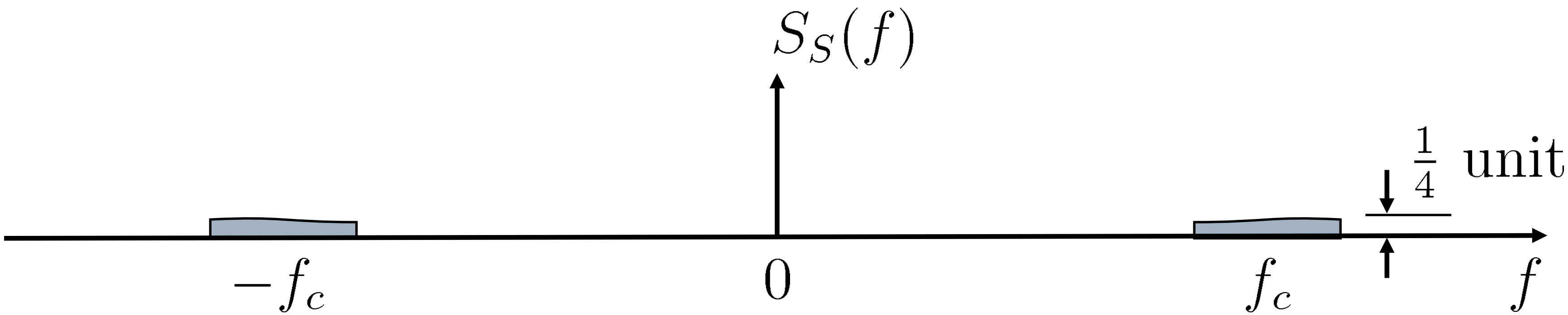
Relation between Passband and Baseband Signals

$$
\begin{aligned}
\boxed{S_S(f)} &= \int_{-\infty}^{\infty} R_{ss}(\tau)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty} \left\{\frac{1}{2}\text{Re}\left[R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}\right]\right\} e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty} \left\{\frac{1}{4}\left[R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau} + R^*_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi f_c\tau}\right]\right\} e^{-j2\pi f\tau}d\tau \\
&= \frac{1}{4}\int_{-\infty}^{\infty} R_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi(f-f_c)\tau}d\tau + \frac{1}{4}\int_{-\infty}^{\infty} R^*_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi(f+f_c)\tau}d\tau \\
&\boxed{= \frac{1}{4}\left[S_B(f-f_c) + S^*_B(-f-f_c)\right]}
\end{aligned}
$$

Relation between Passband and Baseband Signals

Power Spectral Density





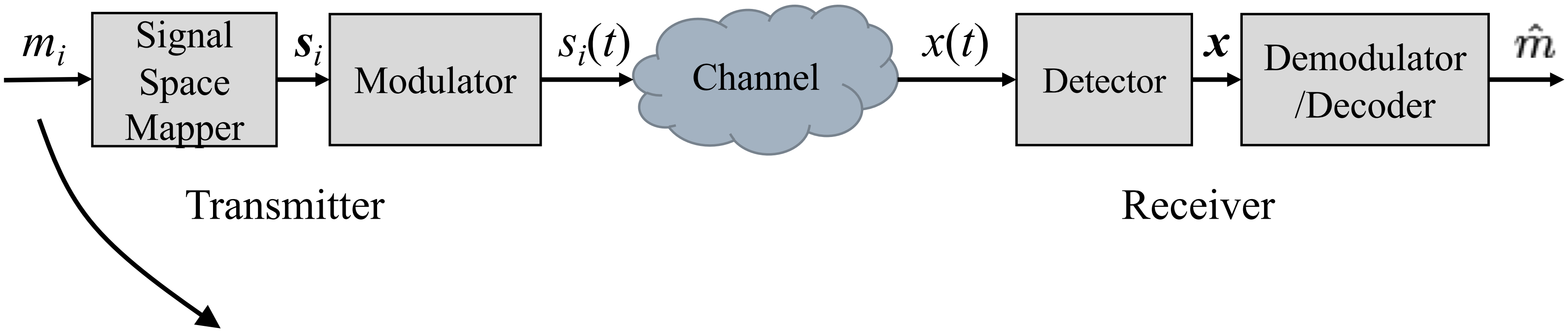
Since $R^*(\tau) = (E[a(t+\tau)a^*(t)])^* = E[a(t)a^*(t+\tau)] = R(-\tau)$, the power spectral density is always real.

Relation between Passband and Baseband Signals

- Integration of the Power Spectral Density gives the Power.
  - Integration of the Prabability Density gives the Prabability.

$$
\begin{aligned}
R_{ss}(0) &= \int_{-\infty}^{\infty} S_S(f)df \\
&= \frac{1}{4}\left[\int_{-\infty}^{\infty} S_B(f-f_c)df + \int_{-\infty}^{\infty} S_B(-f-f_c)df\right] \\
&= \frac{1}{4}\left[R_{\tilde{s}\tilde{s}}(0) + R_{\tilde{s}\tilde{s}}(0)\right] = \frac{1}{2}R_{\tilde{s}\tilde{s}}(0)
\end{aligned}
$$

(Real-valued) Passband Model – Message Source

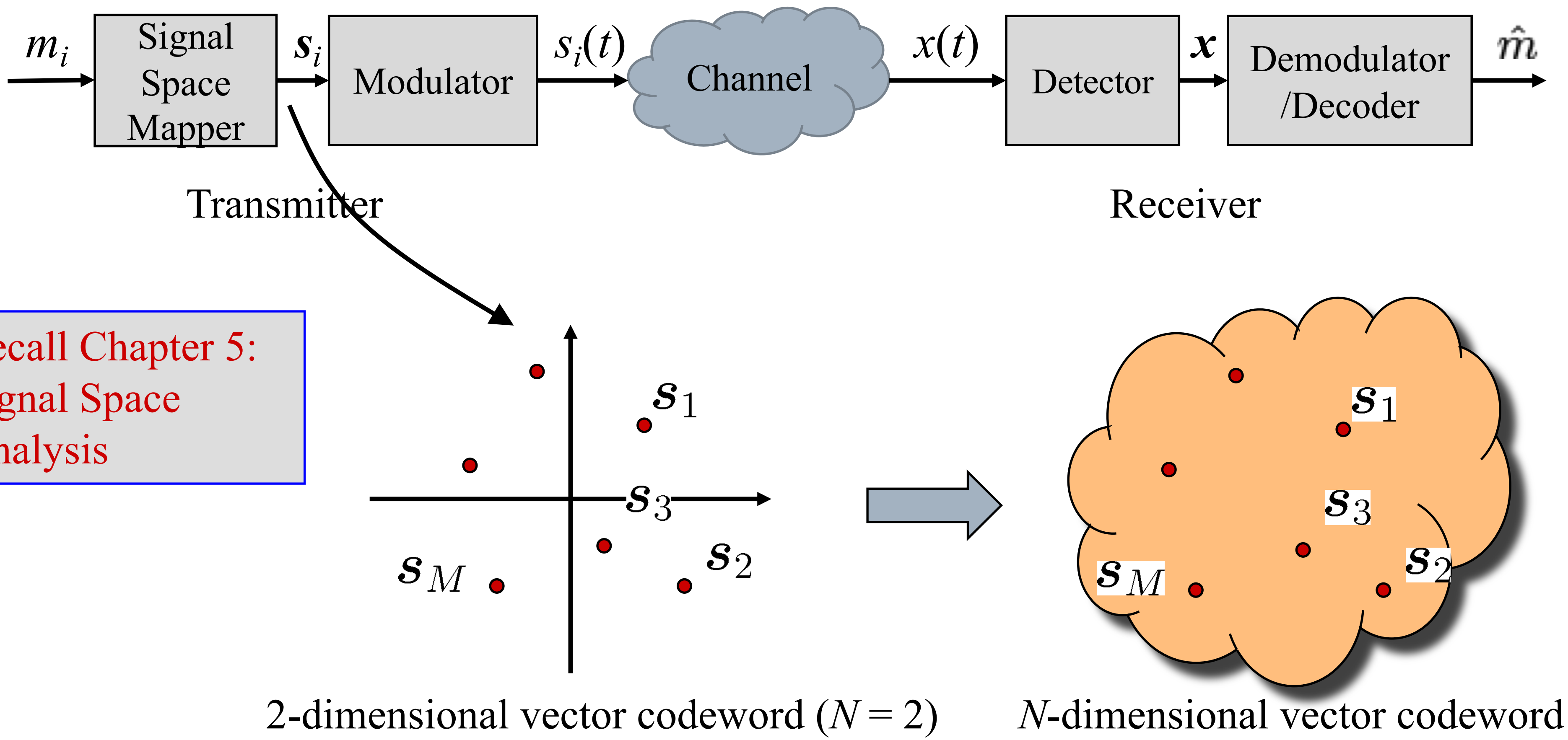


$$m_i \in \{m_1, m_2, \cdots, m_M\}$$

prior probability $p_i = P(m_i)$

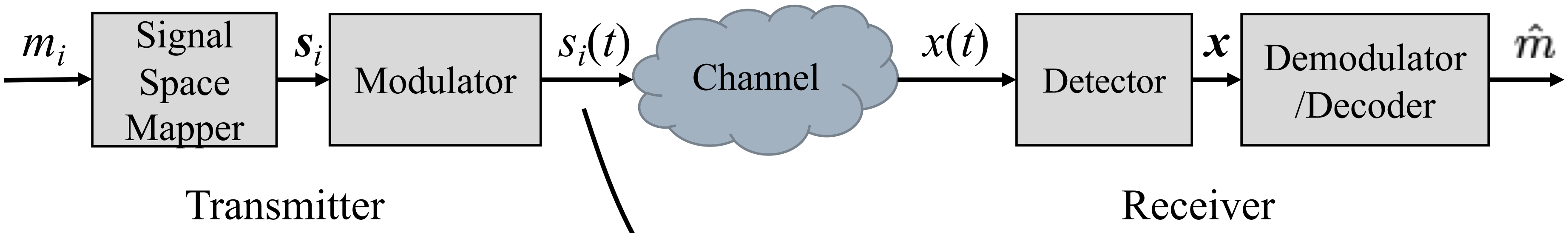
equal prior $p_i = \frac{1}{M}$

(Real-valued) Passband Model – Signal Space Analysis



Recall Chapter 5: Signal Space Analysis

(Real-valued) Passband Model – Signal Space Analysis



Recall Chapter 5: Signal Space Analysis

$$\boldsymbol{s}_i = \begin{bmatrix} s_{i,1} \\ s_{i,2} \\ \vdots \\ s_{i,N} \end{bmatrix} \Rightarrow s_i(t) = \sum_{k=1}^{N} s_{i,k} \cdot \phi_k(t)$$

$\{\phi_k(t)\}_{k=1}^{N}$ is an orthonormal basis.

(Real-valued) Passband Model – Signal Space Analysis

- $s_i(t)$ is an (finite) energy signal of duration T.
  - What is an energy signal?

    Define the inner product of two signals $f(t)$ and $g(t)$ as

    $$\langle f(t), g(t) \rangle = \int_0^T f(t)g(t)dt.$$

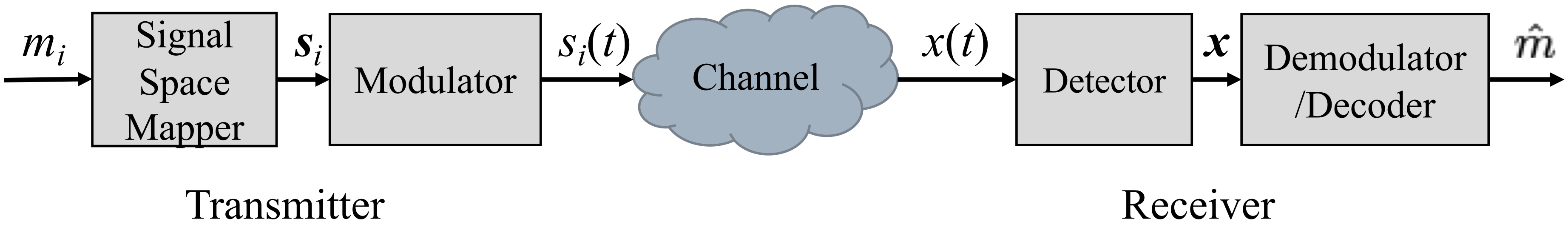
    Then

    $$\begin{aligned} \text{energy of signal } s_i(t) &= \langle s_i(t), s_i(t) \rangle \quad \left(= \|s_i(t)\|^2 \right) \\ &= \int_0^T s_i^2(t)dt < \infty \end{aligned}$$

(Real-valued) Passband Model – Signal Space Analysis

$$
\begin{aligned}
\langle s_i(t), s_i(t)\rangle &= \left\langle \sum_{k=1}^{N} s_{i,k}\cdot\phi_k(t), \sum_{\ell=1}^{N} s_{i,\ell}\cdot\phi_\ell(t)\right\rangle \\
&= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle s_{i,k}\cdot\phi_k(t), s_{i,\ell}\cdot\phi_\ell(t)\rangle \\
&= \sum_{k=1}^{N}\sum_{\ell=1}^{N} s_{i,k}s_{i,\ell}\langle \phi_k(t), \phi_\ell(t)\rangle \\
&= \sum_{k=1}^{N} s_{i,k}^2
\end{aligned}
$$

(Real-valued) Passband Model – Signal Space Analysis

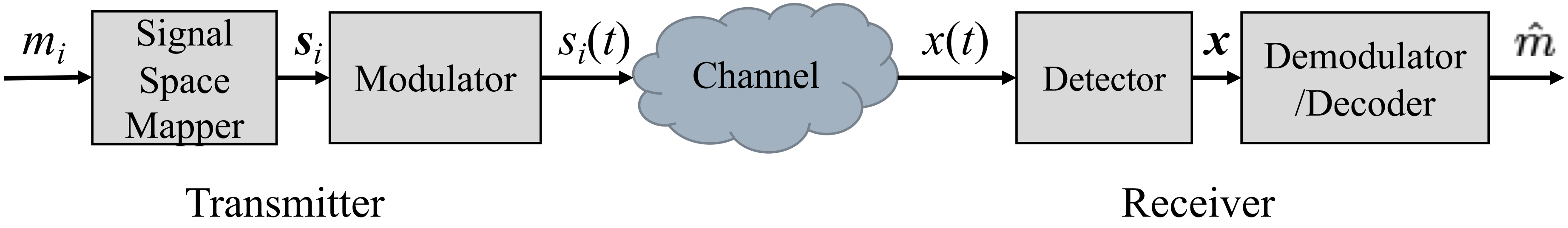


- Example of orthonormal baiss $\{\phi_k(t)\}_{k=1}^{N}$ with $N = 2$ for ASK/PSK signals

$$\left\{\sqrt{\frac{2}{T}}\cos(2\pi f_c t), \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\right\}$$

with T being a multiple of $\dfrac{1}{f_c}$

(Real-valued) Passband Model – Signal Space Analysis

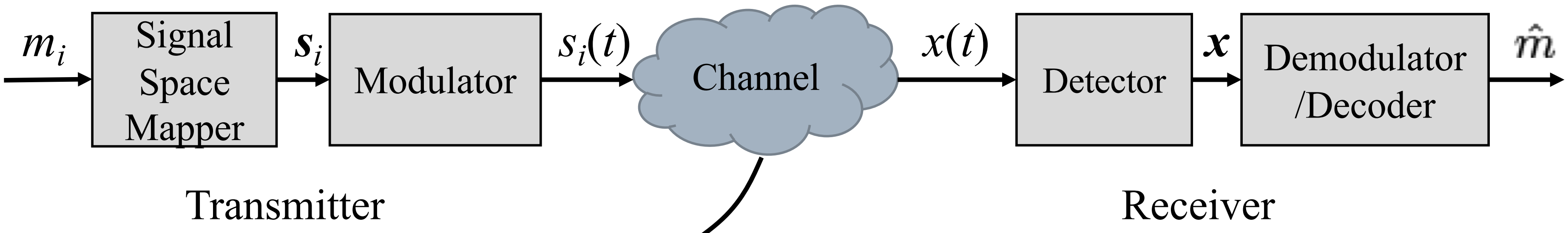


- Example of orthonormal baiss $\{\phi_k(t)\}_{k=1}^N$ with $N = 2$ for FSK signals

$$\left\{\sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c - \frac{1}{2T}\right)t\right), \sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c + \frac{1}{2T}\right)t\right)\right\}$$

with T being a multiple of $\dfrac{1}{f_c}$

(Real-valued) Passband Model – Channel



- Channel
  - Linear: Principle of superposition

    $$s_1(t) \mapsto x_1(t) \text{ and } s_2(t) \mapsto x_2(t) \Rightarrow as_1(t) + bs_2(t) \mapsto ax_1(t) + bx_2(t)$$

