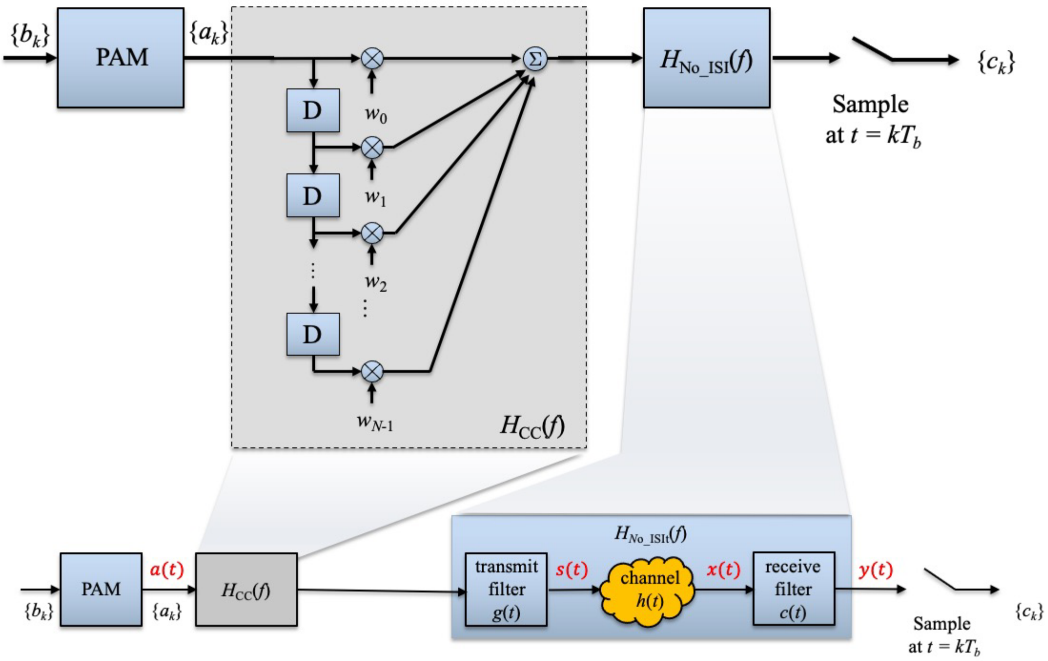
Sample Problems for the 11th Quiz

The figure and equations below will be used in all sample problems, where $H_{\text{CC}}(f)$ and $H_{\text{No\_ISI}}(f) = G(f)H(f)C(f)$ denote the correlative-level coding filter and the filter that satisfies Nyquist's criterion, respectively.



$$a(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b)$$

$$s(t) = \left(\sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \right) \star h_{\text{CC}}(t) \star g(t)$$

$$x(t) = \left(\sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \right) \star h_{\text{CC}}(t) \star g(t) \star h(t)$$

$$y(t) = \left(\sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \right) \star h_{\text{CC}}(t) \star g(t) \star h(t) \star c(t)$$

$$y(iT_b) = \left(\sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \right) \star h_{\text{CC}}(t) \star g(t) \star h(t) \star c(t) \Big|_{t=iT_b}$$

1. (Nyquist's Criterion)

   (a) Let $H_{\text{CC}}(f) = 1$ (i.e., $w_0 = 1$ and $w_1 = \cdots = w_{N-1} = 0$) and suppose $H_{\text{No\_ISI}}(f)$ satisfies

   $$\sum_{k=-\infty}^{\infty} H_{\text{No\_ISI}} \left(f - \frac{k}{T_b} \right) = T_b. \tag{1}$$

   What is the value of $h_{\text{No\_ISI}}(iT_b)$ for each integer i?

(b) Can we uniquely determine the value of $h_{\text{No\_ISI}}(iT_b + \Delta t)$ for $\Delta t = \frac{T_b}{2}$ based on (6)? Justify your answer.

(c) Instead of (6), if we are given

$$\sum_{k=-\infty}^{\infty} H_{\text{No\_ISI}}\left(f - \frac{k}{T_b}\right) = T_b e^{-j2\pi m_0 T_b f}, \tag{2}$$

does ISI occur?

(d) Continue from (c). Give one example of $h_{\text{No\_ISI}}(t)$ that satisfies (2).

(e) Continue from (c). What is $y(iT_b)$ for every integer i based on (2)?

Solution.

