Sample Problems for the 10th Quiz

1. What are the two main limitations of AQF in comparison with AQB?

   Solution. Although AQF is in principle a more accurate estimator, it has two main limitations in comparison with AQB; i.e., it requires:

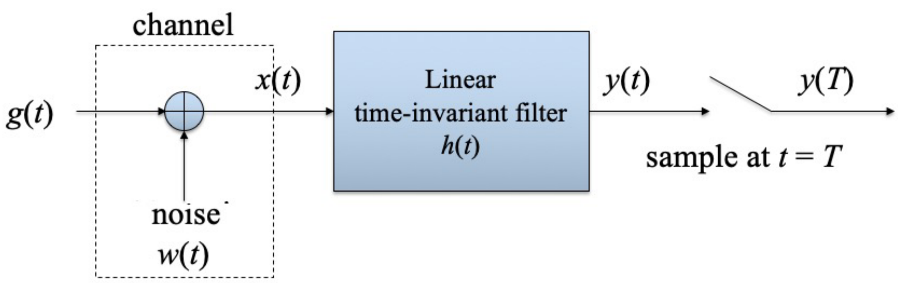
   - an additional buffer to store **unquantized** samples for the learning period (hence a processing delay due to buffering and other operations for AQF is necessary);
   - an explicit transmission of level information $\Delta[n]$ to the receiver (because the receiver only has the quantized samples).

   The above limitations can be relaxed by using AQB.

2. For the one-shot or a single transmission illustrated below (hence, inter-symbol interference is ignored), we have

$$x(t) = a \cdot g(t) + w(t),$$

where $a \in \{\pm 1\}$ is the digital message to be transmitted (i.e., a is the digital message to be carried by $g(t)$).



Note: The WSS $w(t)$ is not necessarily a white noise but could be a colored WSS noise with PSD $S_w(f)$.

(a) Express the input $y(t)$ of the sampler in term of a, $g(t)$, $w(t)$ and $h(t)$.

(b) Express the output $y(T)$ of the sampler in term of a, $G(f)$, $n(T)$ and $H(f)$, where $G(f)$ and $H(f)$ are the Fourier transforms of $g(t)$ and $h(t)$, respectively, and $n(T) = \int_{-\infty}^{\infty} w(\tau)h(T-\tau)d\tau$.

(c) Find $E[n^2(T)]$, provided that $w(t)$ is WSS with PSD $S_W(f)$.
Hint: The PSD of $n(t)$ is $S_n(f) = S_w(f)|H(f)|^2$.

(d) Find the optimal transfer function $H_{\text{opt}}(f)$ such that the output signal-to-noise ratio at the output of the sampler is maximized. In other words, find $H_{\text{opt}}(f)$ such that

$$\eta = \frac{|\,(a \cdot g(t) \star h(t)|_{t=T})|^2}{E[(w(t) \star h(t)|_{t=T})^2]}$$

is maximized.

Note: Since $w(t)$ is not necessarily white, you shall derive

$$\begin{aligned}\eta &= \frac{|a \cdot \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df|^2}{\int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df} \\ &= \frac{|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df|^2}{\int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df} \quad (a \in \{\pm 1\}) \\ &= \frac{\left|\int_{-\infty}^{\infty} \frac{G(f)}{\sqrt{S_w(f)}} \cdot \sqrt{S_w(f)}H(f)e^{j2\pi fT}df\right|^2}{\int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df} \\ &= \cdots\end{aligned}$$

and use the Cauchy-Schwarz inequality.

Solution.

(a)

$$y(t) = a \cdot g(t) \star h(t) + w(t) \star h(t)$$

(b)

$$y(T) = a \cdot \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT}df + n(T)$$

(c) The PSD of $n(t)$ is $S_n(f) = S_w(f)|H(f)|^2$ and $n(t)$ is WSS (because $w(t)$ is WSS). Then, for any t,

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_n(f)df = \int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df.$$

(d)

$$\begin{aligned}\eta &= \frac{\left|\int_{-\infty}^{\infty} \frac{G(f)}{\sqrt{S_w(f)}} \cdot \sqrt{S_w(f)}H(f)e^{j2\pi fT}df\right|^2}{\int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df} \\ &\le \frac{\left(\int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_w(f)}df\right) \cdot \left(\int_{-\infty}^{\infty} S_w(f)|H(f)e^{j2\pi fT}|^2 df\right)}{\int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df} \\ &= \int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_w(f)}df\end{aligned}$$