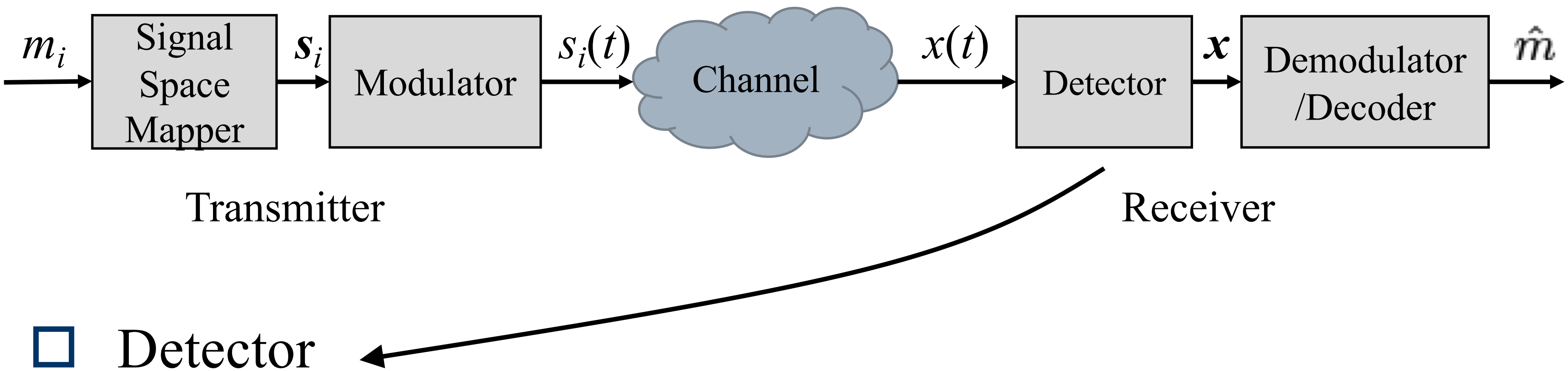
  - Sufficient bandwidth

    No loss of power in $s_i(t)$

  - AWGN

    $x(t) = s_i(t) + n(t)$, where $n(t)$ is a zero-mean white Gaussisn process with two-sided power spectrum density $N_0/2$

(Real-valued) Passband Model – Detector

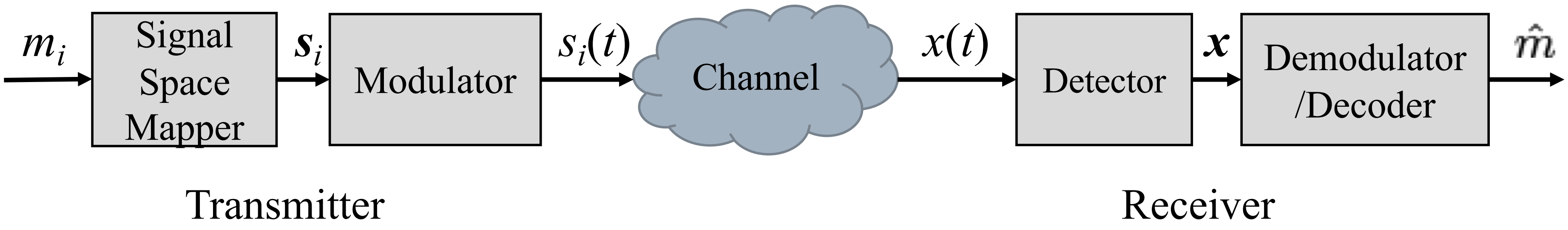


- Detector

$$x(t) = \sum_{k=1}^{N} x_k \cdot \phi_k(t) + \phi'(t) \quad \Rightarrow \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\left\langle \phi'(t), \sum_{k=1}^{N} x_k \cdot \phi_k(t) \right\rangle = 0$$

(Real-valued) Passband Model – Detector

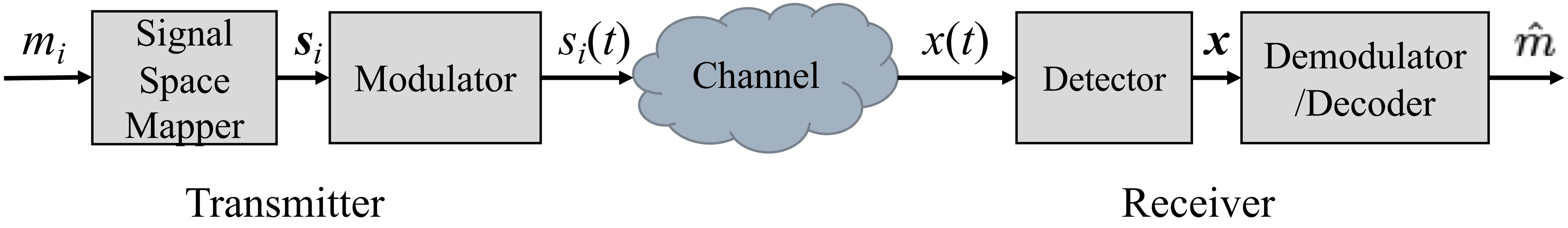


- Demodulator/Decoder

  $\hat{m}$ is the most probable transmitted message in $\{m_1, m_2, \cdots, m_M\}$ given $\boldsymbol{x}$.

$$\hat{m} = \underbrace{\arg \max_{1 \le i \le M} P(m_i | \boldsymbol{x})}_{\substack{\text{maximum a posteriori} \\ \text{(MAP)}}} = \arg \max_{1 \le i \le M} \frac{P(m_i)}{P(\boldsymbol{x})} P(\boldsymbol{x} | m_i) = \underbrace{\arg \max_{1 \le i \le M} P(\boldsymbol{x} | m_i)}_{\text{maximum likelihood (ML)}}$$

Coherent Phase-Shift Keying (PSK) – Antipodal Signaling



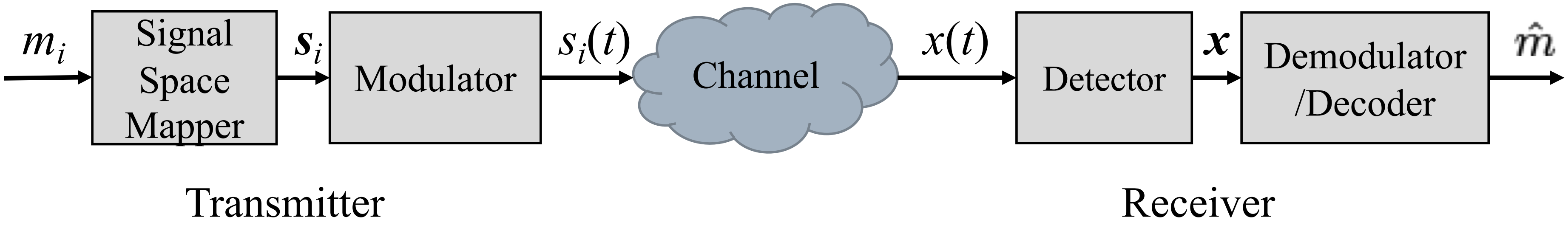
- Binary PSK

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

for $0 \le t < T_b$, where T_b is a multiple of $1/f_c$.

Coherent Phase-Shift Keying (PSK) – Antipodal Signaling



- Vector space analysis of binary PSK
  - Antipodal signal

$$s_1(t) = +\sqrt{E_b}\cdot\phi_1(t) \qquad s_{11} = \langle s_1(t), \phi_1(t)\rangle = +\sqrt{E_b}$$

$$s_2(t) = -\sqrt{E_b}\cdot\phi_1(t) \qquad s_{12} = \langle s_2(t), \phi_1(t)\rangle = -\sqrt{E_b}$$

where $\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$.

Coherent Phase-Shift Keying (PSK) – Antipodal Signaling

- Error probability of binary PSK

$$
\begin{aligned}
& x(t) = s(t) + w(t) \\
\Rightarrow\ & \langle x(t), \phi_1(t)\rangle = \langle s(t), \phi_1(t)\rangle + \langle w(t), \phi_1(t)\rangle \\
\Rightarrow\ & x = \pm\sqrt{E_b} + w \\
\Rightarrow\ & \hat{m} = \arg\max\left\{P\left(x \,\middle|\, -\sqrt{E_b}\right), P\left(x \,\middle|\, +\sqrt{E_b}\right)\right\} \\
\Rightarrow\ & \hat{m} = \arg\max\left\{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+\sqrt{E_b})^2/2\sigma^2}, \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\sqrt{E_b})^2/2\sigma^2}\right\} \\
\Rightarrow\ & x \underset{+\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0
\end{aligned}
$$

$\sigma^2 = N_0/2$ is the variance of w

$$\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Recall:

$$\begin{aligned}
\sigma^2 = E[w^2] &= E\left[\langle w(t), \phi_1(t)\rangle^2\right] \\
&= E\left[\int_0^{T_b}\int_0^{T_b} w(t)\phi_1(t)\cdot w(s)\phi_1(s)dtds\right] \\
&= \int_0^{T_b}\int_0^{T_b} E[w(t)w(s)]\phi_1(t)\phi_1(s)dtds \\
&= \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-s)\phi_1(t)\phi_1(s)dtds \\
&= \frac{N_0}{2}\int_0^{T_b}\phi_1(t)\phi_1(t)dt \\
&= \frac{N_0}{2}\langle\phi_1(t), \phi_1(t)\rangle = \frac{N_0}{2}
\end{aligned}$$

Coherent Phase-Shift Keying (PSK) – Error probability

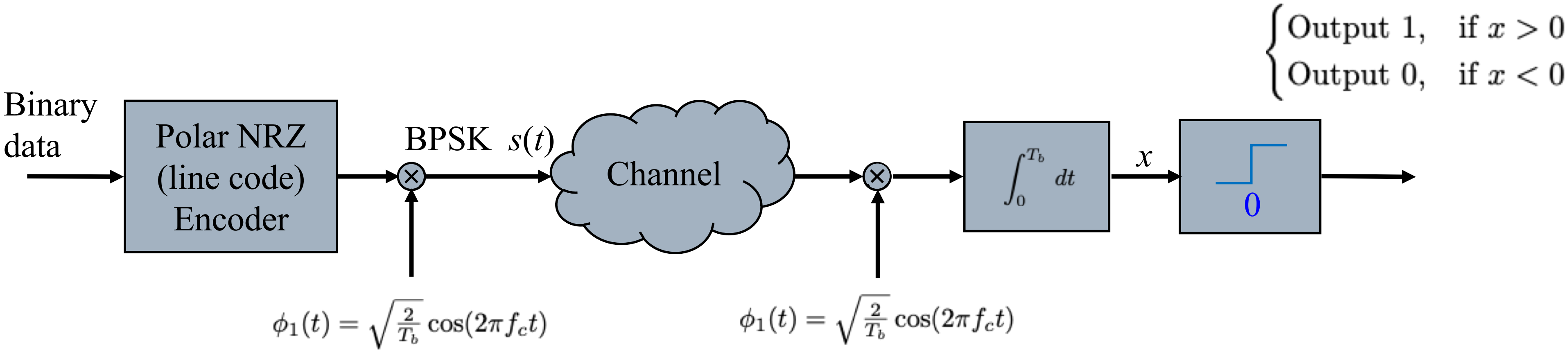
- Error probability of binary PSK
  - Based on the decision rule $x \underset{+\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0$

$$
\begin{aligned}
P(\text{Error}) &= P\left(-\sqrt{E_b}\text{ transmitted}\right) P\left(x > 0 \,\middle|\, -\sqrt{E_b}\text{ transmitted}\right) \\
&\quad + P\left(+\sqrt{E_b}\text{ transmitted}\right) P\left(x < 0 \,\middle|\, +\sqrt{E_b}\text{ transmitted}\right) \\
&= \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+\sqrt{E_b})^2/2\sigma^2} dx + \frac{1}{2}\int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2} dx \\
&= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2} dx = \Phi\left(\frac{0-\sqrt{E_b}}{\sigma}\right) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}
$$

$$
\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)
$$

Coherent Phase-Shift Keying (PSK) – Block diagram

- Block diagram for PSK transmitter and (coherent) receiver



Coherent Phase-Shift Keying (PSK) – Baseband Signal

- Complex-valued baseband signal corresponding to the real-valued BPSK passband signal

$$
\begin{aligned}
s(t) &= \pm\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) \\
&= \text{Re}\left\{\left(\pm\sqrt{\frac{2E_b}{T_b}}\right)e^{j2\pi f_c t}\right\} \\
&= \text{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \\
\Rightarrow \tilde{s}(t) &= \pm\sqrt{\frac{2E_b}{T_b}} \text{ for } 0 \le t < T_b
\end{aligned}
$$

Coherent Phase-Shift Keying (PSK) – Sequential Baseband Signal

- Sequence of complex baseband signals
  - No autocorrelation function for one-shot single random variable
  - Calculation of autocorrelation function requires a random process.

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} I_k \cdot g(t - kT_b),$$

where $I_k = \pm 1$ with equal probability, and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

Coherent Phase-Shift Keying (PSK) – Autocorrelation Function

$$\begin{aligned}
R_{\tilde{s}\tilde{s}}(t+\tau, t) &= E\left[\left(\sum_{k=-\infty}^{\infty} I_k \cdot g(t+\tau-kT_b)\right)\left(\sum_{\ell=-\infty}^{\infty} I_\ell \cdot g(t-\ell T_b)\right)^*\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} E[I_k I_\ell^*] g(t+\tau-kT_b) g^*(t-\ell T_b) \\
&= \sum_{k=-\infty}^{\infty} g(t+\tau-kT_b) g^*(t-kT_b)
\end{aligned}$$

$$
\begin{aligned}
\boxed{\bar{S}_B(f)} &= \int_{-\infty}^{\infty} \bar{R}_{\tilde{s}\tilde{s}}(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T_b} \int_0^{T_b} R_{\tilde{s}\tilde{s}}(t+\tau, t) dt \right) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T_b} \int_0^{T_b} \sum_{k=-\infty}^{\infty} g(t+\tau-kT_b) g^*(t-kT_b) dt \right) e^{-j2\pi f\tau} d\tau \\
&= \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \int_0^{T_b} \left(\int_{-\infty}^{\infty} g(t+\tau-kT_b) e^{-j2\pi f\tau} d\tau \right) g^*(t-kT_b) dt \quad (s = t+\tau-kT_b) \\
&= \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \int_0^{T_b} \left(\int_{-\infty}^{\infty} g(s) e^{-j2\pi f(s-t+kT_b)} d\tau \right) g^*(t-kT_b) dt \\
&= \frac{1}{T_b} G(f) \sum_{k=-\infty}^{\infty} \int_0^{T_b} g^*(t-kT_b) e^{j2\pi f(t-kT_b)} dt \quad (u = t-kT_b) \\
&= \frac{1}{T_b} G(f) \sum_{k=-\infty}^{\infty} \int_{-kT_b}^{(1-k)T_b} g^*(u) e^{j2\pi fu} du \\
&= \frac{1}{T_b} G(f) \int_{-\infty}^{\infty} g^*(u) e^{j2\pi fu} du \\
&= \frac{1}{T_b} G(f) \left(\int_{-\infty}^{\infty} g(u) e^{-j2\pi fu} du \right)^* = \frac{1}{T_b} G(f) G^*(f) = \boxed{\frac{1}{T_b} |G(f)|^2}
\end{aligned}
$$