(a) Sampling theorem states that for any $Q(f) = \mathcal{F}\{q(t)\}$,

$$\frac{1}{T_b}\sum_{n=-\infty}^{\infty} Q\left(f - \frac{n}{T_b}\right) = \sum_{n=-\infty}^{\infty} q(nT_b)e^{-j2\pi nT_b f}.$$

Thus, (6) implies

$$\sum_{n=-\infty}^{\infty} h_{\text{No\_ISI}}(nT_b)e^{-j2\pi nT_b f}\left(= \sum_{n=-\infty}^{\infty} h_{\text{No\_ISI}}(nT_b)\cdot Z^n\right) = 1, \tag{3}$$

where $Z = e^{-j2\pi T_b f}$. Since (3) holds for arbitrary f (i.e., for arbitrary Z satisfying $|Z| = 1$), we obtain

$$h_{\text{No\_ISI}}(nT_b) = \begin{cases} 1, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

(b) The answer is negative since $h_{\text{No\_ISI}}(t)$ can be any function of the form

$$h_{\text{No\_ISI}}(t) = \text{sinc}\left(\frac{t}{T_b}\right)\lambda(t)$$

with $\lambda(0) = 1$.

Note: For example, we can set

$$\lambda(t) = 1 \text{ (as in Slide 8-33)} \quad \text{or} \quad \lambda(t) = \frac{\cos(\pi\alpha\frac{t}{T_b})}{1 - 4\alpha^2\frac{t^2}{T_b^2}} \text{ (as in Slide 8-42).}$$

(c) Sampling theorem states that for any $Q(f) = \mathcal{F}\{q(t)\}$,

$$\frac{1}{T_b}\sum_{n=-\infty}^{\infty} Q\left(f - \frac{n}{T_b}\right) = \sum_{n=-\infty}^{\infty} q(nT_b)e^{-j2\pi nT_b f}.$$

Thus, (2) implies

$$\sum_{n=-\infty}^{\infty} h_{\text{No\_ISI}}(nT_b)e^{-j2\pi nT_b f}\left(= \sum_{n=-\infty}^{\infty} h_{\text{No\_ISI}}(nT_b)\cdot Z^n\right) = e^{-j2\pi m_0 T_b f}\left(= Z^{m_0}\right), \tag{4}$$

where $Z = e^{-j2\pi T_b f}$. Since (4) holds for arbitrary f (i.e., for arbitrary Z satisfying $|Z| = 1$), we obtain

$$h_{\text{No\_ISI}}(nT_b) = \begin{cases} 1, & n = m_0; \\ 0, & n \neq m_0, \end{cases} \tag{5}$$

which guarantees no ISI occurs.

(d)

$$h_{\text{No\_ISI}}(t) = \text{sinc}\left(\frac{t - m_0 T_b}{T_b}\right) \lambda(t - m_0 T_b)$$

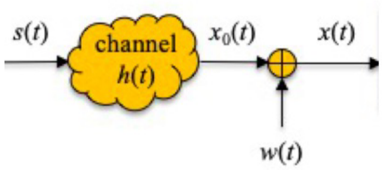
for any $\lambda(t)$ mentioned in the solution for (b).

(e)

$$\begin{aligned} y(iT_b) &= \left(\sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b)\right) \star \underbrace{h_{\text{CC}}(t)}_{\delta(t)} \star \underbrace{g(t) \star h(t) \star c(t)}_{h_{\text{No\_ISI}}(t)}\Bigg|_{t=iT_b} \\ &= \sum_{k=-\infty}^{\infty} a_k\, h_{\text{No\_ISI}}(t - kT_b)\Bigg|_{t=iT_b} \\ &= \sum_{k=-\infty}^{\infty} a_k\, h_{\text{No\_ISI}}(iT_b - kT_b) \quad (h_{\text{No\_ISI}}(iT_b - kT_b) = 1 \text{ when } i - k = m_0.) \\ &= a_{i-m_0} \end{aligned}$$

Note: Thus, there is a decision delay $m_0 T_b$.

2. (Matched filter and Nyquist's Criterion) Let $H_{\text{CC}}(f) = 1$ and suppose a noise $w(t)$ is added in-between $h(t)$ and $c(t)$.



(a) Based on the solution in Sample Problem 2(d) for Quiz 10 (Let the constant $k = 1$ for simplicity), give the matched filter $C_{\text{match}}(f)$ corresponding to $G(f)H(f)$, provided that $S_w(f)$ is the PSD of the noise process $w(t)$.

(b) Give the criterion that $G(f)$ must fulfill such that no ISI occurs, subject to that the matched filter $C_{\text{match}}(f)$ in (a) is employed at the receiver.