The upper bound can be achieved by

$$\frac{G^*(f)}{\sqrt{S_w(f)}} = k\sqrt{S_w(f)}H_{\text{opt}}(f)e^{j2\pi fT} \text{ for some constant } k$$

which implies

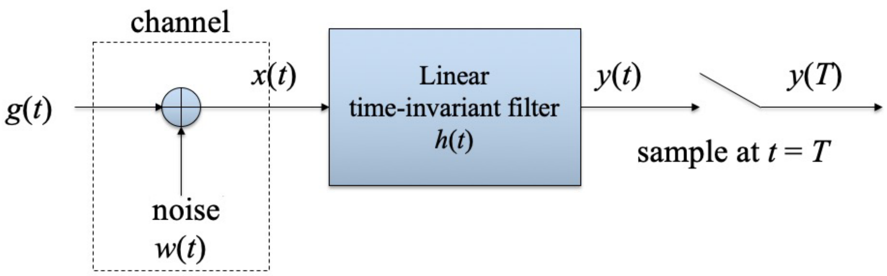
$$H_{\text{opt}}(f) = \frac{1}{k}\frac{G^*(f)}{S_w(f)}e^{-j2\pi fT}.$$

Note: When the noise is not white (and we take $k = 1$ for simplicity),

$$H_{\text{opt}}(f) S_w(f) = G^*(f) e^{-j2\pi f T},$$

i.e., a larger noise power at a specific frequency shall make $H_{\text{opt}}(f)$ smaller such that their product equals $G^*(f)e^{-j2\pi fT}$.

3. We now re-do what we did in our lecture by considering a general color noise.



The diagram above illustrates a one-shot transmission for message a, modeled as

$$x(t) = a \cdot g(t) + w(t) \text{ with } a \in \{\pm 1\}.$$

From the diagram, we know

$$y(t) = a \cdot g(t) \star h(t) + w(t) \star h(t).$$

When sampling at time instance $t = T$, we have

$$\underbrace{y(T)}_{\text{received value } y} = \underbrace{a \cdot g(t) \star h(t)|_{t=T}}_{\text{signal } a \cdot \mathcal{E}_h} + \underbrace{w(t) \star h(t)|_{t=T}}_{\text{noise } n},$$

where $\mathcal{E}_h = g(t) \star h(t)|_{t=T}$ and $\sigma_h^2 = E[n^2]$.

(a) If $w(t)$ is a zero-mean WSS Gaussian noise process with PSD $S_w(f)$, find the mean and variance of n.

(b) What is the probability density function (pdf) of $y = y(T)$ given $a = 1$, denoted as $f(y|a = 1)$?

(c) What is the probability density function (pdf) of $y = y(T)$ given $a = -1$, denoted as $f(y|a = -1)$?

(d) Show that the log-likelihood ratio, $\log\left(\frac{f(y|a=1)}{f(y|a=-1)}\right)$, is a linear function of y.

(e) Let Ψ be the set, where a decision favoring $a = +1$ is made. Suppose $\Pr[a = 1] = p$ and $\Pr[a = -1] = 1 - p$. Then, the detection error can be expressed as

$$\begin{aligned} P_e &= \Pr[a = 1] \cdot \int_{\Psi^c} f(y|a = 1) dy + \Pr[a = -1] \cdot \int_{\Psi} f(y|a = -1) dy \\ &= p \cdot \int_{\Psi^c} f(y|a = 1) dy + (1 - p) \cdot \int_{\Psi} f(y|a = -1) dy. \end{aligned}$$

Reformulate the detection error and argue that the optimal Ψ^* that minimizes the detection error is a likelihood ratio test. Also, give the optimal Ψ^*.

(f) Under equal prior probabilities (i.e., $p = \frac{1}{2}$), show that the optimal detection can be achieved without the knowledge of the noise PSD $S_w(f)$ and regardless of the design of the receive filter $h(t)$.

(g) Express the error probability in (f) in terms of Q-function.

(h) From (g), justify that the matched filter derived in the previous problem minimizes the error probability.

Solution.