Coherent Phase-Shift Keying (PSK) – Autocorrelation Function

- PSD of sequence BPSK

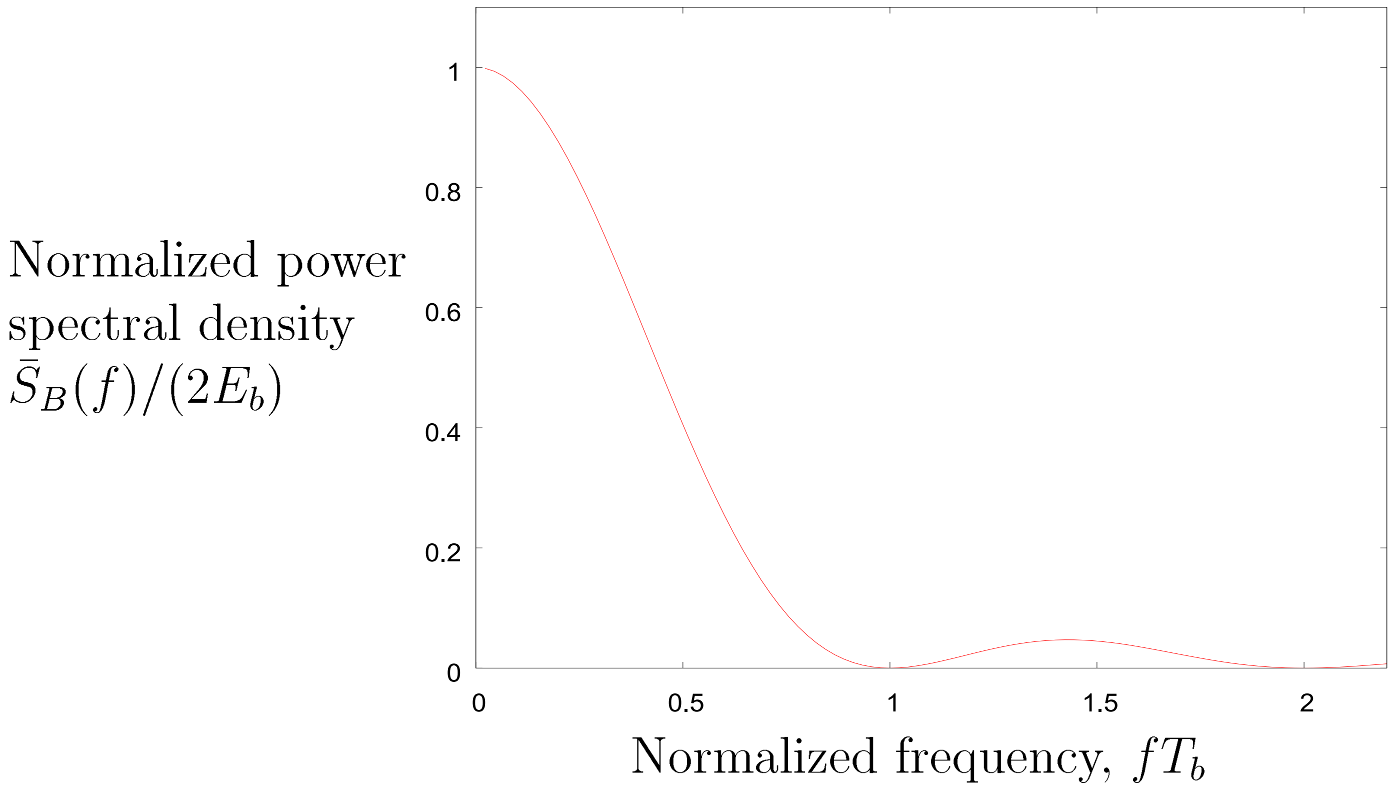
$$G(f) = \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} e^{-j2\pi f t} dt = \sqrt{\frac{2E_b}{T_b}} T_b \, \mathrm{sinc}(T_b f) \, e^{-j\pi f T_b}$$

$$\Rightarrow \boxed{\bar{S}_B(f) = \frac{1}{T_b} |G(f)|^2} = \frac{1}{T_b} \frac{2E_b}{T_b} T_b^2 \, \mathrm{sinc}^2(T_b f)$$

$$= 2E_b \, \mathrm{sinc}^2(T_b f)$$

Coherent Phase-Shift Keying (PSK) – Autocorrelation Function

- PSD of sequence BPSK



Coherent Phase-Shift Keying (PSK) – Quadrature PSK

- QPSK

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right], & 0 \le t < T \\ 0, & \text{elsewhere} \end{cases}$$

where $i = 1, 2, 3, 4$, f_c is a multiple of $1/T$,

E is the transmitted energy per QPSK symbol, and

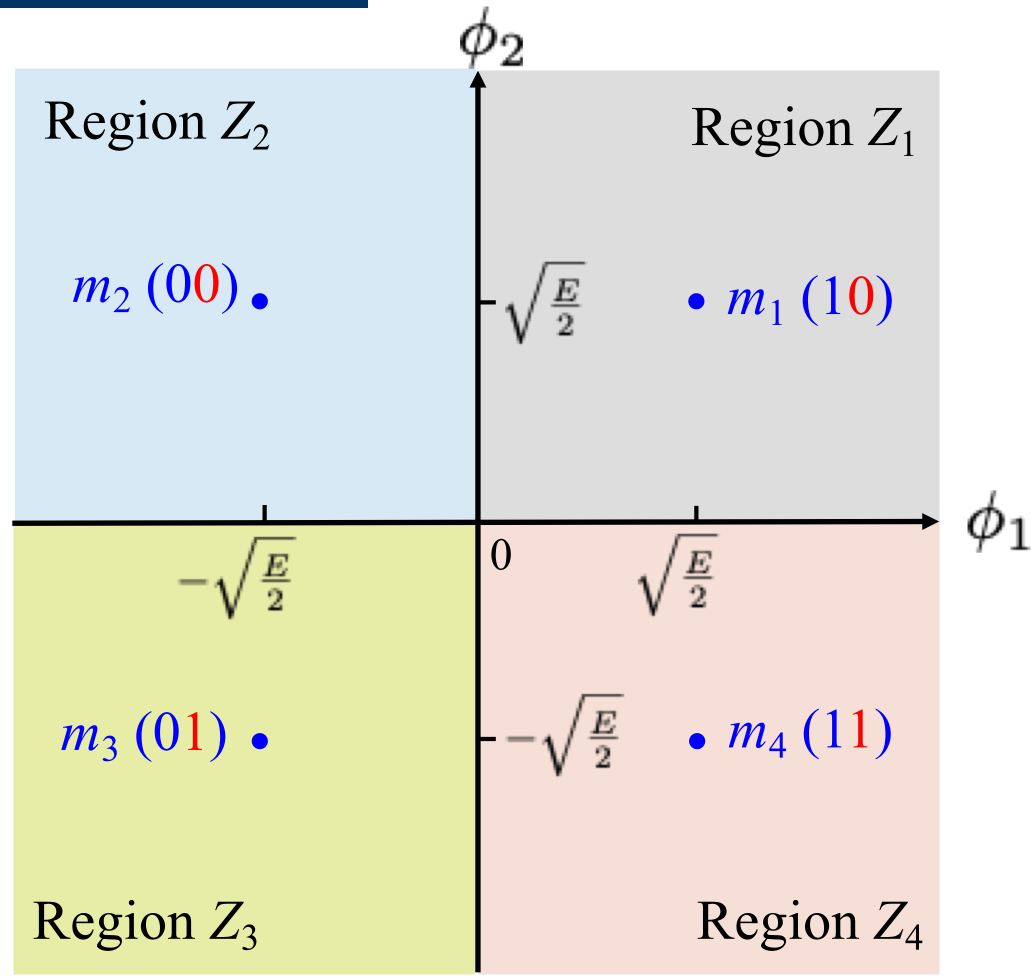
T is the symbol duration.

- Vector space analysis of QPSK

$$\boldsymbol{s}_i = \begin{bmatrix} \sqrt{E}\cos((2i-1)\frac{\pi}{4}) \\ \sqrt{E}\sin((2i-1)\frac{\pi}{4}) \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

Coherent Phase-Shift Keying (PSK) – Quadrature PSK

- Two-dimensional signal space diagram of QPSK



Coherent Phase-Shift Keying (PSK) – Quadrature PSK

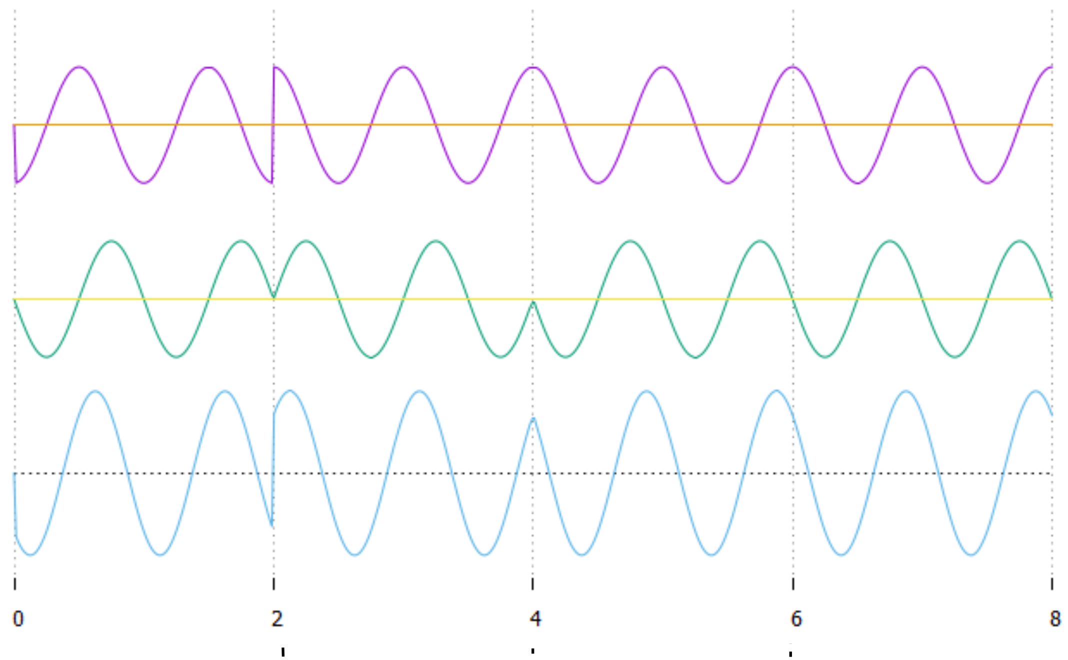
- Example ($E = T = 2$ and $f_c = 1$)

Input data stream	0	0	1	1	1	0	1	0
$(s_{i1},\ s_{i2})$	-1	$+1$	$+1$	-1	$+1$	$+1$	$+1$	$+1$

$$\phi_1(t) = \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$



Error probability of QPSK

$$x(t) = s(t) + w(t)$$

$$\Rightarrow \begin{cases} \langle x(t), \phi_1(t)\rangle = \langle s(t), \phi_1(t)\rangle + \langle w(t), \phi_1(t)\rangle \\ \langle x(t), \phi_2(t)\rangle = \langle s(t), \phi_2(t)\rangle + \langle w(t), \phi_2(t)\rangle \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \pm\sqrt{\frac{E}{2}} + w_1 \\ x_2 = \pm\sqrt{\frac{E}{2}} + w_2 \end{cases}$$

$$\Rightarrow \hat{m} = \arg\max\left\{P\left(x_1, x_2 \middle| \pm\sqrt{E/2}, \pm\sqrt{E/2}\right)\right\}$$

$$\Rightarrow \hat{m} = \arg\max\left\{\frac{1}{2\pi\sigma^2}e^{-[(x_1\mp\sqrt{E/2})^2+(x_2\mp\sqrt{E/2})^2]/(2\sigma^2)}\right\}$$

$$\Rightarrow x_1 \underset{+\sqrt{E/2}}{\overset{-\sqrt{E/2}}{\lessgtr}} 0 \text{ and } x_2 \underset{+\sqrt{E/2}}{\overset{-\sqrt{E/2}}{\lessgtr}} 0$$

$$\sigma^2 = N_0/2$$

Coherent Phase-Shift Keying (PSK) – Error Probability of QPSK

- Following the same derivation as that in Slide IDC1-30

$$\Pr(s_1 \text{ error}) = \Phi\left(-\sqrt{2\frac{E/2}{N_0}}\right) = \Phi\left(-\sqrt{\frac{E}{N_0}}\right)$$

$$\Pr(s_2 \text{ error}) = \Phi\left(-\sqrt{2\frac{E/2}{N_0}}\right) = \Phi\left(-\sqrt{\frac{E}{N_0}}\right)$$

Since $E = 2E_b$,

$$\text{Bit Error Rate} = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)$$

if s_1 and s_2 respectively decide one information bit as indicated in Slide IDC1-41.

$$\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Coherent Phase-Shift Keying (PSK) – Error Probability of QPSK

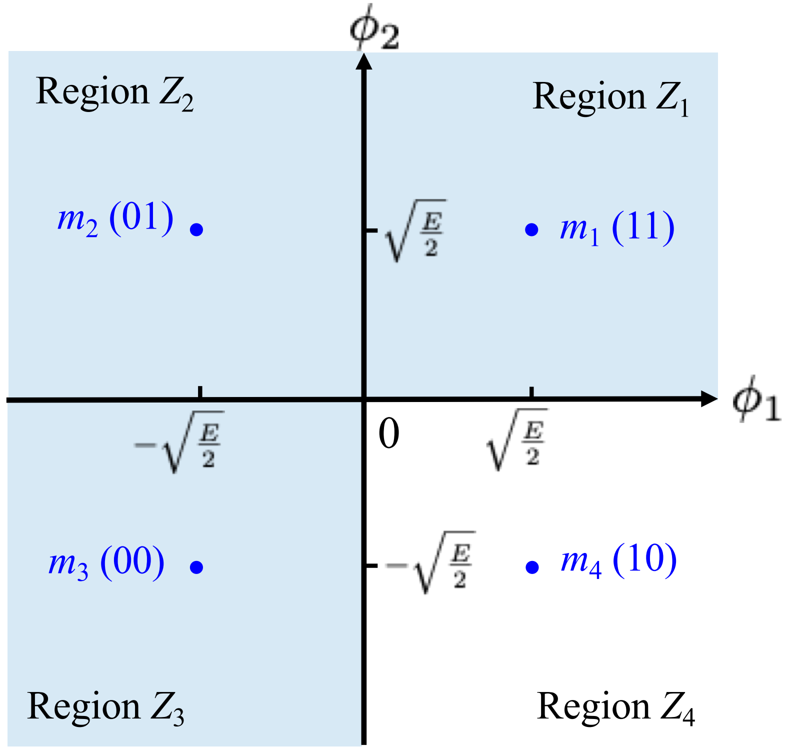
- Symbol error rate of QPSK

$$
\begin{aligned}
&\Pr(\text{Symbol Error}) \\
&\quad = 1 - \Pr(\text{Symbol Correct}) \\
&\quad = 1 - \Pr(s_1 \text{ Correct}) \cdot \Pr(s_2 \text{ Correct}) \\
&\qquad (\text{Because the noises affecting the two decisions are independent}) \\
&\quad = 1 - \left[1 - \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\right]^2 \\
&\quad = 2\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right) - \left[\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\right]^2
\end{aligned}
$$

Coherent Phase-Shift Keying (PSK) – Error Probability of QPSK

- Alternative approach to derive the symbol error rate of QPSK

$$\begin{aligned}
&\Pr(\text{Symbol Error}|s_1 = \sqrt{E/2}, s_2 = -\sqrt{E/2}) \\
&= \int_{\substack{\text{shaded}\\ \text{area}}} p(x_1, x_2|\sqrt{E/2}, -\sqrt{E/2})dx_1dx_2 \\
&= \int_{\substack{\text{shaded}\\ \text{area}}} \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1-\sqrt{E/2})^2+(x_2+\sqrt{E/2})^2}{2\sigma^2}} dx_1dx_2 \\
&= \cdots \text{(omit)} \\
&= 2\,\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right) - \Phi^2\left(-\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}$$

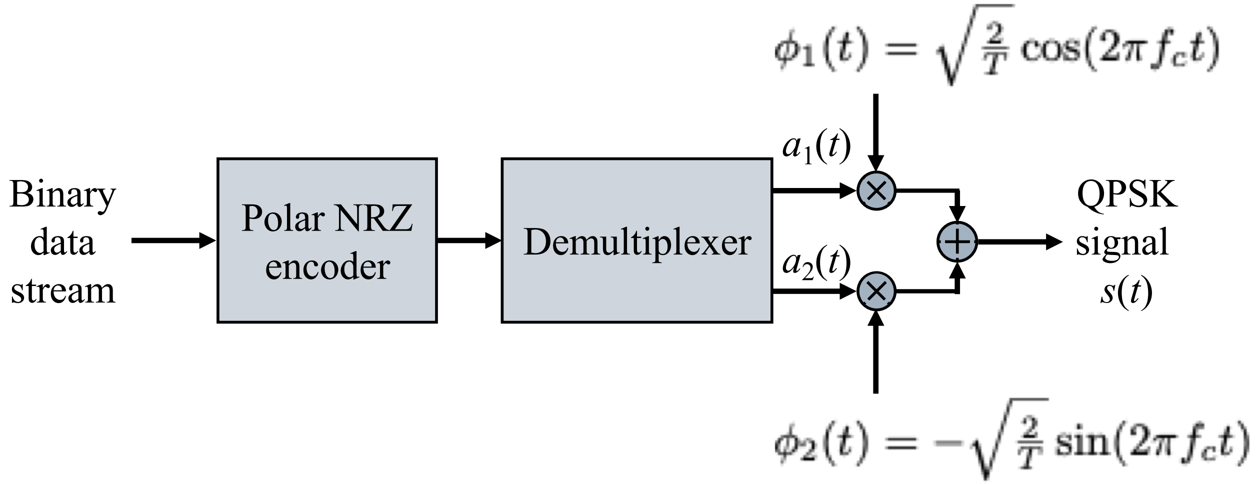


Coherent Phase-Shift Keying (PSK) – Partial Summary

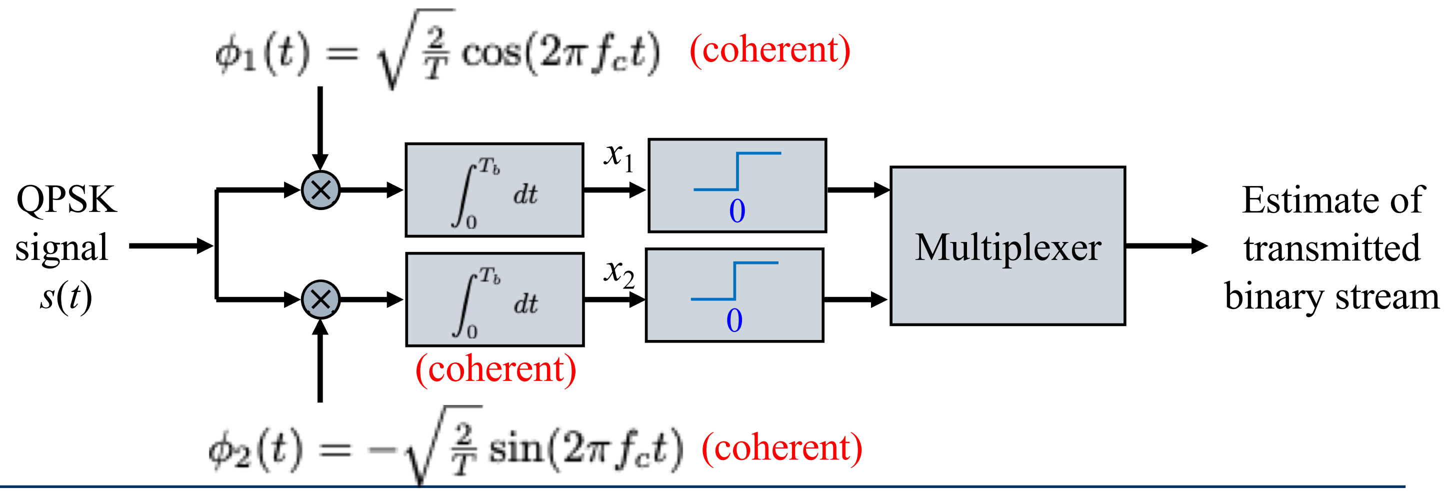
- Partial summary
  - QPSK (with Gray code mapping) and BPSK have the same BER under the same E_b/N_0.
  - QPSK, however, doubles the transmission bit rate per second (or uses half the bandwidth under the same bit rate) by introducing another quadrature.
  - In its implementation, QPSK is more complex since it involves two quadratures.