(c) A $G(f)$ that fulfill the criterion in (b) satisfies

$$|G(f)|^2 = \frac{S_w(f)}{|H(f)|^2}\left[\mathbf{1}\left\{|f| < \frac{1}{2T_b}\right\} \star \Lambda(f)\right] e^{-j2\pi f(m_0-1)T_b}.$$

If $\lambda(t) = 1$ for $t \in \mathbb{R}$, find m_0 that makes $G(f)$ theoretically tractable.

(d) Continue from (c). Will the phase $\angle H(f)$ affects the design of $G(f)$ for the fulfillment of both matched filter and Nyquist's criterion? Justify your answer.

Solution.

(a) $C_{\text{match}}(f) = \frac{G^*(f)H^*(f)e^{-j2\pi f T_b}}{S_w(f)}$

(b) From (a), we obtain

$$G(f)H(f)C_{\text{match}}(f) = G(f)H(f)\frac{G^*(f)H^*(f)e^{-j2\pi f T_b}}{S_w(f)} = \frac{|G(f)|^2|H(f)|^2}{S_w(f)}e^{-j2\pi f T_b}.$$

Hence, $G(f)$ must fulfill

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{\left|G\left(f-\frac{n}{T_b}\right)\right|^2 \left|H\left(f-\frac{n}{T_b}\right)\right|^2}{S_w\left(f-\frac{n}{T_b}\right)} e^{-j2\pi\left(f-\frac{n}{T_b}\right)T_b} \\
&= \sum_{n=-\infty}^{\infty} \frac{\left|G\left(f-\frac{n}{T_b}\right)\right|^2 \left|H\left(f-\frac{n}{T_b}\right)\right|^2}{S_w\left(f-\frac{n}{T_b}\right)} e^{-j2\pi f T_b} = p_{\text{nz}}\, e^{-j2\pi m_0 T_b}
\end{aligned}$$

for some (real-valued) non-zero constant p_{nz} and some (non-negative) integer m_0.
Note: From Sample Problem 1(d),

$$h_{\text{No\_ISI}}(t) = \text{sinc}\left(\frac{t-m_0 T_b}{T_b}\right)\lambda(t-m_0 T_b)$$

for some $\lambda(t)$ satisfying $\lambda(0) = p_{\text{nz}}$. This implies

$$\frac{|G(f)|^2|H(f)|^2}{S_w(f)}e^{-j2\pi f T_b} = \left(\mathbf{1}\left\{|f| < \frac{1}{2T_b}\right\} \star \Lambda(f)\right) e^{-j2\pi m_0 T_b},$$

where $\Lambda(f) = \mathcal{F}\{\lambda(t)\}$. Equivalently,

$$|G(f)|^2 = \frac{S_w(f)}{|H(f)|^2}\left[\mathbf{1}\left\{|f| < \frac{1}{2T_b}\right\} \star \Lambda(f)\right] e^{-j2\pi f(m_0-1)T_b}.$$

(c) $\lambda(t) = 1$ implies $\Lambda(f) = \delta(f)$. Hence,

$$\begin{aligned}
|G(f)|^2 &= \frac{S_w(f)}{|H(f)|^2}\left[\mathbf{1}\left\{|f| < \frac{1}{2T_b}\right\} \star \delta(f)\right] e^{-j2\pi f(m_0-1)T_b} \\
&= \frac{S_w(f)}{|H(f)|^2}\mathbf{1}\left\{|f| < \frac{1}{2T_b}\right\} e^{-j2\pi f(m_0-1)T_b}
\end{aligned}$$

Since $|G(f)|$ is real-valued, we must have $m_0 = 1$ in order to make the right-hand-size real.

(d) From the solution in (b), it is clear that $\angle H(f)$ does not impact the design of $G(f)$ in (c). In other words, $\angle H(f)$ does not impact the choice of $\angle G(f)$ because $C_{\text{match}}(f)$ will compensate both angles.

3. For sequence transmission, the (timing) error margin becomes an important practical consideration at the receiver. Let Δt be the time difference in the sampling manipulation between transmitter and receiver. Then,

$$y(iT_b + \Delta t) = \sum_{k=-\infty}^{\infty} a_k \cdot h_{\text{No\_ISI}}((i-k)T_b + \Delta t)$$

(a) Set $i = 0$, $T_b = 1$ (hence, $W = \frac{1}{2}$) and $\Delta t = 0.01$. Show that the ISI will become

$$\sum_{k=-\infty,\ k\neq 0}^{\infty} a_k \cdot h_{\text{No\_ISI}}(-k + 0.01).$$

(b) Find $h_{\text{No\_ISI}}(0.01)$ and $|h_{\text{No\_ISI}}(-k + 0.01)|$ for $k = -1$, -2 and -3, if $h_{\text{No\_ISI}}(t) = \text{sinc}(t)$.

