Sample Problems for the 7th Quiz (May 3)

Corrections

- Slide 6-16: The equation at the bottom is perhaps better rewritten as

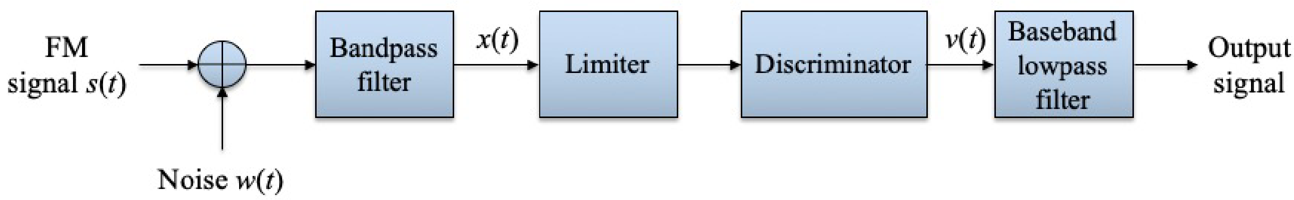
$$g_\delta(t) \star g_{\text{realizable}}(t) \rightleftharpoons G_\delta(f) G_{\text{realizable}}(f) = G_\delta(f) G_{\text{ideal}}(f) \rightleftharpoons g_\delta(t) \star g_{\text{ideal}}(t)$$

- Slide 6-26: "Section 3.9" can be replaced by "Part 7."

1. For an FM receiver below, we have $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau$. Denote the passband noise process as

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) = r(t) \cos(2\pi f_c t + \Psi(t)),$$

where $n_I(t) = r(t) \cos[\Psi(t)]$ and $n_Q(t) = r(t) \sin[\Psi(t)]$.



Due to the limiter, the amplitude of $x(t)$ is of no influence on its output but only the phase remains. Hence, we can assume the amplitude is equal to 1 for simplicity. Then, from Sample Problem 4 for the midterm, the output of the limiter $x_{\text{limiter}}(t)$ is given by

$$x_{\text{limiter}}(t) = \cos[2\pi f_c t + \theta(t)]$$

where

$$\theta(t) = \phi(t) + \tan^{-1}\left(\frac{r(t)\sin[\Psi(t) - \phi(t)]}{A_c + r(t)\cos[\Psi(t) - \phi(t)]}\right). \tag{1}$$

(a) Since $r(t)\cos[\Psi(t) - \phi(t)]$ and $r(t)\sin[\Psi(t) - \phi(t)]$ have exactly the same joint distribution as $n_I(t) = r(t)\cos[\Psi(t)]$ and $n_Q(t) = r(t)\sin[\Psi(t)]$, we rewrite (1) as

$$\theta(t) = \phi(t) + \tan^{-1}\left(\frac{n_Q(t)}{A_c + n_I(t)}\right).$$

The discriminator then outputs the derivative of the phase, i.e., $v(t) = \theta'(t)$. Show that

$$v(t) \approx \phi'(t) + \frac{n_Q'(t)}{A_c},$$

provided $A_c \gg n_I(t)$ and $A_C \gg n_Q(t)$ with high probability.

Hint:

$$\left(\tan^{-1}\left(\frac{f(t)}{g(t)}\right)\right)' = \frac{f'(t)g(t) - f(t)g'(t)}{f^2(t) + g^2(t)}.$$

(b) With

$$v(t) \approx \phi'(t) + \frac{n'_Q(t)}{A_c} = 2\pi k_f m(t) + 2\pi n_d(t),$$

where $2\pi A_c n_d(t) = n'_Q(t)$, show that the SNR_O of the FM system is given by

$$\text{SNR}_\text{O} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3},$$

provided $E[m^2(t)] = P$ and the bandwidth W of the ideal baseband lowpass filter satisfies $W < B_T$.

Hint: The PSD of $n_d(t)$ is equal to

$$\text{PSD}_{n_d}(f) = \left|\frac{f}{A_c}\right|^2 \text{PSD}_{n_Q}(f) = \frac{f^2}{A_c^2} N_0 \quad \text{for } |f| < \frac{B_T}{2}.$$

(c) Based on the the effective noise power of $2\pi n_d(t)$ at the demodulator output in (b), explain what the noise quieting effect of an FM system is.