Coherent Phase-Shift Keying (PSK) – Block Diagram

Transmitter



Receiver



Coherent Phase-Shift Keying (PSK) – Sequential Baseband Signal

- Sequence of complex baseband signals
  - No autocorrelation function of one-shot (namely, single) random variable.
  - Calculation of autocorrelation function requires a random process.

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} e^{j[(\pi/2)I_k + \pi/4]} g(t - kT),$$

where $I_k = 0, 1, 2, 3$ with equal prob., and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E}{T}}, & 0 \le t < T = 2T_b \\ 0, & \text{otherwise} \end{cases}$$

Coherent Phase-Shift Keying (PSK) – Sequential Baseband Signal

$$
\begin{aligned}
& R_{\tilde{s}\tilde{s}}(t+\tau, t) \\
= \; & E\left[\left(\sum_{k=-\infty}^{\infty} e^{j[(\pi/2)I_k+\pi/4]} g(t+\tau-kT)\right)\left(\sum_{\ell=-\infty}^{\infty} e^{j[(\pi/2)I_\ell+\pi/4]} g(t-\ell T)\right)^*\right] \\
= \; & \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} g(t+\tau-kT)g(t-\ell T)E\left[e^{j(\pi/2)(I_k-I_\ell)}\right] \\
= \; & \sum_{k=-\infty}^{\infty} g(t+\tau-kT)g(t-kT)
\end{aligned}
$$

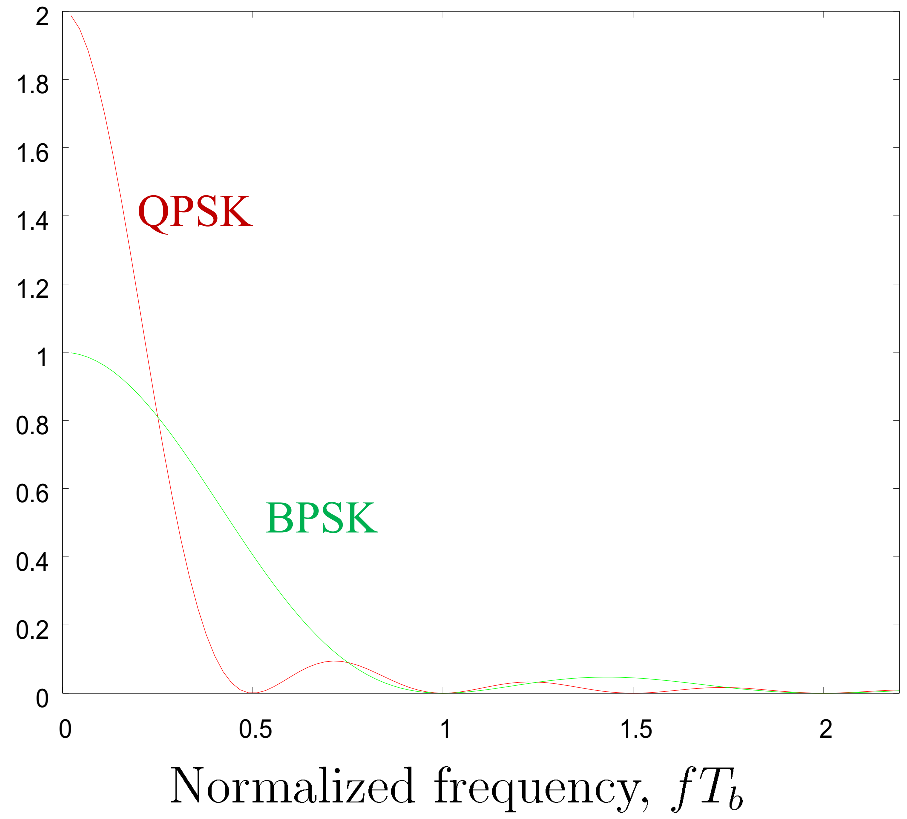
$E[e^{j(\pi/2)I_k}] = 0$ for uniform prior

$$
\Rightarrow \bar{S}_B(f) = \frac{1}{T}|G(f)|^2 = 2E\,\mathrm{sinc}^2(Tf) = 4E_b\,\mathrm{sinc}^2(2T_b f)
$$

Coherent Phase-Shift Keying (PSK) – PSD

- Time-averaged PSDs of BPSK and QPSK under the same E_b and T_b

Normalized power spectral density $\bar{S}_B(f)/(2E_b)$

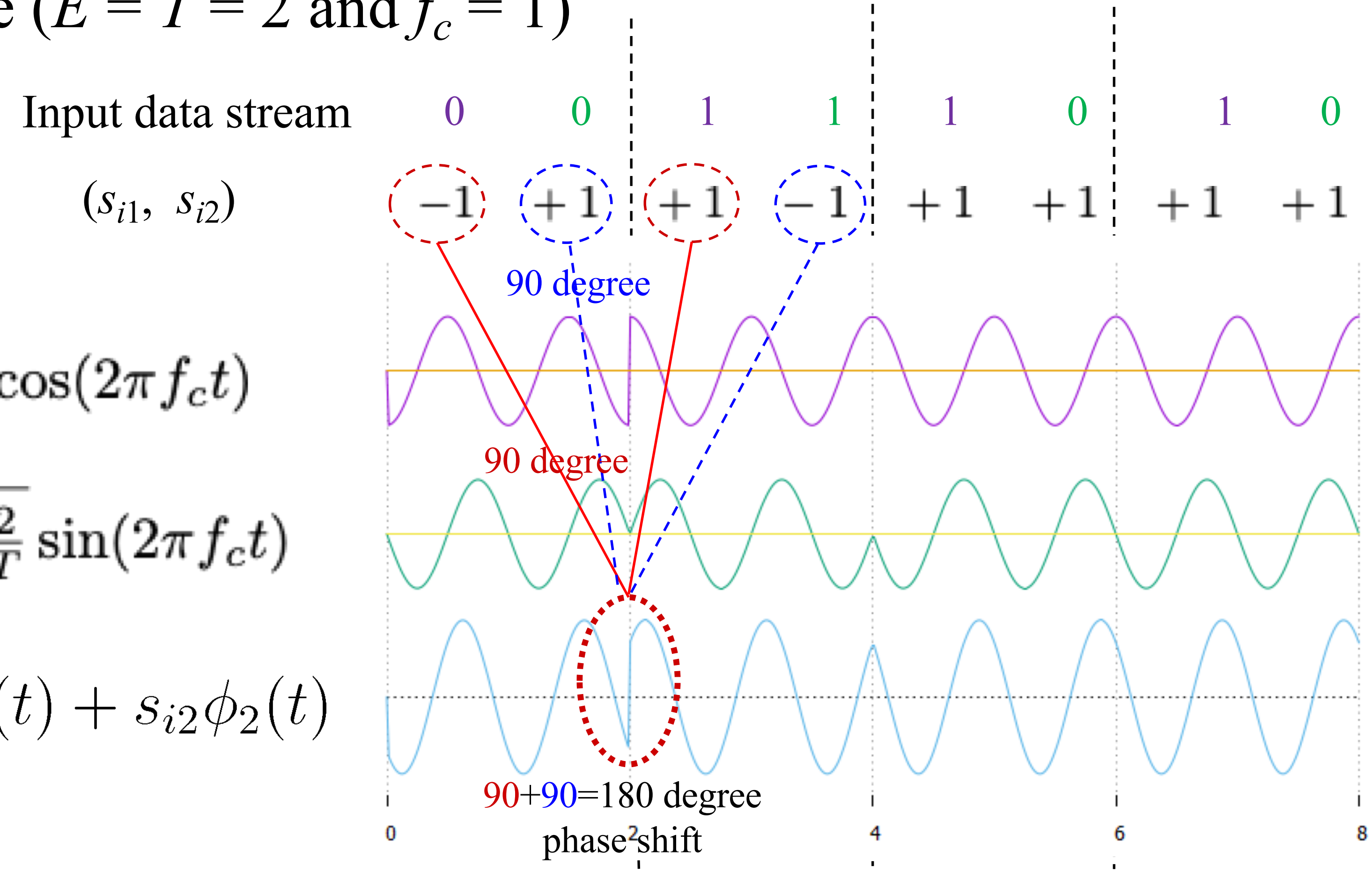


$$s(t) = \text{Re}\left\{e^{j[(\pi/2)I_k+\pi/4]}e^{j2\pi f_c t}g(t)\right\} \text{ for } I_k = 0(++), 1(-+), 2(--), 3(+-)$$

Single sign change = 90 degree shift
Double sign change = 180 degree shift

Coherent PSK – Offset QPSK

- Example ($E = T = 2$ and $f_c = 1$)



$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$

Coherent PSK – Offset QPSK

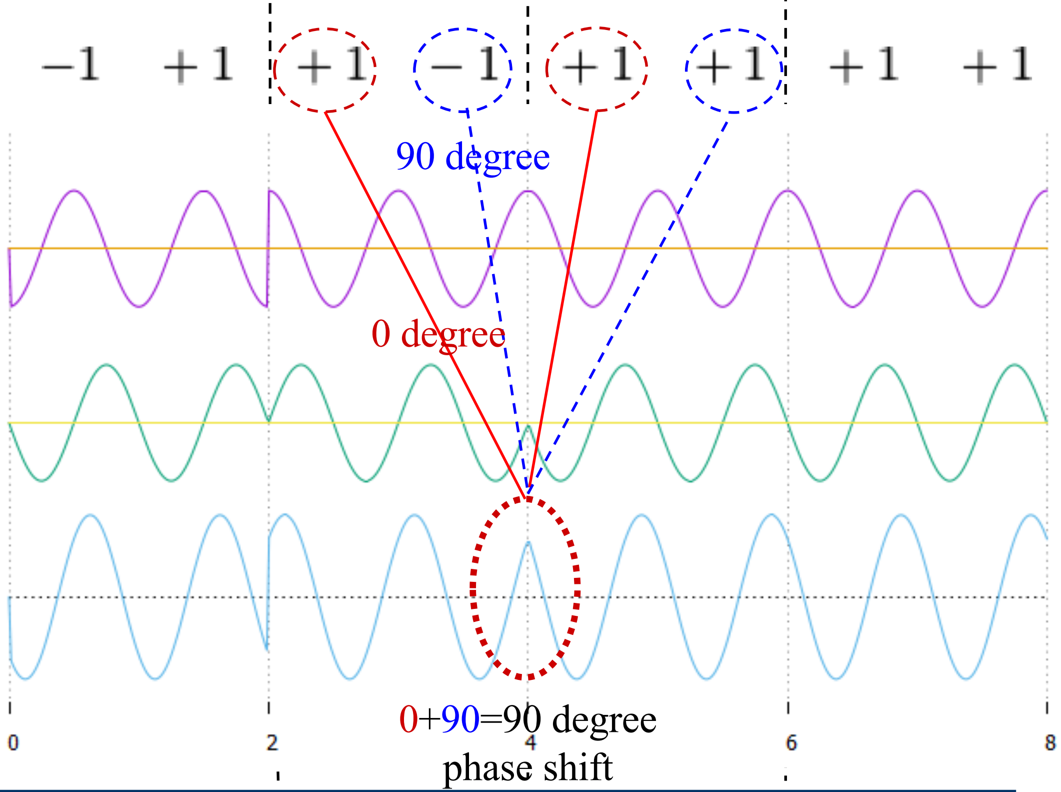
- Example ($E = T = 2$ and $f_c = 1$)

Input data stream	0	0	1	1	1	0	1	0
$(s_{i1},\ s_{i2})$	-1	$+1$	$+1$	-1	$+1$	$+1$	$+1$	$+1$

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$



Coherent PSK – Offset QPSK

- Example ($E = T = 2$ and $f_c = 1$)

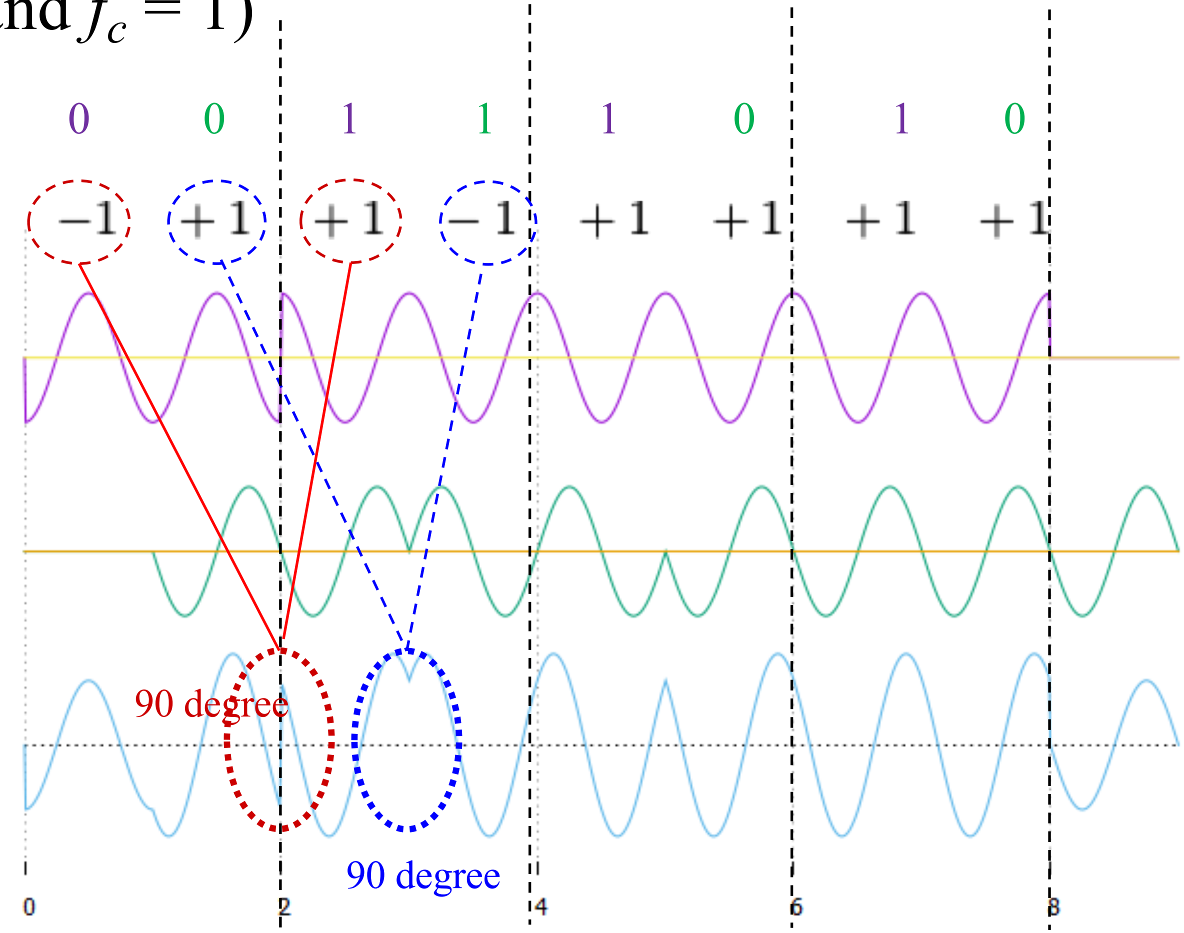
Input data stream

$(s_{i1},\ s_{i2})$

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c (t - \tfrac{T}{2}))$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$



Coherent PSK – Offset QPSK

- Offset QPSK
  - By "offseting" the quadrature component by half a symbol interval with respect to the in-phase component, Offset QPSK limits the amplitude fluctuation to 90 degree.
  - The 90 degree phase transition in OQPSK occurs twice as frequently encountered in QPSK.
    - Personal comment: One 180 degree phase transition in QPSK becomes two 90 degree phase transitions in OQPSK. Hence, "twice" is an over-estimate.
  - Under AWGN and coherent receiver, the error rate of OQPSK is exactly the same as that of QPSK.

