Sample Problems for the 8th Quiz (May 10)

- Slides 6-46∼58 will not be covered by quizzes or the final exam.
- Correction for Sample Problem 3(b): In the solution, a "2" is missing in the note.

  Note: In such case, no improvement can be obtained by pre- and de-emphasis filters and

$$|H_{\text{de,opt}}(f)|^2 = \left(\frac{2W}{P}\sqrt{M_0 N_0}\right)\sqrt{\frac{M_0}{N_0}} = 1.$$

- Correction for Sample Problem 4(b): The first equation in the solution should be

$$\Phi_e(f) = \frac{j2\pi f}{j2\pi f + 2\pi k_0}\Phi_1(f) = (j2\pi f\, \Phi_1(f)) \cdot \frac{1}{j2\pi f + 2\pi k_0}$$

1. (Sampling Theorem) When performing *sampling* on a signal $g(t)$, we obtain the sampled signal

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot \delta(t - nT_s),$$

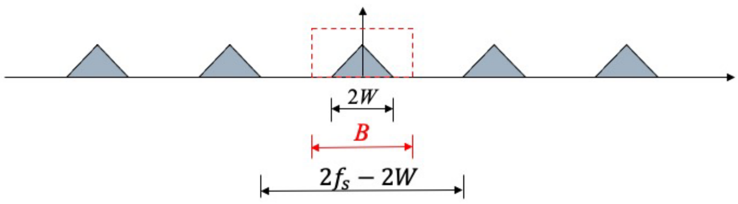
where T_s is the sampling period. Let the sampling rate be denoted as $f_s = \frac{1}{T_s}$.

(a) Show that the Fourier transform of $g_\delta(t)$ is

$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G\left(f - nf_s\right),$$

where $G(f) = \mathcal{F}\{g(t)\}$.

(b) Suppose $G(f)$ is bandlimited with bandwidth W. Then, $G_\delta(f)$ can be depicted as:



Show that we can reconstruct $g(t)$ from its samples $\{g(nT_s)\}_{n=-\infty}^{\infty}$ via

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot BT_s \operatorname{sinc}(B(t - nT_s))$$

for any B satisfying $2W \le B \le 2(f_s - W)$.

(c) Continue from (b). Can we reconstruct $g(t)$ from its samples $\{g(nT_s)\}_{n=-\infty}^{\infty}$ via

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot h(t - nT_s),$$

where the Fourier transform of $h(t)$ satisfies

$$H(f) = \mathcal{F}\{h(t)\} = \begin{cases} 1, & |f| \leq W; \\ 0, & |f| \geq f_s - W; \\ \text{arbitrary}, & \text{otherwise.} \end{cases}$$

(d) Continue from (b). Show that $g(t) = g_\delta(t) \star h(t)$, where "$\star$" denotes the convolution operation, and $h(t) = BT_s \operatorname{sinc}(Bt)$.

Solution.

(a) First, we note that

$$\begin{aligned} G_\delta(f) &= \mathcal{F}\{g_\delta(t)\} \\ &= \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} g(nT_s) \cdot \delta(t - nT_s) \right\} \\ &= \sum_{n=-\infty}^{\infty} g(nT_s) \cdot \mathcal{F}\left\{\delta(t - nT_s)\right\} \\ &= \sum_{n=-\infty}^{\infty} g(nT_s) \cdot e^{-j2\pi nT_s f} \end{aligned} \tag{1}$$