(a) Since $w(t)$ is a zero-mean WSS Gaussian noise process with PSD $S_w(f)$, n must be Gaussian with mean

$$E[n] = E\left[\int_{-\infty}^{\infty} h(\tau)w(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} h(\tau)E\left[w(t-\tau)\right]d\tau = 0$$

and variance

$$\sigma_h^2 = E[n^2] = \int_{-\infty}^{\infty} S_w(f)|H(f)|^2 df.$$

(b) $f(y|a=1) = \text{Normal}(\mathcal{E}_h, \sigma_h^2) = \frac{1}{\sqrt{2\pi\sigma_h^2}} \exp\left\{-\frac{(y-\mathcal{E}_h)^2}{2\sigma_h^2}\right\}$

(c) $f(y|a=-1) = \text{Normal}(-\mathcal{E}_h, \sigma_h^2) = \frac{1}{\sqrt{2\pi\sigma_h^2}} \exp\left\{-\frac{(y+\mathcal{E}_h)^2}{2\sigma_h^2}\right\}$

(d) The derivation

$$\begin{aligned}
\log \frac{f(y|a=1)}{f(y|a=-1)} &= \log \frac{\frac{1}{\sqrt{2\pi\sigma_h^2}} \exp\left\{-\frac{(y-\mathcal{E}_h)^2}{2\sigma_h^2}\right\}}{\frac{1}{\sqrt{2\pi\sigma_h^2}} \exp\left\{-\frac{(y+\mathcal{E}_h)^2}{2\sigma_h^2}\right\}} \\
&= \log \exp\left\{\frac{2\mathcal{E}_h}{\sigma_h^2} y\right\} \\
&= \frac{2\mathcal{E}_h}{\sigma_h^2} y
\end{aligned}$$

immediately shows that $\log \frac{f(y|a=1)}{f(y|a=-1)}$ is a linear function of y.

(e)

$$\begin{aligned}
P_e &= p \cdot \int_{\Psi^c} f(y|a=1)dy + (1-p)\cdot \int_{\Psi} f(y|a=-1)dy \\
&= p\left(1 - \int_{\Psi} f(y|a=1)dy\right) + (1-p)\cdot \int_{\Psi} f(y|a=-1)dy \\
&= p - p\int_{\Psi} f(y|a=1)dy + (1-p)\cdot \int_{\Psi} f(y|a=-1)dy \\
&= p + \int_{\Psi} \left((1-p)f(y|a=-1) - pf(y|a=1)\right) dy.
\end{aligned}$$

Thus, in order to minimize P_e, the value of the integration should be made as small as possible, which implies that a choice that can achieve this objective is

$$\begin{aligned}
\Psi^* &= \{y \in \Re : (1-p)f(y|a=-1) - pf(y|a=1) < 0\} \\
&= \left\{y \in \Re : \underbrace{\frac{f(y|a=1)}{f(y|a=-1)}}_{\text{likelihood ratio}} > \frac{1-p}{p}\right\},
\end{aligned}$$

which is apparently a likelihood ratio test.

(f) Under equal prior probabilities,

$$\begin{aligned}\Psi^* &= \left\{ y \in \Re : \underbrace{\frac{f(y|a=1)}{f(y|a=-1)}}_{\text{likelihood ratio}} > 1 \right\} \\ &= \left\{ y \in \Re : \underbrace{\log \frac{f(y|a=1)}{f(y|a=-1)}}_{\text{log-likelihood ratio}} > 0 \right\} \\ &= \left\{ y \in \Re : \frac{\mathcal{E}_h}{\sigma_h^2} y > 0 \right\} \\ &= \left\{ y \in \Re : y > 0 \right\};\end{aligned}$$

Apparently, Ψ^* has nothing to do with $S_w(f)$ and the design of $h(t)$.

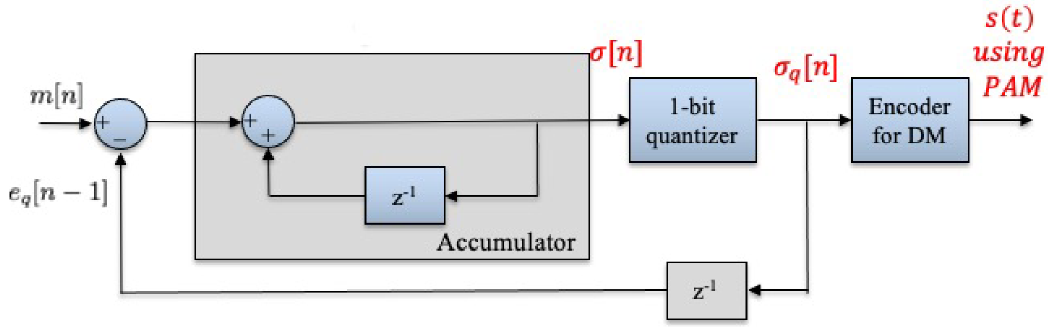
Note: In order to know σ_h^2, we may need to know the PSD of $w(t)$, which is hard to be accurately estimated in many applications.