Coherent PSK – π/4-shifted DQPSK

- π/4-shifted DQPSK (D=Differential)
  - The input dibit does not determine the absolute phase, but the phase change.

$$\text{in-phase} \Rightarrow \cos(\theta_k) = \cos(\theta_{k-1} + \Delta\theta_k)$$

$$\text{quadrature} \Rightarrow \sin(\theta_k) = \sin(\theta_{k-1} + \Delta\theta_k)$$

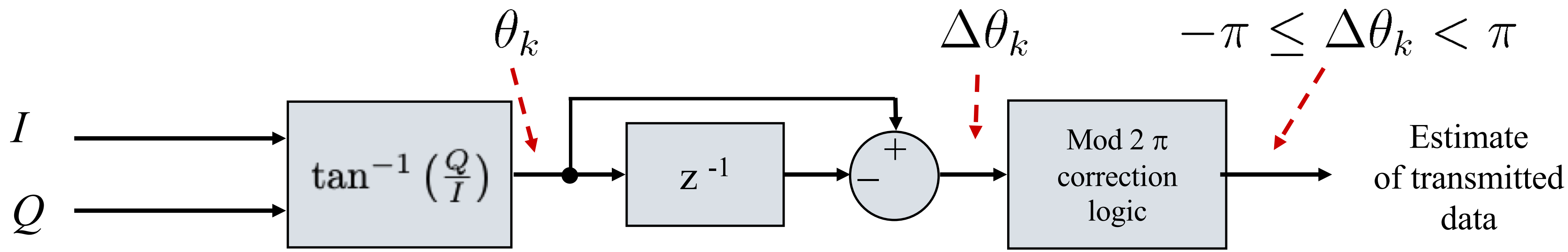
$$\Delta\theta_k = \begin{cases} +\pi/4, & 00 \\ +3\pi/4, & 01 \\ -3\pi/4, & 11 \\ -\pi/4, & 10 \end{cases}$$

Coherent PSK – π/4-shifted DQPSK

- π/4-shifted DQPSK
  - The phase transition is restricted to either 45 or 135 degree.
    - No 0 degree phase transition occurs now!
  - Noncoherent receiver is feasible.

Coherent PSK – Detection of π/4-shifted DQPSK

- Noncoherent receiver
  - Differential detector



$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \Rightarrow I = \langle x(t), \phi_1(t)\rangle$$

$$\phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t) \Rightarrow Q = \langle x(t), \phi_2(t)\rangle$$

Coherent *M*-ary PSK

- *M*-ary PSK

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (i-1)\frac{2\pi}{M}\right], & 0 \le t < T \\ 0, & \text{elsewhere} \end{cases}$$

where $i = 1, 2, \ldots, M$, f_c is a multiple of $1/T$,

E is the transmitted energy per *M*-ary PSK symbol, and

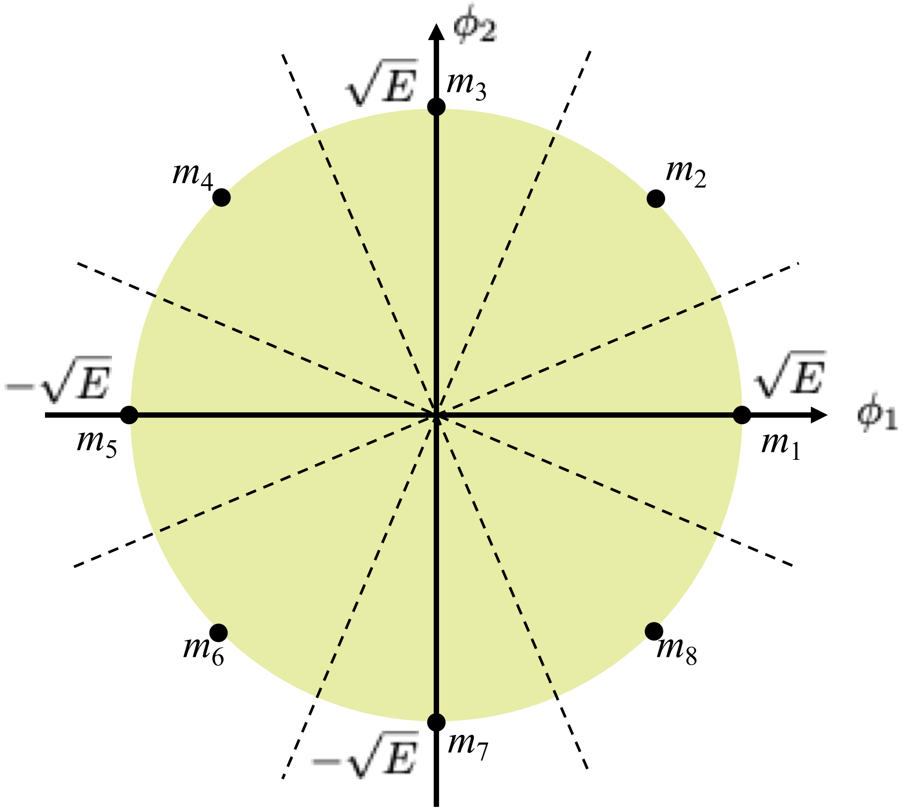
T is the symbol duration.

- Vector space analysis of *M*-ary PSK

$$\boldsymbol{s}_i = \begin{bmatrix} \sqrt{E}\cos((i-1)\frac{2\pi}{M}) \\ \sqrt{E}\sin((i-1)\frac{2\pi}{M}) \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

Coherent *M*-ary PSK

- Example – 8PSK (Octaphase-shift keying)



Union Bound for Coherent M-ary PSK Error

$$
\begin{aligned}
P_{e,\text{symbol}} &= \sum_{i=1}^{M} \Pr(m_i \text{ transmitted}) \Pr(\text{decision} \neq m_i | m_i \text{ transmitted}) \\
&= \sum_{i=1}^{M} \frac{1}{M} \Pr\left(\begin{array}{l} \text{decision} = m_1 \\ \ \text{or } \cdots \\ \ \text{or decision} = m_{i-1} \\ \ \text{or decision} = m_{i+1} \\ \ \text{or } \cdots \\ \ \text{or decision} = m_M \end{array} \middle| \, m_i \text{ transmitted} \right) \\
&\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\ell=1, \ell \neq i}^{M} \Pr\left(\text{decision} = m_\ell | m_i \text{ transmitted}\right)
\end{aligned}
$$

Union Bound for Coherent *M*-ary PSK Error

- If the signal constellation is symmetric in the sense that

$$\sum_{\ell=1,\ell\neq i}^{M} \Pr\left(\text{decision} = m_\ell | m_i \text{ transmitted}\right) = \text{constant in } i$$

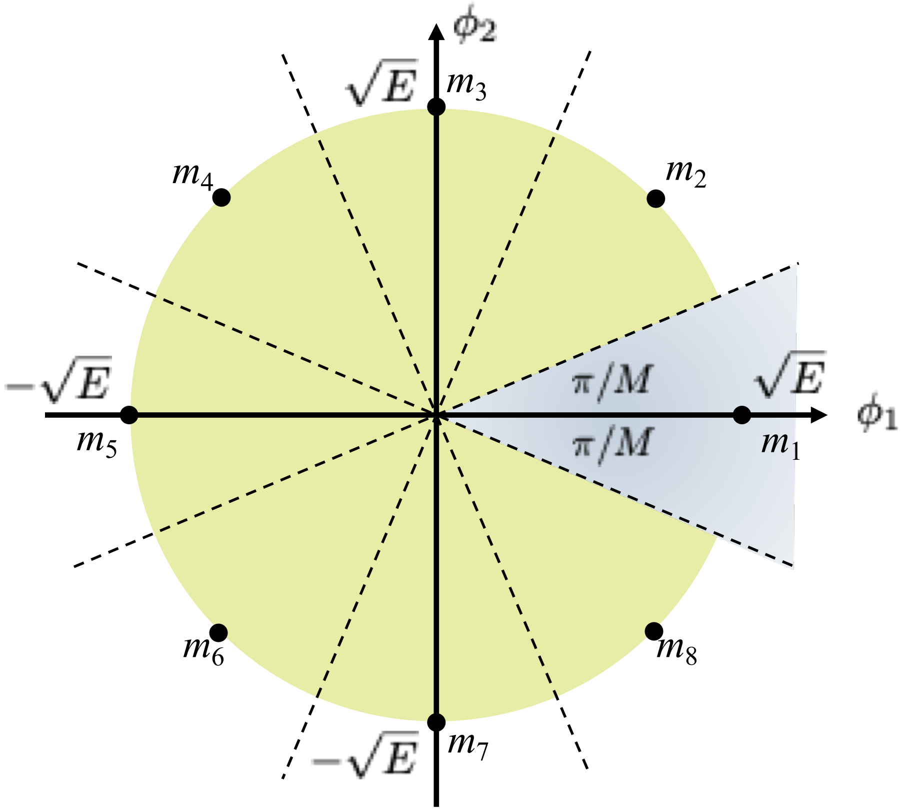
then

$$\begin{aligned} P_{e,\text{symbol}} &\leq \sum_{\ell=1,\ell\neq i}^{M} \Pr(\text{decision} = m_\ell | m_i \text{ transmitted}) \\ &= \sum_{\ell=1,\ell\neq i}^{M} \Phi\left(-\frac{d_{\ell,i}}{\sqrt{2N_0}}\right) \text{ under AWGN (See (5.89) in text)} \end{aligned}$$

$$\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Union Bound for Coherent *M*-ary PSK Error

- Example – 8PSK



Union Bound for Coherent M-ary PSK Error

$$
\begin{aligned}
P_{e,\text{symbol}} &\leq \sum_{\ell=1,\ell\neq 1}^{8} \Phi\left(-\frac{d_{\ell,1}}{\sqrt{2N_0}}\right) \text{ under AWGN (See (5.89) in text)} \\
&= \Phi\left(-\frac{d_{2,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{3,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{4,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{5,1}}{\sqrt{2N_0}}\right) \\
&+ \Phi\left(-\frac{d_{8,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{7,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{6,1}}{\sqrt{2N_0}}\right) \\
&= 2\Phi\left(-\frac{d_{2,1}}{\sqrt{2N_0}}\right) + 2\Phi\left(-\frac{d_{3,1}}{\sqrt{2N_0}}\right) + 2\Phi\left(-\frac{d_{4,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{5,1}}{\sqrt{2N_0}}\right) \\
&\approx 2\Phi\left(-\frac{d_{2,1}}{\sqrt{2N_0}}\right)
\end{aligned}
$$

This is the lower bound of the upper bound.
So, it is not really an upper bound!

$$d_{2,1} = 2\sqrt{E}\sin\left(\frac{\pi}{M}\right)$$

PSD of Coherent *M*-ary PSK

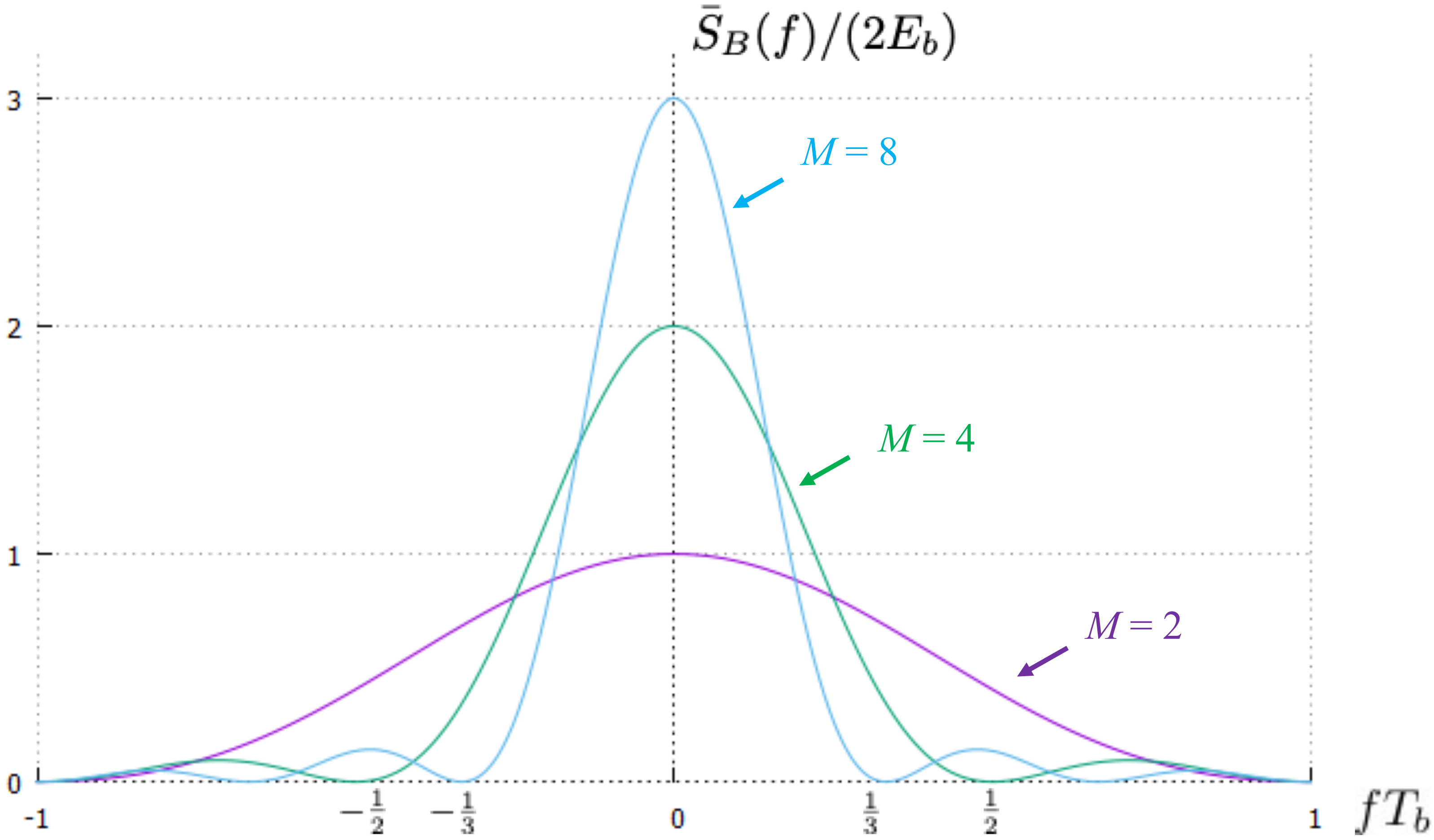
- Same as Slide IDC1-48

$$\bar{S}_B(f) = \frac{1}{T}|G(f)|^2 = 2E\,\text{sinc}^2(Tf)$$

With $T = T_b \log_2(M)$ and $E = E_b \log_2(M)$

$$\Rightarrow \bar{S}_B(f) = 2E_b \log_2(M)\,\text{sinc}^2(T_b \log_2(M) f)$$

PSD of Coherent M-ary PSK



Bandwidth Efficiency of Coherent *M*-ary PSK

- Bandwidth efficiency of *M*-ary PSK signals
  - Null-to-null bandwidth

$$B = \frac{2}{T} = \frac{2}{T_b \log_2(M)} = \frac{2R_b}{\log_2(M)}$$

  - Bandwidth efficiency

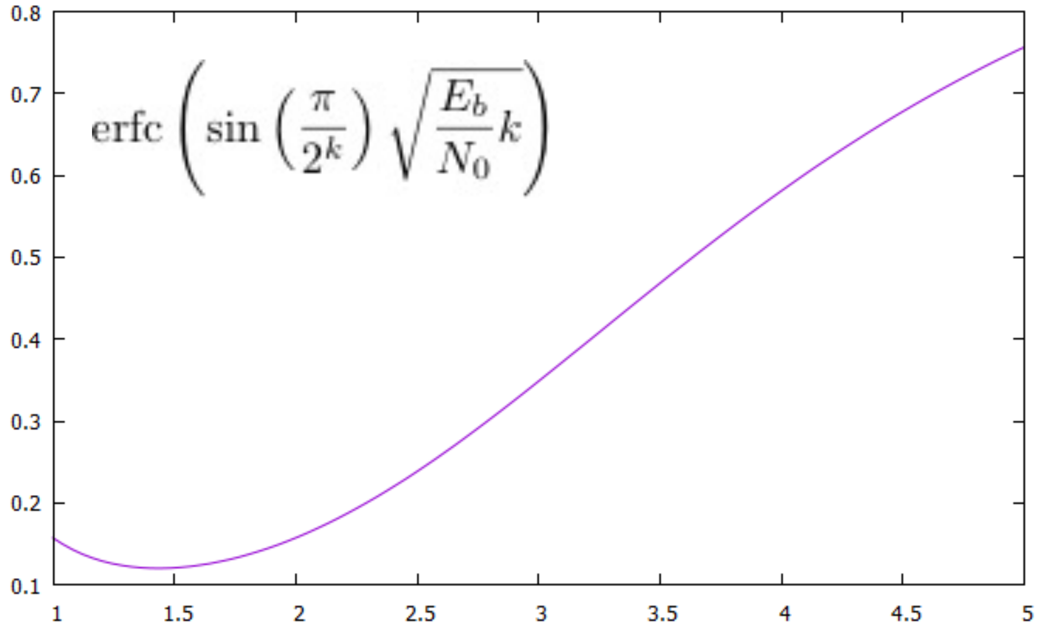
$$\rho \text{ (bits/s/Hz)} = \frac{R_b}{B} = \frac{1}{2}\log_2(M)$$

$$P_{e,\text{symbol}} \approx 2\Phi\left(-\sin\left(\frac{\pi}{M}\right)\sqrt{\frac{2E_b \log_2(M)}{N_0}}\right)$$