Second, it is clear that $f_s \sum_{n=-\infty}^{\infty} G\left(f - nf_s\right)$ is a periodic function (in the frequency domain) with period f_s. By Fourier series expansions,

$$f_s \sum_{n=-\infty}^{\infty} G\left(f - nf_s\right) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi \frac{n}{f_s} f}, \tag{2}$$

where

$$\begin{aligned}
c_n &= \frac{1}{f_s}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}}\left(f_s\sum_{n=-\infty}^{\infty}G\left(f-nf_s\right)\right)e^{-j2\pi\frac{n}{f_s}f}df\\
&= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}}\left(\sum_{n=-\infty}^{\infty}G\left(f-nf_s\right)\right)e^{-j2\pi\frac{n}{f_s}f}df\\
&= \sum_{n=-\infty}^{\infty}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}}G\left(f-nf_s\right)e^{-j2\pi\frac{n}{f_s}f}df \quad (s=f-nf_s)\\
&= \sum_{n=-\infty}^{\infty}\int_{-\frac{f_s}{2}-nf_s}^{\frac{f_s}{2}-nf_s}G\left(s\right)e^{-j2\pi\frac{n}{f_s}(s+nf_s)}ds\\
&= \sum_{n=-\infty}^{\infty}\int_{-\frac{f_s}{2}-nf_s}^{\frac{f_s}{2}-nf_s}G\left(s\right)e^{-j2\pi\frac{n}{f_s}s}ds\\
&= \int_{-\infty}^{\infty}G\left(s\right)e^{-j2\pi\frac{n}{f_s}s}ds\\
&= g\left(-\frac{n}{f_s}\right)=g(-nT_s).
\end{aligned}$$

Consequently, equating (1) and (2) completes the proof.

(b)

$$\begin{aligned}
g(t) &= \int_{-\infty}^{\infty}G(f)e^{j2\pi ft}\mathrm{d}f\\
&= \int_{-B/2}^{B/2}G(f)e^{j2\pi ft}\mathrm{d}f \quad (\text{Because } B\geq 2W)\\
&= \int_{-B/2}^{B/2}\left(\frac{1}{f_s}\sum_{n=-\infty}^{\infty}g(nT_s)e^{-j2\pi nT_s f}\right)e^{j2\pi ft}\mathrm{d}f\\
&\qquad \left(\text{Because } G(f)=\frac{1}{f_s}G_\delta(f) \text{ for } |f|\leq W\leq\frac{B}{2}\right.\\
&\qquad\qquad \left.\text{and } G_\delta(f)=\sum_{n=-\infty}^{\infty}g(nT_s)e^{-j2\pi nT_s f}\right)\\
&= \frac{1}{f_s}\sum_{n=-\infty}^{\infty}g(nT_s)\int_{-B/2}^{B/2}e^{j2\pi f(t-nT_s)}\mathrm{d}f\\
&= \sum_{n=-\infty}^{\infty}g(nT_s)\cdot BT_s\,\mathrm{sinc}(B(t-nT_s))
\end{aligned}$$

Note: In Slide 6-11, we simply take $B(=B_{\min})=2W$.

(c) Yes.

(d)

$$\begin{aligned}
g(t) &= \sum_{n=-\infty}^{\infty} g(nT_s) \cdot BT_s \operatorname{sinc}(B(t-nT_s)) \\
&= \sum_{n=-\infty}^{\infty} g(nT_s) \cdot \int_{-\infty}^{\infty} \delta(\tau - nT_s) \underbrace{BT_s \operatorname{sinc}(B(t-\tau))}_{=h(t-\tau)} d\tau \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} g(nT_s) \cdot \delta(\tau - nT_s) \right) h(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} g_\delta(\tau) h(t-\tau) d\tau \\
&= g_\delta(t) \star h(t).
\end{aligned}$$

2. The sample-and-hold (s/h) output can be expressed as

$$m_{\text{s/h}}(t) = m_\delta(t) \star h(t),$$

where

$$m_\delta(t) = \sum_{k=-\infty}^{\infty} m(kT_s)\delta(t - kT_s), \quad h(t) = \begin{cases} 1, & 0 \le t < T; \\ 0, & \text{otherwise.} \end{cases}$$

and $T_s > T$ is the sampling period. Denote by $H(f)$, $M(f)$ and $M_\delta(f)$ the Fourier transforms of $h(t)$, $m(t)$ and $m_\delta(t)$, respectively.