(c) Re-do (b), if $h_{\text{No\_ISI}}(t) = \text{sinc}(t)\frac{\cos(\pi t)}{(1-4t^2)}$.

Solution.

(a)

$$\begin{aligned} y(\Delta t) &= \sum_{k=-\infty}^{\infty} a_k \cdot h_{\text{No\_ISI}}(-kT_b + \Delta t) \\ &= a_0 \cdot h_{\text{No\_ISI}}(\Delta t) + \underbrace{\sum_{k=-\infty, k\neq 0}^{\infty} a_k \cdot h_{\text{No\_ISI}}(-kT_b + \Delta t)}_{\text{ISI}} \\ &= a_0 \cdot h_{\text{No\_ISI}}(0.01) + \underbrace{\sum_{k=-\infty, k\neq 0}^{\infty} a_k \cdot h_{\text{No\_ISI}}(-k + 0.01)}_{\text{ISI}} \end{aligned}$$

(b) $h_{\text{No\_ISI}}(0.01) = \frac{\sin(0.01\cdot\pi)}{0.01\cdot\pi} \approx 0.9998$ and

$$\begin{aligned} |\text{sinc}(-k + 0.01) &= \left|\frac{\sin(-k\pi + 0.01\cdot\pi)}{-k\pi + 0.01\cdot\pi}\right| \\ &= \frac{\sin(0.01\cdot\pi)}{-k\pi + 0.01\cdot\pi} = \begin{cases} \frac{\sin(0.01\cdot\pi)}{1.01\pi}, & k = -1 \\ \frac{\sin(0.01\cdot\pi)}{2.01\pi}, & k = -2 \\ \frac{\sin(0.01\cdot\pi)}{3.01\pi}, & k = -3 \end{cases} \\ &\approx \begin{cases} 0.0098994, & k = -1 \\ 0.0049743, & k = -2 \\ 0.0033217, & k = -3 \end{cases} \end{aligned}$$

Note: The sum of the three ISI terms corresponding to $k = -1$, $k = -2$ and $k = -3$ is as large as 0.018195.

(c) $h_{\text{No\_ISI}}(0.01) = \frac{\sin(0.01\pi)}{0.01\pi}\frac{\cos(0.01\pi)}{(1-4\cdot 0.01^2)} \approx 0.999974$ and

$$|h_{\text{No\_ISI}}(-k+0.01) = \left|\frac{\sin(0.02\pi)}{2\pi(-k+0.01)}\frac{1}{(1-4\times(-k+0.01)^2)}\right| \approx \begin{cases} 0.0032121, & k=-1 \\ 0.00032795, & k=-2 \\ 0.000094212, & k=-3 \end{cases}$$

Note: The sum of the three ISI terms corresponding to $k=-1$, $k=-2$ and $k=-3$ is at most 0.0036.

4. (a) Define $H_{\text{ISI}}(f) = H_{\text{CC}}(f)H_{\text{No\_ISI}}(f)$. With

$$H_{\text{CC}}(f) = \sum_{k=0}^{N-1} w_k e^{-j2\pi f k T_b}$$

and

$$\sum_{k=-\infty}^{\infty} H_{\text{No\_ISI}}\left(f - \frac{k}{T_b}\right) = T_b. \tag{6}$$

What is the value of $h_{\text{ISI}}(iT_b)$ for each integer i?

(b) A benefit for adding ISI via $H_{\text{CC}}(f)$ is that a better spectrum efficiency can be reached. It can be derived that the time-averaged PSD of the signal to the channel (i.e., $H(f)$) is given by

$$\overline{\text{PSD}}(f) = \frac{1}{T}\cdot \bar{S}_a(f)\cdot |H_{\text{CC}}(f)|^2 \cdot |G(f)|^2$$

where from Sample Problem 1(a) for Quiz 9, we have

$$\bar{S}_a(f) = \sum_{k=-\infty}^{\infty} \bar{\phi}_a(k)e^{-j2\pi f k T_b} \quad \text{and} \quad \bar{\phi}_a(k) = \lim_{N\to\infty}\frac{1}{2N}\sum_{k=-N}^{N-1} E[a_{m+k}a_m^*].$$

Now with the duo-binary correlative level coding, we have

$$H_{\text{CC}}(f) = 1 + e^{-j2\pi f T}.$$

Determine $|H_{\text{CC}}(f)|^2$.