Bandwidth Efficiency of Coherent *M*-ary PSK

- Final note
  - There is a trade-off between symbol error rate and bandwidth efficiency for *M*-ary PSK signals.

$$\frac{E_b}{N_0} = 1 \qquad \Phi(-x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$



$$k = 2\rho$$

$$\rho = \frac{k}{2} \text{ for } M = 2^k$$

$$P_{e,\text{symbol}} \approx 2\Phi\left(-\sin\left(\frac{\pi}{2^k}\right)\sqrt{\frac{2E_b k}{N_0}}\right) \approx 2\Phi\left(-\frac{\pi}{2^k}\sqrt{\frac{2E_b k}{N_0}}\right) \text{ for } M = 2^k$$

Hybrid Amplitude/Phase Modulations

- For M-ary PSK signals, the in-phase and quadrature components are "dependent".
  - How about making them "independent" (to increase the data rate)?
  - Answer: M-ary quadrature amplitude modulation (QAM)

$$s_k(t) = a_k\sqrt{E_0}\phi_1(t) + b_k\sqrt{E_0}\phi_2(t)$$

$$\boldsymbol{s}_k = \begin{bmatrix} a_k\sqrt{E_0} \\ b_k\sqrt{E_0} \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

Hybrid Amplitude/Phase Modulations

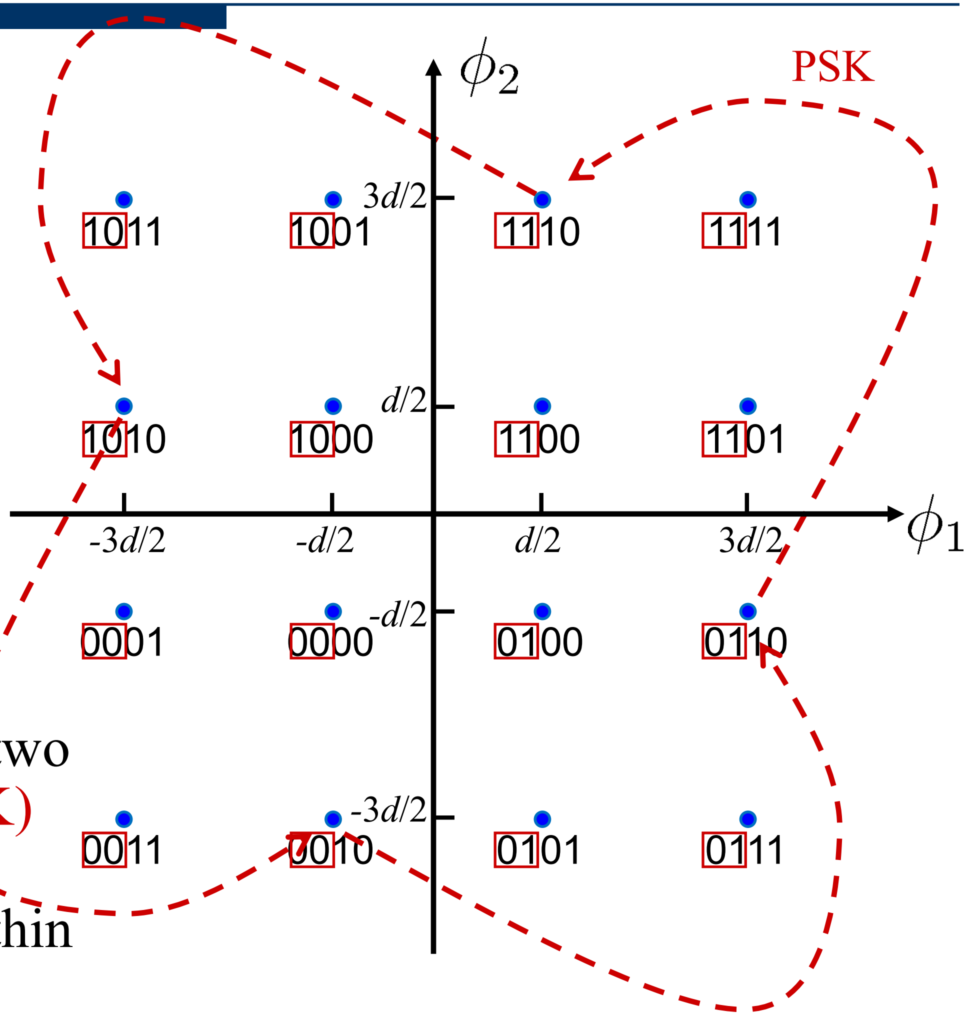
- Square constellation

$$\sqrt{E_0} = \frac{d}{2}$$

$$a_k \in \{-3, -1, +1, +3\}$$

$$b_k \in \{-3, -1, +1, +3\}$$

- Gray-encoded quadbits
  - Gray-encode the first two bits by quadrants (PSK)
  - Gray-encode the remaining two bits within quadrants (ASK)

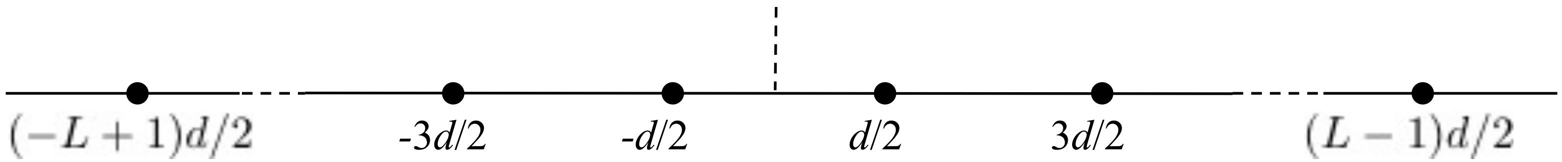


Hybrid Amplitude/Phase Modulations

- Symbol error rate of square QAM $(M = L^2)$

$$
\begin{aligned}
P_{e,M\text{-}QAM} &= 1 - P_{c,M\text{-}QAM} \\
&= 1 - \left(1 - P_{e,\sqrt{M}\text{-}ASK}\right)^2 \\
&= 2P_{e,\sqrt{M}\text{-}ASK} - P^2_{e,\sqrt{M}\text{-}ASK} \\
&\approx 2P_{e,\sqrt{M}\text{-}ASK}
\end{aligned}
$$

- Symbol error rate of equal-prior L-ary ASK



Symbol error rate of equal-prior L-ary ASK

$$
\begin{aligned}
P_{e,L\text{-ASK}} &= \frac{1}{L}\Pr\left(x-(-L+1)\frac{d}{2} > \frac{d}{2}\,\middle|\, s=(-L+1)\frac{d}{2}\right)\\
&\quad + \sum_{k=2}^{L-1}\frac{1}{L}\Pr\left(\left|x-(-L+2k-1)\frac{d}{2}\right| > \frac{d}{2}\,\middle|\, s=(-L+2k-1)\frac{d}{2}\right)\\
&\quad + \frac{1}{L}\Pr\left(x-(L-1)\frac{d}{2} < -\frac{d}{2}\,\middle|\, s=(L-1)\frac{d}{2}\right)\\
&= \frac{1}{L}\left[1-\Phi\left(\frac{d/2}{\sqrt{N_0/2}}\right)\right] + \sum_{k=2}^{L-1}\frac{1}{L}2\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right) + \frac{1}{L}\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right)\\
&= 2\frac{(L-1)}{L}\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right)\\
&= 2\left(1-\frac{1}{L}\right)\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right)
\end{aligned}
$$

$$
\left(= 2\left(1-\frac{1}{L}\right)\Phi\left(-\frac{d}{\sqrt{2N_0}}\right)\right)
$$

Hybrid Amplitude/Phase Modulations

- Average transmitted energy of M-ary QAM

$$\sqrt{E_0} = \frac{d}{2}$$

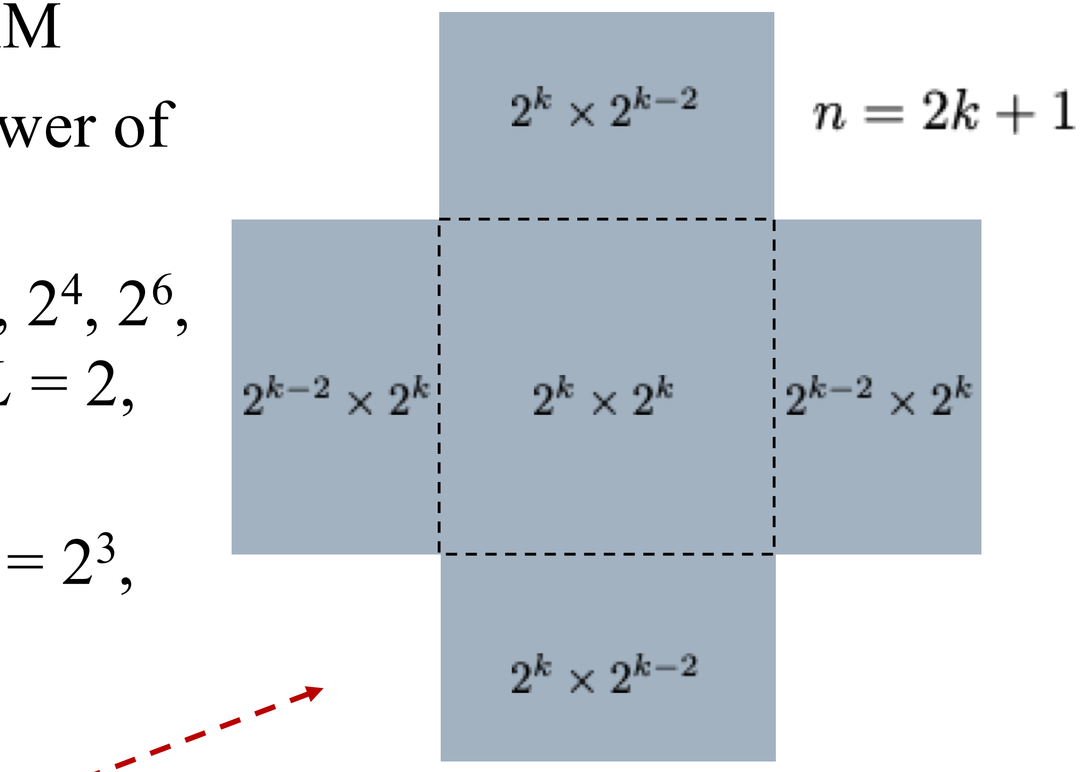
$$L = \sqrt{M}$$

$$\begin{aligned} E_{av} &= \sum_{k=1}^{M} \frac{1}{M} \|\boldsymbol{s}_k\|^2 \\ &= \sum_{k=1}^{M} \frac{1}{M} E_0 (a_k^2 + b_k^2) \\ &= \frac{E_0}{M} \sum_{k=1}^{L} \sum_{k'=1}^{L} [(-L + 2k - 1)^2 + (-L + 2k' - 1)^2] \\ &= \frac{2}{3}(M-1)E_0 \quad \Rightarrow \frac{d}{2} = \sqrt{E_0} = \sqrt{\frac{3E_{av}}{2(M-1)}} \end{aligned}$$

$$P_{e,M\text{-}QAM} \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) \Phi\left(-\sqrt{\frac{3E_{av}}{N_0(M-1)}}\right)$$

Hybrid Amplitude/Phase Modulations

- Square constellation QAM
  - M is usually even power of two.
  - For example, $M = 2^2, 2^4, 2^6, 2^8, \ldots$ (in such case $L = 2, 2^2, 2^3, 2^4, \ldots$)
- Question: How about $M = 2^3, 2^5, 2^7, \ldots, 2^n$
  - Answer: Cross constellation QAM



$$M = 2^n = 2^{2k+1} = 2^{2k} + 4 \times 2^{2k-2}$$

Hybrid Amplitude/Phase Modulations

- Symbol error rate of cross-constellation QAM

$$\sqrt{E_0} = \frac{d}{2}$$

$$P_{e,M\text{-cross-QAM}} \approx 4\left(1 - \frac{1}{\sqrt{2M}}\right)\Phi\left(-\sqrt{\frac{2E_0}{N_0}}\right)$$

$$P_{e,M\text{-square-QAM}} \approx 4\left(1 - \frac{1}{\sqrt{M}}\right)\Phi\left(-\sqrt{\frac{2E_0}{N_0}}\right)$$

Hybrid Amplitude/Phase Modulations – Carrierless Amplitude/Phase Modulation (CAP)

- QAM can be viewed as one of the family members in carrierless amplitude/phase modulation (CAP)

$$\begin{aligned} s_k(t) &= a_k\sqrt{E_0}\phi_1(t) + b_k\sqrt{E_0}\phi_2(t) \\ &= a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t) \end{aligned}$$

where $g(t) = \begin{cases} \sqrt{\frac{2E_0}{T}}, & 0 \le t < T; \\ 0, & \text{otherwise} \end{cases}$ and $\begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$

Re-express QAM into CAP form

$$
\begin{aligned}
s(t) &= \sum_{k=-\infty}^{\infty} s_k(t) \\
&= \sum_{k=-\infty}^{\infty} \left(a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t)\right) \\
&= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} (a_k + j b_k) g(t-kT) e^{j2\pi f_c t}\right\} \\
&= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \left((a_k + j b_k) e^{j2\pi f_c kT}\right) \cdot \left(g(t-kT) e^{j2\pi f_c (t-kT)}\right)\right\} \\
&= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \tilde{A}_k \cdot g_+(t-kT)\right\}
\end{aligned}
$$

Carrierless since no carrier f_c appears in this formula.

where $\tilde{A}_k = (a_k + j b_k) e^{j2\pi f_c kT}$ and $g_+(t) = g(t) e^{j2\pi f_c t}$.

Hybrid Amplitude/Phase Modulations – CAP

- Properties of the passband in-phase and quadrature pulses in CAP

$$g_+(t) = g(t)\cos(2\pi f_c t) + j g(t)\sin(2\pi f_c t) = p(t) + j\hat{p}(t)$$

$$\text{where } \begin{cases} p(t) = g(t)\cos(2\pi f_c t) \\ \hat{p}(t) = g(t)\sin(2\pi f_c t) \end{cases}$$

- **Property 1**: $\hat{p}(t)$ is the Hilbert transform of $p(t)$.

(If $G(f) = 0$ for $|f| > f_c$.)