(a) Show that the spectrum of $m_{\text{s/h}}(t)$ is equal to

$$M_{\text{s/h}}(f) = \frac{1}{T_s} H(f) \sum_{k=-\infty}^{\infty} M\left(f - \frac{k}{T_s}\right).$$

Hint: From Sampling theorem,

$$M_\delta(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} M\left(f - \frac{k}{T_s}\right).$$

(b) Suppose $M(t)$ is band-limited to W, which satisfies $\frac{1}{T_s} > 2W$, and the receiver passes $M_{\text{s/h}}(f)$ through an ideal lowpass filter of bandwidth W. We obtain

$$\left[M_{\text{s/h}}(f)\right]_{\text{Lowpass}} = \frac{1}{T_s} H(f) M(f).$$

Find the ratio of

$$\frac{\min_{|f| \le W} |H(f)|}{\max_{|f| \le W} |H(f)|},$$

and check the above ratio when $T = 0.1T_s$.
Hint: $H(f) = T\text{sinc}(fT)e^{-j\pi fT}$

(c) When $T \leq 0.1T_s$, the ratio in (b) is approximately one. Thus,

$$\left[M_{\rm s/h}(f)\right]_{\rm Lowpass} \approx \frac{1}{T_s}|H(0)|e^{-j\pi fT}M(f) = \frac{T}{T_s}e^{-j\pi fT}M(f).$$

Find the inverse Fourier transform of the $\frac{T}{T_s}e^{-j\pi fT}M(f)$.

Solution.

(a) First, we have $M_{\rm s/h}(f) = M_\delta(f)H(f)$. Hence,

$$M_{\rm s/h}(f) = \frac{1}{T_s}H(f)\sum_{k=-\infty}^{\infty} M\left(f - \frac{k}{T_s}\right).$$

(b) $T < T_s < \frac{1}{2W}$ implies $2W < \frac{1}{T}$. Thus,

$$\max_{|f|\leq W}|H(f)| = \max_{|f|\leq W} T{\rm sinc}(Tf) = T$$

and

$$\min_{|f|\leq W}|H(f)| = |H(W)| = T{\rm sinc}\left(TW\right) = T\cdot\frac{\sin(\pi TW)}{\pi TW} = \frac{\sin(\pi TW)}{\pi W}.$$

Therefore, the ratio is equal to

$$\frac{\sin(\pi TW)}{\pi TW}.$$

When $T = 0.1T_s < 0.1\cdot\frac{1}{2W}$, we have $TW < 0.05$ and

$$\frac{\sin(\pi TW)}{\pi TW} > \frac{\sin(\pi\cdot 0.05)}{\pi\cdot 0.05} \approx 0.9959.$$

(c)

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{T}{T_s}e^{-j\pi fT}M(f)e^{j2\pi ft}df &= \frac{T}{T_s}\int_{-\infty}^{\infty}M(f)e^{j2\pi f(t-T/2)}df \\ &= \frac{T}{T_s}m\left(t-\frac{T}{2}\right).\end{aligned}$$

Thus, the receiver recovers a delayed version of $m(t)$.

Note: The equalizer $\frac{1}{H(f)}$ is non-causal due to the phase requirement $e^{j\pi fT}$. From filter design aspect, a non-causal filter is not feasible. However, we do not really need to "implement" $e^{j\pi fT}$ to recover the signal but simply to accept $m(t-\frac{T}{2})$ as the output, knowing that the received signal will begin at $t = \frac{T}{2}$ (or will have a delay of $\frac{T}{2}$).

3. Let the range of the message m be $[-n_{\max}, n_{\max}] = [-1, 1)$. Define a uniform quantizer $g(\cdot)$ as

$$g(m) = -A + k\Delta - \frac{\Delta}{2} \quad \text{for } -A+(k-1)\Delta \leq m < -A + k\Delta$$

where $\Delta = \frac{2A}{L}$ and $1 \leq k \leq L$.

(a) Under L even, is $g(\cdot)$ of midtread type, or of midrise type?

(b) Under L odd, is $g(\cdot)$ of midtread type, or of midrise type?