(g)

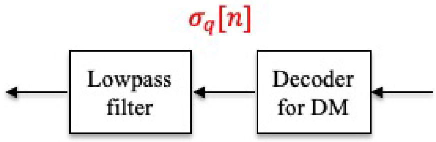
$$\begin{aligned}P_e &= \frac{1}{2} \cdot \int_{-\infty}^{0} f(y|a=1)dy + \frac{1}{2} \cdot \int_{0}^{\infty} f(y|a=-1)dy \\ &= \frac{1}{2} \cdot \int_{-\infty}^{0} \text{Normal}(\mathcal{E}_h, \sigma_h^2)dy + \frac{1}{2} \cdot \int_{0}^{\infty} \text{Normal}(-\mathcal{E}_h, \sigma_h^2)dy \\ &= \int_{-\infty}^{0} \text{Normal}(\mathcal{E}_h, \sigma_h^2)dy \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma_h^2}} \exp\left\{ -\frac{(y-\mathcal{E}_h)^2}{2\sigma_h^2} \right\} dy \quad (\text{Let } z = \frac{y-\mathcal{E}_h}{\sigma_h}) \\ &= \int_{-\infty}^{\frac{-\mathcal{E}_h}{\sigma_h}} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{z^2}{2} \right\} dz \\ &= Q\left(\frac{\mathcal{E}_h}{\sigma_h} \right)\end{aligned}$$

(h) The matched filter maximizes the output SNR $\frac{\mathcal{E}_h^2}{\sigma_h^2}$. Since $Q(\cdot)$ function is monotonically decreasing, to maximize $\mathcal{E}_h^2/\sigma_h^2$ is equivalent to minimizing the error probability in (g). Consequently, the filter design that maximizes the output SNR at the sampler output indirectly minimizes the error probability.

4. This problem demonstrates a situation that the delta-sigma modulation recovers the message signal perfectly. This example also demonstrates the importance of the design of the lowpass filter at the receiver.



Delta-sigma modulator



Delta-sigma demodulator

The above delta-sigma modulation follows the below equations:

$$\begin{aligned}\sigma_q[n-1] &= \begin{cases}\Delta, & \sigma[n-1] \geq 0\\ -\Delta, & \sigma[n-1] < 0\end{cases}\\ \sigma[n] &= \sigma[n-1] + (m[n] - \sigma_q[n-1])\end{aligned}$$

(a) Let $m(t) = \sin(2\pi t)$ and $T_s = \frac{1}{4}$. Find the sequence of $\{m[n] = m(nT_s), n \geq 0\}$.

(b) Initialize $\sigma[0] = \sigma_q[0] = -\Delta$ for $0 < \Delta < 1$. Find the sequence of $\{\sigma_q[n], n \geq 1\}$.

(c) Let the transmitting waveform $s(t)$ corresponding to $\{\sigma_q[n], n \geq 1\}$ be

$$s(t) = \sum_{n=-\infty}^{\infty} \sigma_q[n+1] \cdot a(t - nT_s),$$

where

$$a(t) = \begin{cases}1, & 0 \leq t < T_s\\ 0, & \text{otherwise}\end{cases}$$

Find the Fourier transform of $s(t)$.

Note: Since $\{\sigma_q[n], n \geq 1\}$ is periodic, we set $\{\sigma_q[n], n \leq 0\}$ to be a periodic extension of $\{\sigma_q[n], n \geq 1\}$ for analytical convenience.

Hint: $\sum_{k=-\infty}^{\infty} e^{-j2\pi fk} = \sum_{k=-\infty}^{\infty} \delta(f-k)$

(d) The ideal lowpass filter removes all frequencies of $S(f)$ larger than or equal to 1.5 Hz (and keeps all frequencies of $S(f)$ below 1.5 Hz). Check whether the output of the lowpass filter is proportional to $m(t)$.

Solution.