Hybrid Amplitude/Phase Modulations – CAP

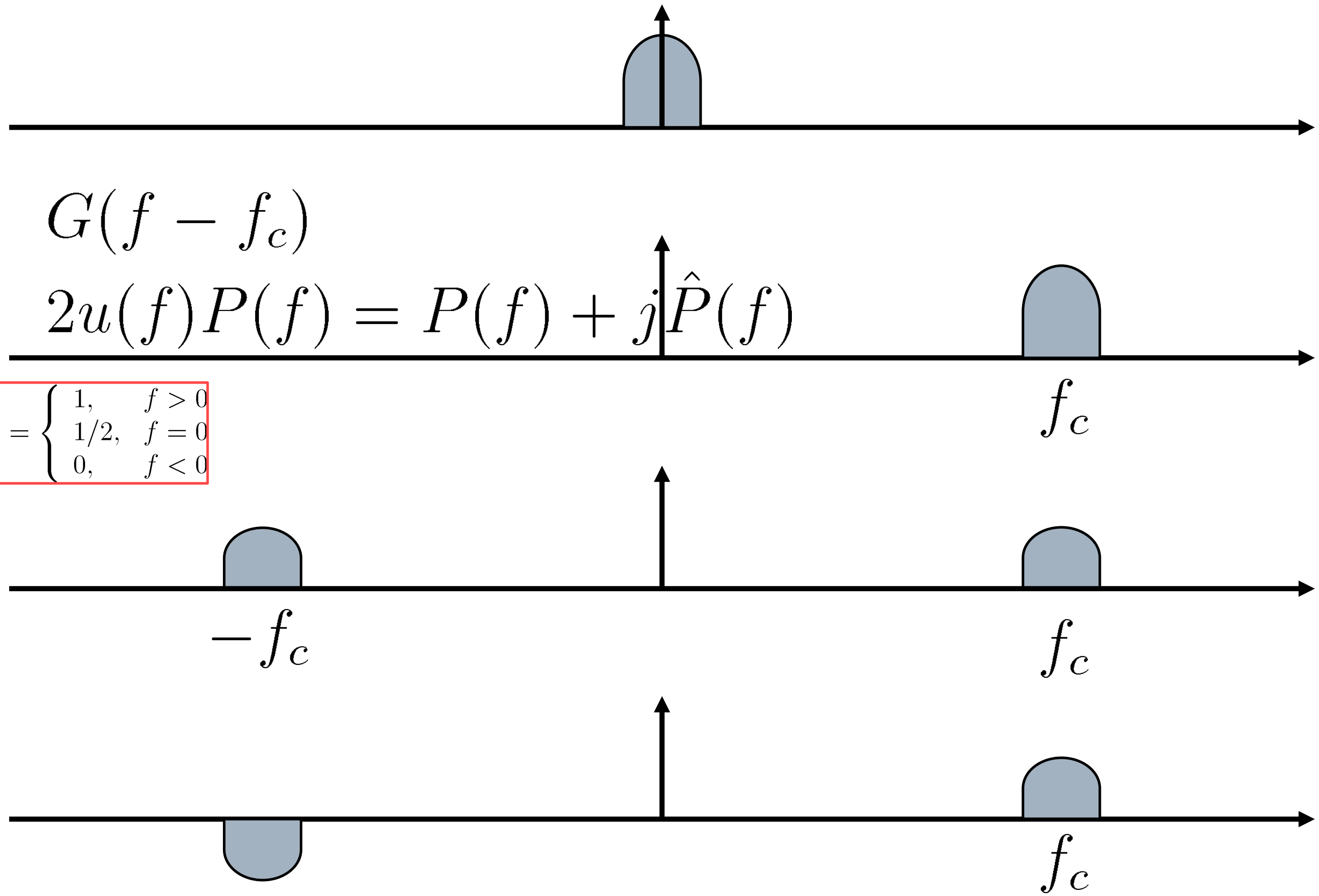
$G(f)$

$$\begin{aligned} G_+(f) &= G(f - f_c) \\ &= 2u(f)P(f) = P(f) + j\hat{P}(f) \end{aligned}$$

$$u(f) = \begin{cases} 1, & f > 0 \\ 1/2, & f = 0 \\ 0, & f < 0 \end{cases}$$

$P(f)$

$j\hat{P}(f)$



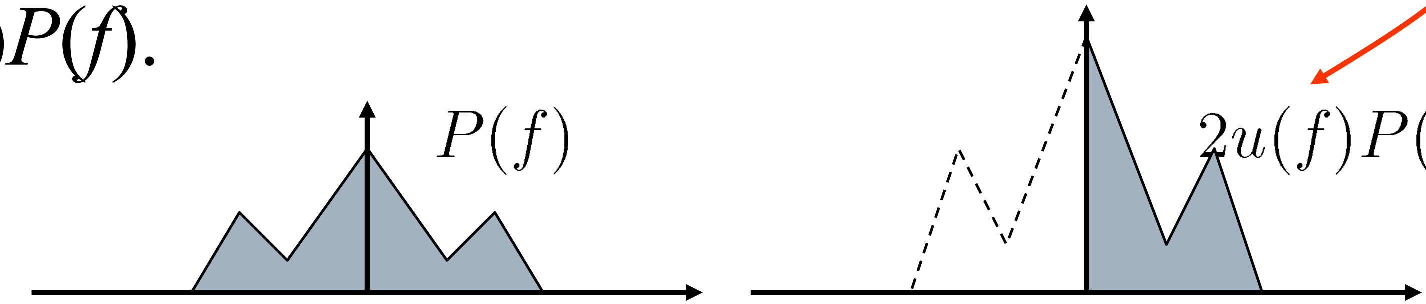
Appendix: Hilbert transform

- Let $P(f)$ be the spectrum of a real function $p(t)$.
  - By convention, denote by $u(f)$ the unit step function, i.e.,

$$u(f) = \begin{cases} 1, & f > 0 \\ 1/2, & f = 0 \\ 0, & f < 0 \end{cases}$$

Multiply by 2 to unchange the area.

- Put $g_+(t)$ to be the function corresponding to $2u(f)P(f)$.



Appendix: Hilbert transform

- How to obtain $g_+(t)$?
- Answer: *Hilbert Transformer*.

Proof: Observe that

$$2u(f) = 1 + \text{sgn}(f), \text{ where } \text{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

Then by the next slide, we learn that

$$2u(f) \stackrel{\text{Inverse\_Fourier}}{=} \delta(t) + j\frac{1}{\pi t} \cdot \mathbf{1}\{t \neq 0\}$$

By extended (inverse) Fourier transform,

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sgn}(f) e^{-a|f|+j2\pi ft} df &= \int_{0}^{\infty} e^{-a|f|+j2\pi ft} df - \int_{-\infty}^{0} e^{-a|f|+j2\pi ft} df \\ &= \int_{0}^{\infty} e^{-(a-j2\pi t)f} df - \int_{0}^{\infty} e^{-(a+j2\pi t)f} df \\ &= \frac{1}{a-j2\pi t} - \frac{1}{a+j2\pi t} \\ &= \frac{j4\pi t}{a^2+4\pi^2 t^2}\end{aligned}$$

$$\operatorname{sgn}(f) \stackrel{\text{Inverse\_Fourier}}{=} \lim_{a\downarrow 0} j\frac{4\pi t}{a^2+4\pi t^2} = \begin{cases} \frac{j}{\pi t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

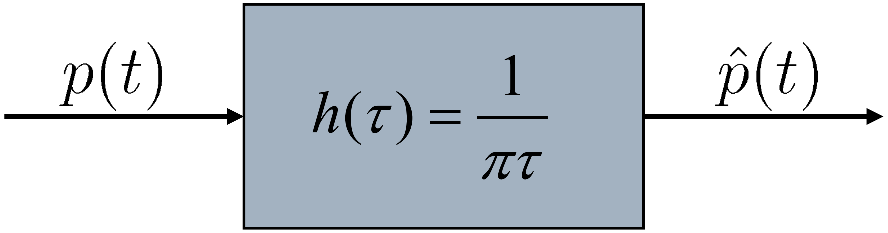
$$2u(f) = 1 + \operatorname{sgn}(f) \stackrel{\text{Inverse\_Fourier}}{=} \delta(t) + \frac{j}{\pi t} \cdot \mathbf{1}\{t \neq 0\}$$

Appendix: Hilbert transform

$$\begin{aligned} g_+(t) &= \text{Fourier}^{-1}\{2u(f)P(f)\} \\ &= \text{Fourier}^{-1}\{2u(f)\} * \text{Fourier}^{-1}\{P(f)\} \\ &= \left(\delta(t) + j\frac{1}{\pi t}\mathbf{1}\{t \neq 0\}\right) * p(t) \\ &= p(t) + j\frac{1}{\pi t}\cdot\mathbf{1}\{t \neq 0\} * p(t) \\ &= p(t) + j\hat{p}(t), \end{aligned}$$

where $\hat{p}(t) = \int_{-\infty}^{\infty} p(\tau)\frac{1}{\pi(t-\tau)}d\tau$ is the Hilbert transform of $p(t)$.

Appendix: Hilbert transform

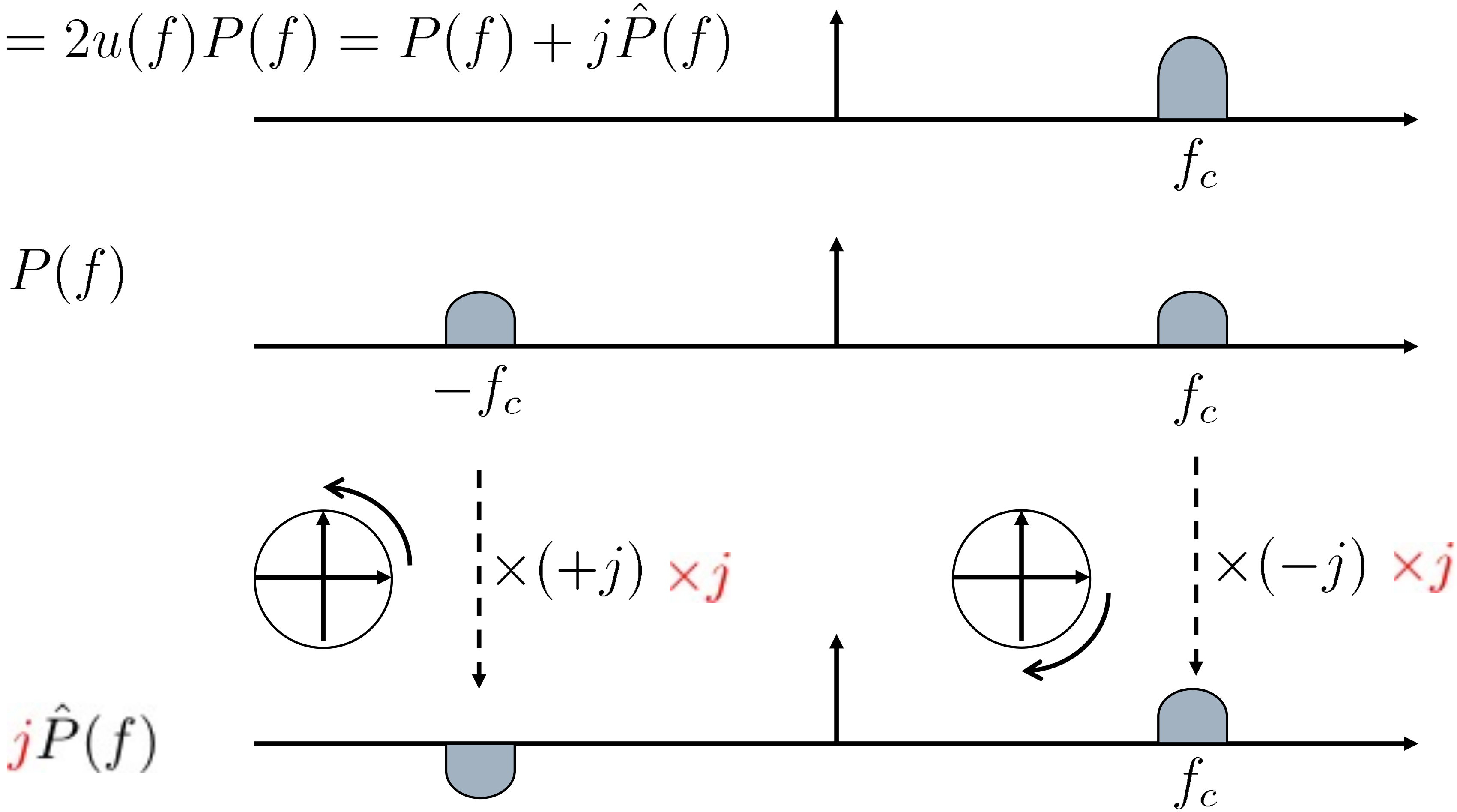


$$h(\tau) = \frac{1}{\pi\tau} \Rightarrow H(f) = -j\text{sgn}(f), \text{ where } \text{sgn}(f) = \begin{cases} +1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

$$\Rightarrow \hat{P}(f) = (-j\text{sgn}(f)) \cdot P(f) = \begin{cases} |P(f)| \exp\{j[\angle P(f) - \pi/2]\}, & f > 0 \\ 0, & f = 0 \\ |P(f)| \exp\{j[\angle P(f) + \pi/2\}, & f < 0 \end{cases}$$

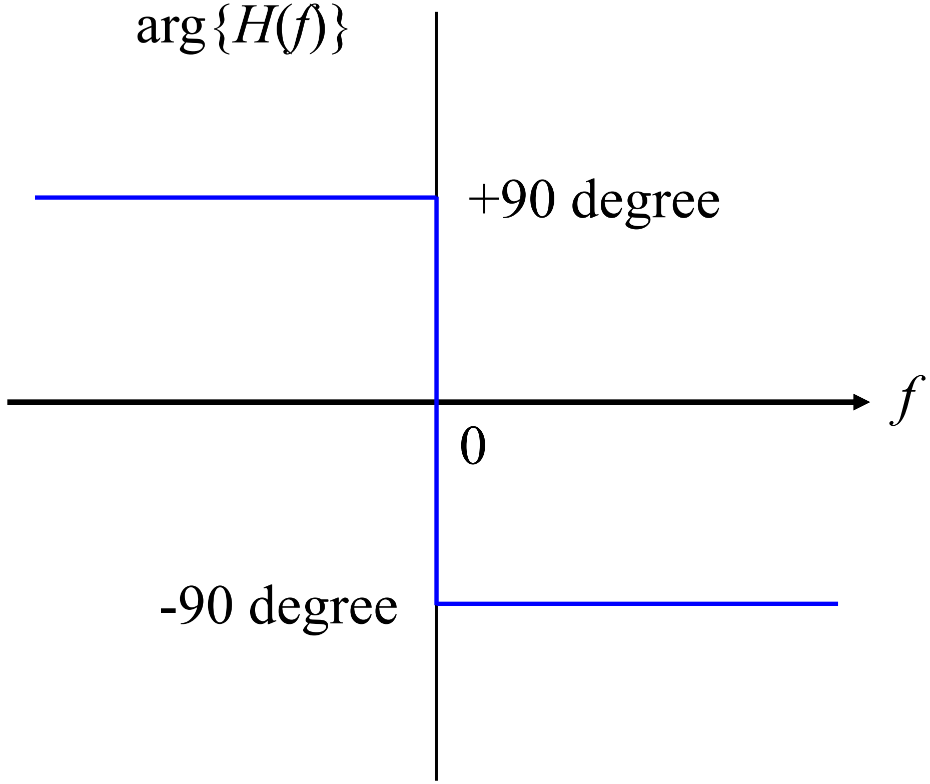
Hybrid Amplitude/Phase Modulations – CAP

$$G_+(f) = 2u(f)P(f) = P(f) + j\hat{P}(f)$$



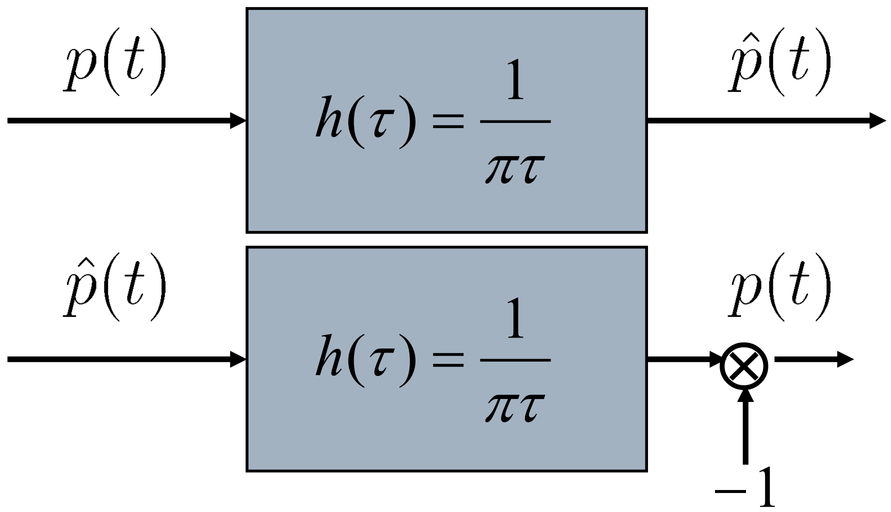
Appendix: Hilbert transform

- Hence, Hilbert transform is basically a *90 degree phase shifter*.