(c) Under the condition that m is uniformly distributed over $[-1, 1)$, determine the output SNR under $A = 2$.

Hint: $g(M) = M - Q$, where the (imaginarily equivalent) quantization noise Q is still uniformly distributed over $[-\Delta/2, \Delta/2)$.

(d) Under the condition that m is uniformly distributed over $[-1, 1)$, determine the output SNR under $A = 0.5$.

Solution.

(a) $g(\cdot)$ is a midrise quantizer if there exists an integer k such that $-A+k\Delta = -A+k\frac{2A}{L} = 0$, which implies $k = \frac{L}{2}$. Thus, whenever L is even, the uniform $g(\cdot)$ is of midrise type.

(b) Based on "$g(\cdot)$ is a midrise quantizer if there exists an integer k such that $-A+k\Delta = -A + k\frac{2A}{L} = 0$," we can similarly justify that the uniform $g(\cdot)$ is of midtread type if L is odd.

(c) With $A = 2$, we have $\Delta = \frac{2A}{L} = \frac{2\cdot m_{\text{max}}}{L/2} = \frac{2\cdot m_{\text{max}}}{L'}$, where we set $L' = L/2$. Hence, we can follow exactly the same derivation on Slide 6-37∼38 to obtain

$$\text{SNR}_O = \frac{3P}{m_{\text{max}}^2}(L')^2 = \frac{3P}{4m_{\text{max}}^2}L^2 = \frac{3P}{4}L^2 = \frac{1}{4}L^2.$$

where $P = E[m^2] = \int_{-1}^{1} m^2\left(\frac{1}{2}\right)dm = \frac{1}{3}$.

Note: With only 50% load (i.e., $\frac{m_{\text{max}}}{A} = 50\%$) in this example, we can still obtain 6 dB gain whenever the resolution is increased by 1 bit. However, the output signal-to-noise ratio of a 50%-load quantizer will be 6 dB worse than that of a full-load quantizer, which is $3PL^2 = L^2$ from Slide 6-37.

(d) When $A = 0.5$, we have

$$Q = M - g(M) = \begin{cases} M \bmod \Delta - \frac{\Delta}{2}, & |M| < \frac{1}{2}; \\ M - (\frac{1}{2} - \frac{\Delta}{2}), & M \geq \frac{1}{2} \\ M - (-\frac{1}{2} + \frac{\Delta}{2}), & M \leq -\frac{1}{2} \end{cases}$$

Hence,

$$Q \text{ is uniformly distributed over } \begin{cases} [-\frac{\Delta}{2}, \frac{\Delta}{2}), & \text{given } |M| < \frac{1}{2}; \\ [\frac{\Delta}{2}, \frac{1}{2} + \frac{\Delta}{2}), & \text{given } \frac{1}{2} \leq M < 1; \\ [-\frac{\Delta}{2}, -\frac{1}{2} - \frac{\Delta}{2}), & \text{given } -1 \leq M \leq \frac{1}{2}. \end{cases}$$

(i.e., $|Q|$ is uniformly distributed over $[\frac{\Delta}{2}, \frac{1}{2} + \frac{\Delta}{2})$, given that $|M| \geq \frac{1}{2}$). Thus, we derive

$$\begin{aligned}
E[Q^2] &= \Pr\left[|M| < \frac{1}{2}\right] E\left[Q^2 \middle| |M| < \frac{1}{2}\right] \\
&\quad + \Pr\left[|M| \geq \frac{1}{2}\right] E\left[Q^2 \middle| |M| \geq \frac{1}{2}\right] \\
&= \frac{1}{2}\int_{-\Delta/2}^{\Delta/2} q^2 \left(\frac{1}{\Delta}\right) dq + \frac{1}{2}\int_{\Delta/2}^{\frac{1}{2}+\frac{\Delta}{2}} q^2 \cdot 2\, dq \\
&= \frac{1}{24}\Delta^2 + \frac{1}{24}\left(3\Delta^2 + 3\Delta + 1\right) \\
&= \frac{1}{6}\Delta^2 + \frac{1}{8}\Delta + \frac{1}{24},
\end{aligned}$$

which implies

$$\text{SNR}_O = \frac{P}{\frac{1}{6}\Delta^2 + \frac{1}{8}\Delta + \frac{1}{24}} = \frac{24PL^2}{4 + 3L + L^2} \quad \Big(\leq 24P = 8 \approx 9.03 \text{ dB}\Big).$$