(d) Recall from Slides 4-91 and 4-92 that for the discriminator to work properly, we require

$$\left|\frac{2k_f}{B_T} m(t)\right| \leq 1.$$

Thus, we can derive $4k_f^2 m^2(t) \leq B_T^2$, implying

$$4k_f^2 E[m^2(t)] = 4k_f^2 P \leq B_T^2.$$

Find the largest k_f such that SNR_O in (b) is maximized.

Solution.

(a)

$$\begin{aligned}
v(t) &= \phi'(t) + \left(\frac{n'_Q(t)(A_c + n_I(t)) - n_Q(t) n'_I(t)}{(A_c + n_I(t))^2 + n_Q^2(t)}\right) \\
&\approx \phi'(t) + \left(\frac{n'_Q(t) A_c - n_Q(t) n'_I(t)}{A_c^2}\right) \\
&\quad \text{Because } A_c \gg n_I(t) \text{ and } A_c \gg n_Q(t) \text{ with high probability} \\
&= \phi'(t) + \frac{n'_Q(t)}{A_c} - \underbrace{\left(\frac{n_Q(t)}{A_c}\right)}_{\substack{A_c \gg n_Q(t) \\ \text{with high probability}}} \left(\frac{n'_I(t)}{A_c}\right) \\
&\approx \phi'(t) + \frac{n'_Q(t)}{A_c}.
\end{aligned}$$

Note: $A_c \gg n_Q(t)$ does not imply $A_c \gg n'_Q(t)$.

(b) We can ignore the 2π factor and compute the SNR_O based on $k_f m(t) + n_d(t)$, from which the average signal power is equal to

$$E[(k_f m(t))^2] = k_f^2 E[m^2(t)] = k_f^2 P.$$

The average noise power is given by

$$\int_{-W}^{W} \text{PSD}_{n_d}(f)df = \int_{-W}^{W} \frac{f^2}{A_c^2} N_0 df = \frac{2N_0 W^3}{3A_c^2}.$$

This implies

$$\text{SNR}_\text{O} = \frac{k_f^2 P}{\frac{2N_0 W^3}{3A_c^2}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}.$$

(c) For fixed noise power level N_0 and fixed message bandwidth W, increasing carrier power A_c^2 will decrease the effective noise power at the demodulator output (by a factor of $1/A_c^2$). This is called the noise quieting effect.

(d) It is obvious that taking $k_f^2 = B_T^2/(4P)$ maximizes

$$\text{SNR}_\text{O} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3} = \frac{3A_c^2 B_T^2}{8N_0 W^3}.$$

Note: This indicates the maximum SNR_O is proportional to B_T^2 and hence can be improved by increasing B_T.

2. Continuing from Problem 1, we know that the exact relation of $v(t)$, input $\phi'(t)$ and noise term $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$ should be:

$$v(t) \;=\; \phi'(t) + \underbrace{\left(\frac{n'_Q(t)(A_c + n_I(t)) - n_Q(t)n'_I(t)}{(A_c + n_I(t))^2 + n_Q^2(t)}\right)}_{=2\pi\cdot n_d(t)}.$$

(a) Let $n_I(t) = \lambda A_c \cos(\psi(t))$ and $n_Q(t) = \lambda A_c \sin(\psi(t))$. Show that

$$2\pi n_d(t) = \lambda\psi'(t)\frac{\cos(\psi(t)) + \lambda}{1 + 2\lambda\cos(\psi(t)) + \lambda^2}$$

(b) Find the value of $n_d(t)$ when $\psi(t) = \pi$, and argue that there is a discontinuity at $\lambda = 1$.