Appendix: Hilbert transform

Hilbert transform pair $\begin{cases} \hat{p}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} p(\tau)\frac{1}{t-\tau}d\tau \\ p(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \hat{p}(\tau)\frac{1}{t-\tau}d\tau \end{cases}$



Appendix: Hilbert Transform

- An important property of Hilbert transform is that:

$p(t)$ and $\hat{p}(t)$ are ortgogonal in the sense of integration.

In other words, $\int_{-\infty}^{\infty} p(t)\hat{p}(t)dt = 0$.

(See the proof in the next slide.)

The real and imaginary parts of $g_+(t) = p(t) + j\hat{p}(t)$ are orthogonal to each other.

(Examples of Hilbert transform pairs can be found in Table A6.4.)

$$
\begin{aligned}
\int_{-\infty}^{\infty} p(t)\hat{p}(t)dt &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} P(f)e^{j2\pi ft}df\right)\hat{p}(t)dt \\
&= \int_{-\infty}^{\infty} P(f)\left(\int_{-\infty}^{\infty} \hat{p}(t)e^{j2\pi ft}dt\right)df \\
&= \int_{-\infty}^{\infty} P(f)\hat{P}(-f)df \\
&= \int_{-\infty}^{\infty} P(f)\left[-j\text{sgn}(-f)P(-f)\right]df \\
&= j\left(\int_{0}^{\infty} P(f)P(-f)df - \int_{-\infty}^{0} P(f)P(-f)df\right) \\
&= j\left(\int_{0}^{\infty} P(f)P(-f)df - \int_{0}^{\infty} P(-f)P(f)df\right) \\
&= 0, \text{ if } \int_{0}^{\infty} P(f)P(-f)df < \infty.
\end{aligned}
$$

Hybrid Amplitude/Phase Modulations – CAP

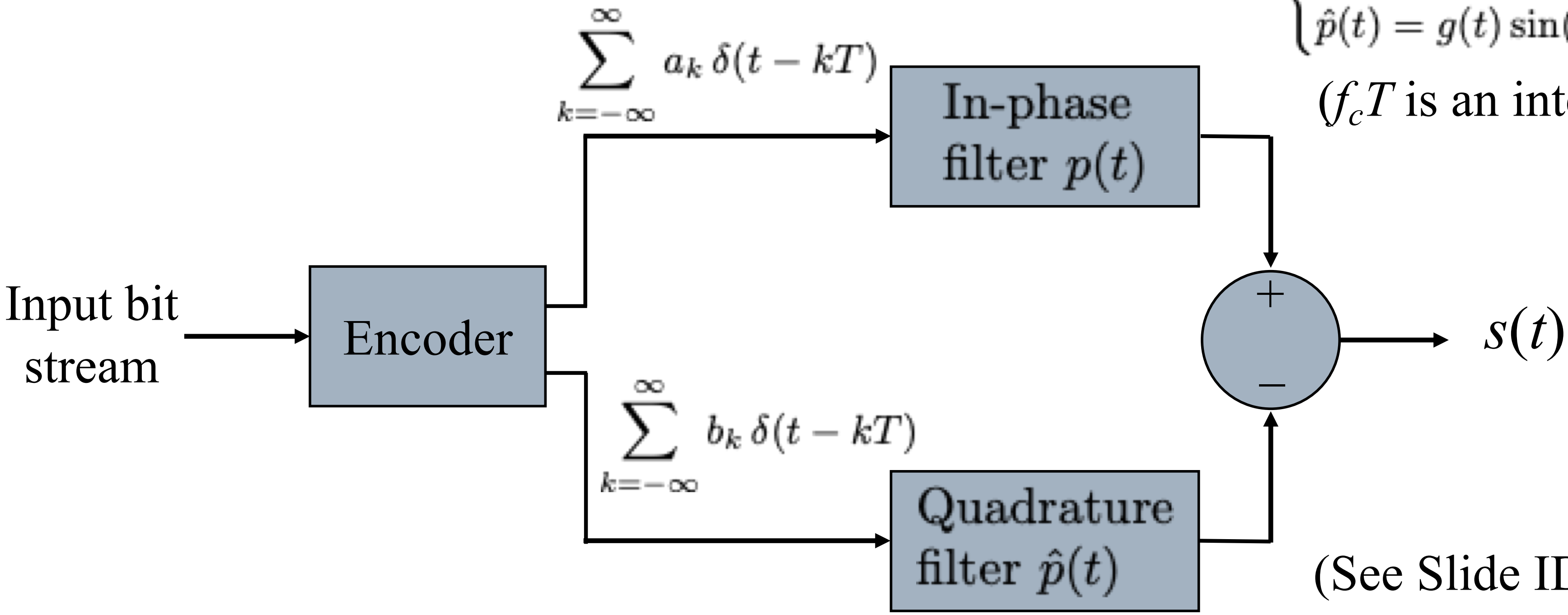
- Properties of the passband in-phase and quadrature pulses in CAP
  - **Property 2**: $\hat{p}(t)$ is orthogonal to $p(t)$.
  - **Property 3:** $\hat{p}(t) * \lambda(t)$ is orthogonal to $p(t) * \lambda(t)$ for any linear filter $\lambda(t)$.
    - This is similar to use another pulse shaping function as $g(t)*\lambda(t)$. (Recall that $p(t) = g(t)\cos(2\pi f_c t)$ and $\hat{p}(t) = g(t)\sin(2\pi f_c t)$.)
    - One is thus free to choose the pulse shaping function (to, e.g., improve the bandwidth efficiency) without affecting the orthogonality of two quadratures.

Hybrid Amplitude/Phase Modulations – CAP

- Block diagram of CAP transmitter

$$\begin{cases} p(t) = g(t)\cos(2\pi f_c t) \\ \hat{p}(t) = g(t)\sin(2\pi f_c t) \end{cases}$$

($f_c T$ is an integer.)



(See Slide IDC1-75)

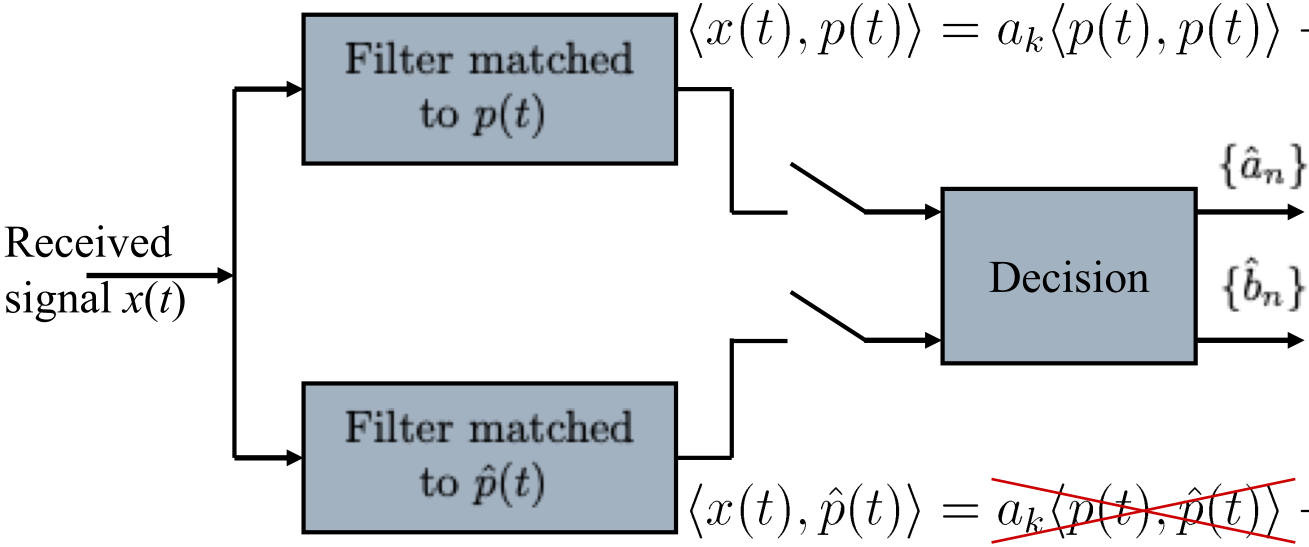
$$s(t) = \sum_{k=-\infty}^{\infty} \left(a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t)\right) = \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \tilde{A}_k \cdot g_+(t-kT)\right\}$$

Hybrid Amplitude/Phase Modulations – CAP

- Block diagram of CAP receiver

$p(t) \perp \hat{p}(t)$

$$\langle x(t), p(t)\rangle = a_k \langle p(t), p(t)\rangle - \cancel{b_k \langle \hat{p}(t), p(t)\rangle} + \langle w(t), p(t)\rangle$$

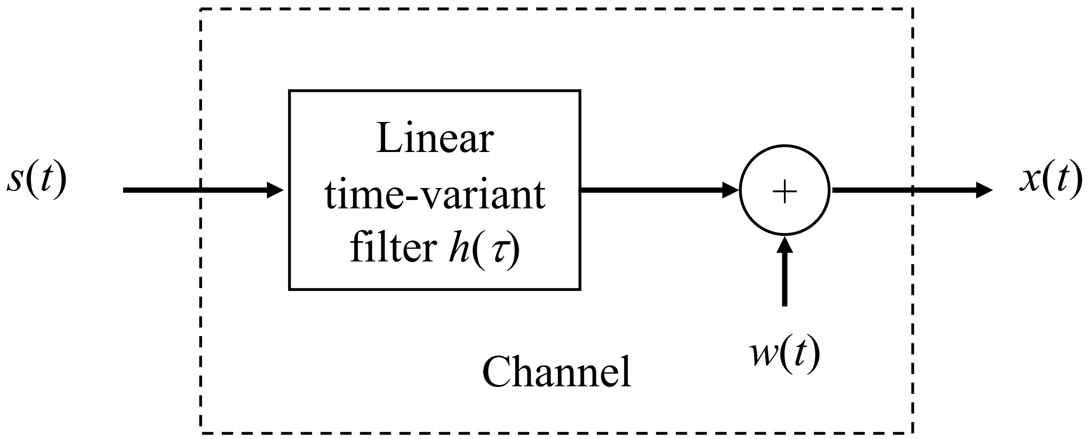


$$\langle x(t), \hat{p}(t)\rangle = \cancel{a_k \langle p(t), \hat{p}(t)\rangle} - b_k \langle \hat{p}(t), \hat{p}(t)\rangle + \langle w(t), \hat{p}(t)\rangle$$

$$\begin{aligned} x(t) &= s(t) + w(t) \\ &= a_k p(t) - b_k \hat{p}(t) + w(t), \text{ where } w(t) \text{ AWGN} \end{aligned}$$

Hybrid Amplitude/Phase Modulations – CAP

- How about channels with **intersymbol interferences**, in addition to AWGN?

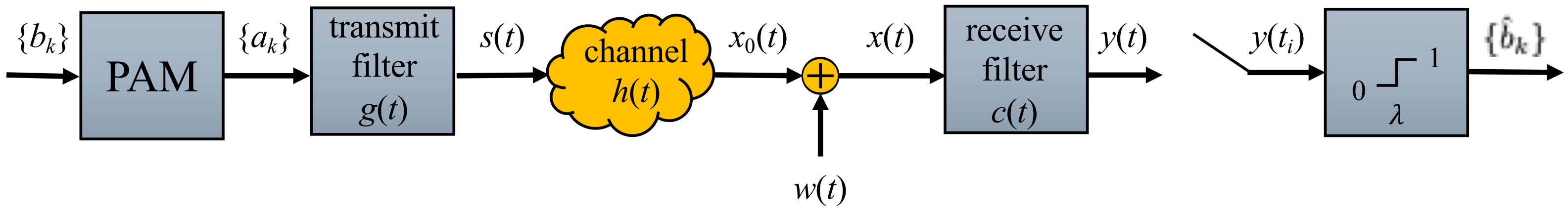


Hybrid Amplitude/Phase Modulations – CAP

Recall the below terms:

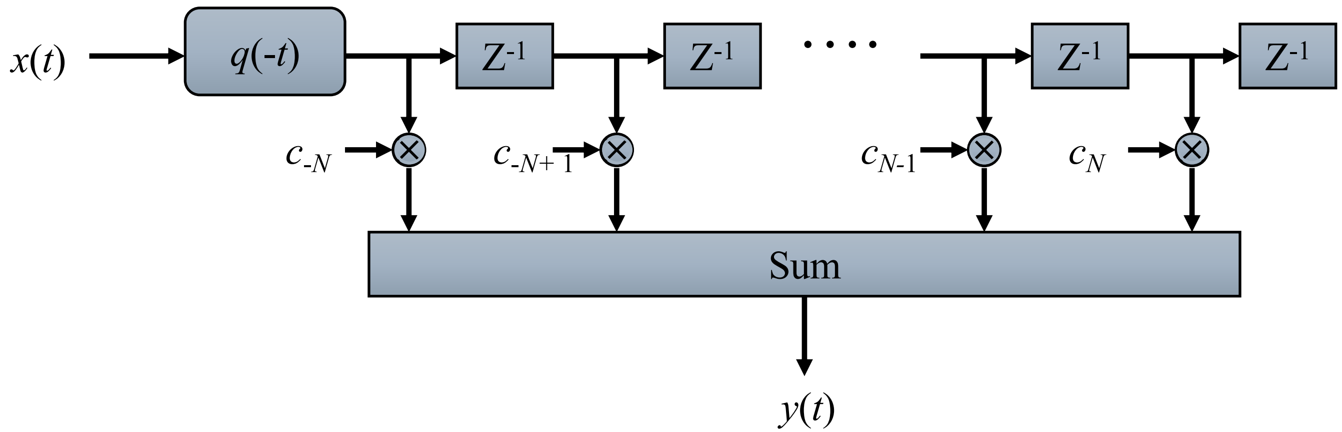
1. Zero-forcing equalizer
2. Nyquist criterion/ISI
3. Noise enhancement
4. MMSE equalizer

- Recall in Section 4.9: Optimum linear receiver.



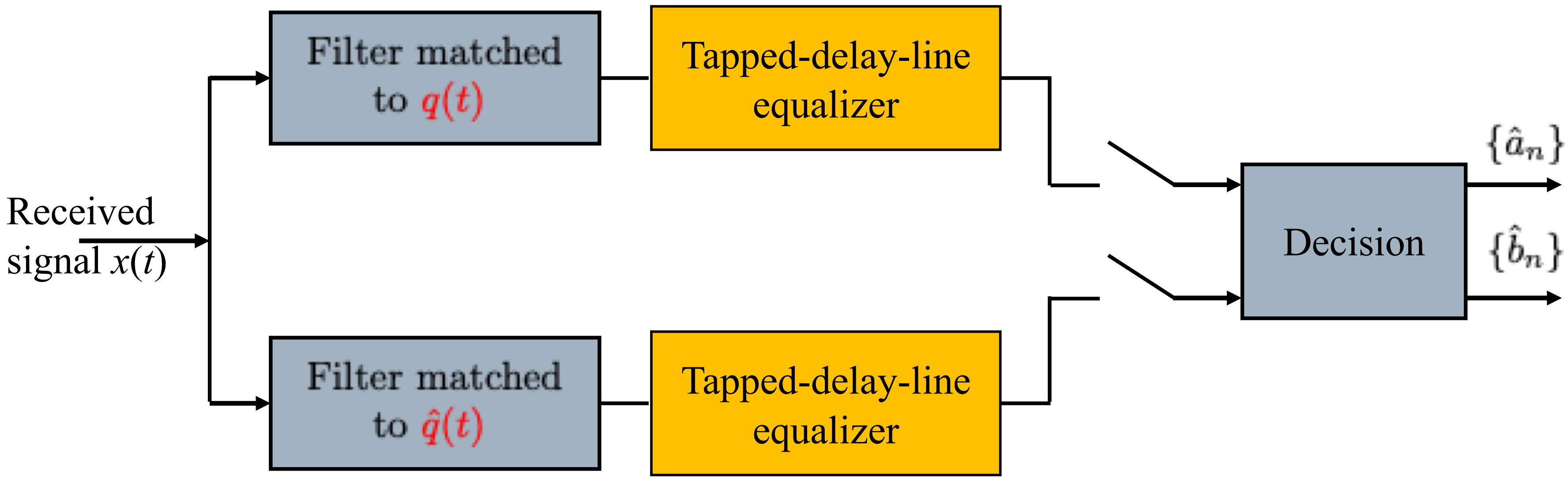
- Design of receiver $c(t)$ as an MMSE equalizer

$$q(t) = g(t) * h(t)$$



Hybrid Amplitude/Phase Modulations – CAP

- This leads to



- Can we implement the above structure in "digital" form?

Hybrid Amplitude/Phase Modulations – CAP