(c) Determine the null-to-null bandwidth of $\overline{\text{PSD}}(f)$ before and after adding the correlative-level coding in (b) if $\bar{S}_a(f) = 1$ and $|G(f)|^2 = T_b^2 \operatorname{sinc}^2(T_b f)$.

(d) Can we further reduce the null-to-null bandwidth in (c) by using a new correlative level coding design? If your answer is positive, give one example. If your answer is negative, prove it.

Solution.

(a) $h_{\text{ISI}}(iT_b) = \sum_{k=0}^{N-1} w_k a_{i-k}\left(= w_0 a_i + \underbrace{\sum_{k=1}^{N-1} w_k a_{i-k}}_{\text{ISI term}}\right)$

(b)

$$|H_{\text{CC}}(f)|^2 = |1+e^{-j2\pi fT}|^2 = |e^{j2\pi f\frac{T}{2}}+e^{-j2\pi f\frac{T}{2}}|^2 = 4\cos^2(\pi fT).$$

(c)

$$\begin{aligned}
\overline{\text{PSD}}(f) &= \begin{cases} \frac{1}{T_b}\cdot 4\cos^2(\pi fT)\cdot T_b^2\cdot\text{sinc}^2(T_bf), & \text{after adding correlative-level coding;}\\ \frac{1}{T_b}\cdot T_b^2\cdot\text{sinc}^2(T_bf), & \text{before adding correlative-level coding}\end{cases}\\
&= \begin{cases} T_b\cdot 4\cos^2(\pi fT)\cdot\frac{\sin^2(\pi T_bf)}{\pi^2T_b^2f^2}, & \text{after adding correlative-level coding;}\\ T_b\cdot\text{sinc}^2(T_bf), & \text{before adding correlative-level coding}\end{cases}\\
&= \begin{cases} 4T_b\cdot\frac{\sin^2(2\pi T_bf)}{4\pi^2T_b^2f^2}, & \text{after adding correlative-level coding;}\\ T_b\cdot\text{sinc}^2(T_bf), & \text{before adding correlative-level coding}\end{cases}\\
&= \begin{cases} 4T_b\cdot\text{sinc}^2(2T_bf), & \text{after adding correlative-level coding;}\\ T_b\cdot\text{sinc}^2(T_bf), & \text{befor adding correlative-level coding}\end{cases}
\end{aligned}$$

Hence, the null-to-null bandwidth of $\overline{\text{PSD}}(f)$ is equal to

$$\begin{cases} \frac{1}{2T_b}, & \text{after adding correlative-level coding;}\\ \frac{1}{T_b}, & \text{before adding correlative-level coding.}\end{cases}$$

Note: We can halve the bandwidth requirement at the price of adding one ISI term (i.e., $w_0 = w_1 = 1$ but $w_k = 0$ for $k \geq 2$).

(d) The answer is positive. If we introduce more controlled ISI terms, we can further lower the bandwidth requirement. For example, with $w_0 = w_1 = w_2 = w_3 = 1$ and $w_k = 0$ for $k \geq 4$, we have

$$\begin{aligned}
H_{\text{CC}}(f) &= 1+e^{-j2\pi fT_b}+e^{-j2\pi f(2T_b)}+e^{-j2\pi f(3T_b)}\\
&= \frac{1-e^{-j2\pi f(4T_b)}}{1-e^{-j2\pi fT_b}}\\
&= \frac{(e^{j\pi f(4T_b)}-e^{-j\pi f(4T_b)})e^{-j\pi f(4T_b)}}{(e^{j\pi fT_b}-e^{-j\pi fT_b})e^{-j\pi fT_b}}\\
&= \frac{\sin(4\pi fT_b)}{\sin(\pi fT_b)}e^{-j3\pi fT_b},
\end{aligned}$$

which implies

$$\begin{aligned}
\overline{\text{PSD}}(f) &= \frac{1}{T_b}\cdot\frac{\sin^2(4\pi fT_b)}{\sin^2(\pi fT_b)}\cdot T_b^2\cdot\text{sinc}^2(T_bf)\\
&= \frac{1}{T_b}\cdot\frac{\sin^2(4\pi fT_b)}{\sin^2(\pi fT_b)}\cdot T_b^2\cdot\frac{\sin^2(\pi fT_b)}{\pi^2f^2T_b^2}\\
&= 16T_b\cdot\frac{\sin^2(4\pi fT_b)}{16\pi^2f^2T_b^2}\\
&= 16T_b\,\text{sinc}(4fT_b).
\end{aligned}$$

The null-to-null bandwidth is reduced to $\frac{1}{4T_b}$.