(a) $\{m[n], n \geq 0\} = \{0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots\}$.

(b) From

$$\begin{aligned} \sigma_q[n-1] &= \begin{cases} \Delta, & \sigma[n-1] \geq 0 \\ -\Delta, & \sigma[n-1] < 0 \end{cases} \\ \sigma[n] &= \sigma[n-1] + (m[n] - \sigma_q[n-1]) \end{aligned}$$

we build up the below table (Initial values are set at $n = 0$)

n	0	1	2	3	4	5	6	7	8	$\cdots$
$m[n]$	0	1	0	-1	0	1	0	-1	0	$\cdots$
$\sigma[n]$	$-\Delta$	1	$1-\Delta$	-2Δ	$-\Delta$	1	$1-\Delta$	-2Δ	$-\Delta$	$\cdots$
$\sigma_q[n]$	$-\Delta$	Δ	Δ	$-\Delta$	$-\Delta$	Δ	Δ	$-\Delta$	$-\Delta$	$\cdots$
$m[n]-\sigma_q[n]$	0	$1+\Delta$	$-\Delta$	$-1-\Delta$	Δ	$1+\Delta$	$-\Delta$	$-1-\Delta$	Δ	$\cdots$

We therefore have

$$\{\sigma_q[n], n \geq 1\} = \{\Delta, \Delta, -\Delta, -\Delta, \Delta, \Delta, -\Delta, -\Delta, \ldots\}$$

(c) The waveform corresponds to this $\{\sigma_q[n]\}$ can be re-formulated as

$$s(t) = \sum_{k=-\infty}^{\infty} g(t-k)$$

where

$$g(t) = \begin{cases} \Delta, & 0 \leq t < 0.5 \\ -\Delta, & -0.5 \leq t < 0 \end{cases}$$

Derive

$$\begin{aligned} G(f) &= \Delta \cdot \mathcal{F}\big\{\mathbf{1}\{|t| \leq 0.25\}\big\}\big(e^{j2\pi f(-0.25)} - e^{j2\pi f(0.25)}\big) \\ &= \Delta \cdot 0.5\,\text{sinc}(0.5f)\,\big(-j2\sin(0.5\pi f)\big) \\ &= -j\Delta\,\text{sinc}(0.5f)\,\sin(0.5\pi f) \end{aligned}$$

The Fourier transform of $s(t)$ is given by

$$\begin{aligned} S(f) &= \mathcal{F}\{s(t)\} = \sum_{k=-\infty}^{\infty} \mathcal{F}\{g(t-k)\} \\ &= \sum_{k=-\infty}^{\infty} G(f)e^{j2\pi f(-k)} \\ &= G(f)\sum_{k=-\infty}^{\infty} e^{-j2\pi fk} \\ &= G(f)\sum_{k=-\infty}^{\infty} \delta(f-k) \\ &= (-j\Delta\,\text{sinc}(0.5f)\,\sin(0.5\pi f))\sum_{k=-\infty}^{\infty} \delta(f-k) \\ &= -j\frac{2\Delta}{\pi}\sum_{k\text{ odd}} \frac{1}{k}\delta(f-k). \end{aligned}$$

(d) The lowpass filter removes the high frequency components and produces

$$-j\frac{2\Delta}{\pi}\sum_{k=\pm 1}\frac{1}{k}\delta(f-k) = -j\frac{2\Delta}{\pi}[-\delta(f+1)+\delta(f-1)] = \frac{4\Delta}{\pi}\frac{[\delta(f-1)-\delta(f+1)]}{2j},$$

which is exactly the Fourier transform of $\frac{4\Delta}{\pi}\sin(2\pi t)$.

Note: In this example, you may notice that $s(t)$ and $m(t)$ have the same **zero-crossings** at $t = \ldots, -1, -0.5, 0, 0.5, 1, \ldots$. Thus, the delta-sigma modulation keeps the zero-crossings of $m(t)$ but may not retain its amplitude variations. Hence, a signal

$$m_{\text{new}}(t) = \begin{cases} A\sin(2\pi t), & k \le t < k+\frac{1}{2}; \\ \frac{1}{A}\sin(2\pi t), & k+\frac{1}{2} \le t < k+1 \end{cases}$$

may produce the same $\{\sigma_q[n]\}$ as $m(t) = \sin(2\pi t)$ but the recovered waveform will be different from $m_{\text{new}}(t)$.