Note: In comparison with a full-load uniform quantizer without saturation, where $\text{SNR}_O = 3PL^2$ with $m_{\max} = 1$, quantizer saturation will seriously degrade the output SNR and will also "saturate" (i.e., has an upper bound on) the SNR_O.

4. (a) For given intervals $\{I_k\}_{k=1}^L$, show that the optimal representation levels $\{v_k\}_{k=1}^L$ for square error distortion measure $d(m, v_j) = (m - v_j)^2$ is given by

$$v_{k,\text{optimal}} = \frac{\int_{I_k} m \cdot f_M(m)dm}{\int_{I_k} f_M(m)dm},$$

provided that $f_M(\cdot)$ is the pdf of the message M.
Hint: $\min_{\{v_k\}_{k=1}^L} \sum_{k=1}^L \int_{I_k} d(m, v_k) f_M(m)dm$

(b) In Slide 6-44, an exercise is given, asking "What is the best $\{m_k\}$ and $\{v_k\}$ if M is uniformly distributed over $[-A, A)$." Find the solution of it.

Solution.

(a)

$$\begin{aligned}
\frac{\partial}{\partial v_j}\left(\sum_{k=1}^L \int_{I_k} d(m, v_k) f_M(m)dm\right) &= \sum_{k=1}^L \frac{\partial}{\partial v_j}\int_{I_k} d(m, v_k) f_M(m)dm \\
&= \frac{\partial}{\partial v_j}\int_{I_j} d(m, v_j) f_M(m)dm \\
&= \int_{I_j} \frac{\partial (m - v_j)^2}{\partial v_j} f_M(m)dm \\
&= \int_{I_j} [-2(m - v_j)] f_M(m)dm = 0
\end{aligned}$$

implies

$$\begin{aligned}
&\int_{I_j} m f_M(m)dm - \int_{I_j} v_j f_M(m)dm = 0 \\
\Leftrightarrow\quad & \int_{I_j} m f_M(m)dm = v_j \int_{I_j} f_M(m)dm \\
\Leftrightarrow\quad & v_j = \frac{\int_{I_j} m f_M(m)dm}{\int_{I_j} f_M(m)dm}
\end{aligned}$$

With $m_1 = -A$ and $m_{L+1} = A$, we define

$$\begin{aligned}
f(m_2, \ldots, m_L) &\triangleq \frac{1}{2A}\sum_{k=1}^{L}\int_{m_k}^{m_{k+1}} \left(m - \frac{m_k + m_{k+1}}{2}\right)^2 dm \\
&= \frac{1}{2A}\sum_{k=1}^{L}\frac{1}{3}\left(m - \frac{m_k + m_{k+1}}{2}\right)^3\Bigg|_{m_k}^{m_{k+1}} \\
&= \frac{1}{6A}\sum_{k=1}^{L}\left[\left(\frac{m_{k+1} - m_k}{2}\right)^3 - \left(\frac{m_k - m_{k+1}}{2}\right)^3\right] \\
&= \frac{1}{24A}\sum_{k=1}^{L}(m_{k+1} - m_k)^3
\end{aligned}$$

For $2 \leq j \leq L$, the optimal solutions should satisfy

$$\frac{\partial f(m_2, \cdots, m_L)}{\partial m_j} = \frac{1}{8A}(m_j - m_{j-1})^2 - \frac{1}{8A}(m_{j+1} - m_j)^2 = 0,$$

which indicates $m_j - m_{j-1}$ must be a constant for $2 \leq j \leq L$. Accordingly, $m_j = -A + (j-1)\frac{2A}{L}$ for $1 \leq j \leq L+1$.