(c) (Just for your reference. Not a part of quizzes and final exam.) The previous sub-problem indicates that when

$$r^2(t) = n_I^2(t) + n_Q^2(t) \approx A_c^2 \quad \text{and} \quad \psi(t) \approx \pi,$$

a noise spike will occur. Noting that $E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)] = B_T N_0$, we know from Slide 3-46 that

i) the pdf of $r(t)$ is Rayleigh-distributed:

$$f_{r(t)}(x) = \frac{x}{B_T N_0} e^{-\frac{x^2}{2B_T N_0}} \text{ for } x \geq 0,$$

ii) $\psi(t)$ is uniformly distributed over $[0, 2\pi)$, and

iii) $r(t)$ and $\psi(t)$ are independent.

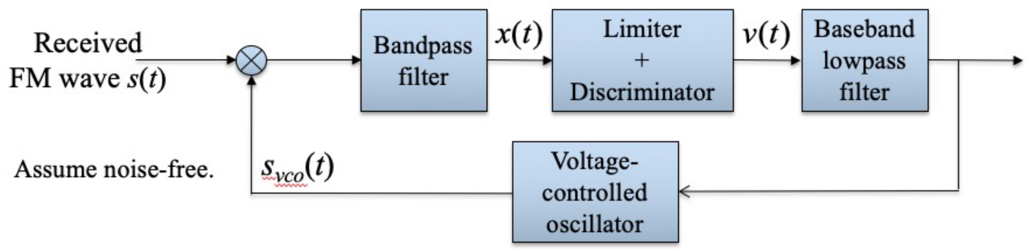
For $0 < \epsilon < 1$, show that

$$\Pr\left[\left|\frac{r^2(t)}{A_c^2} - 1\right| \le \epsilon \text{ and } \left|\frac{\psi(t)}{\pi} - 1\right| \le \epsilon\right] = 2\epsilon e^{-\frac{A_c^2}{2B_T N_0}} \sinh\left(\frac{A_c^2}{2B_T N_0}\epsilon\right).$$

(d) (Just for your reference. Not a part of quizzes and final exam.) A way to reduce the threshold 20 in the requirement

$$\frac{A_c^2}{2B_T N_0} \ge 20$$

for "near-elimination" of clicking-sound effect is to reduce the "effective noise power" from $B_T N_0$ down to $(1-\alpha)B_T N_0$ by introducing a feedback as shown below:



Denote $s(t) = \cos(2\pi f_c t + \phi(t))$, and let $s_{\text{vco}}(f) = 2\cos(2\pi f_{\text{vco}} t + \phi_{\text{vco}}(t))$ with $\phi_{\text{vco}}(t) = \alpha\phi(t)$. Show that

$$x(t) = \cos(2\pi(f_c - f_{\text{vco}})t + (1-\alpha)\phi(t)),$$

provided that the transfer function $H_{\text{vco}}(f)$ of the ideal bandpass filter satisfies

$$H_{\text{vco}}(f) = \begin{cases} 1, & |f - (f_c - f_{\text{vco}})| < (1-\alpha)\frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

where B_T is the transmission bandwidth of $s(t)$.

Solution.

(a) Noting that $n_I'(t) = -n_Q(t)\psi'(t)$, $n_Q'(t) = n_I(t)\psi'(t)$ and $n_I^2(t) + n_Q^2(t) = \lambda^2 A_c^2$, we have

$$\begin{aligned}
2\pi n_d(t) &= \frac{n_Q'(t)(A_c + n_I(t)) - n_Q(t)n_I'(t)}{(A_c + n_I(t))^2 + n_Q^2(t)} \\
&= \frac{n_I(t)\psi'(t)(A_c + n_I(t)) + n_Q^2(t)\psi'(t)}{(A_c + n_I(t))^2 + n_Q^2(t)} \\
&= \frac{n_I(t)A_c + \lambda^2 A_c^2}{A_c^2 + 2n_I(t)A_c + \lambda^2 A_c^2}\psi'(t) \\
&= \frac{\lambda\cos(\psi(t))A_c^2 + \lambda^2 A_c^2}{A_c^2 + 2\lambda\cos(\psi(t))A_c^2 + \lambda^2 A_c^2}\psi'(t) \\
&= \lambda\psi'(t)\frac{\cos(\psi(t)) + \lambda}{1 + 2\lambda\cos(\psi(t)) + \lambda^2}
\end{aligned}$$

(b) When $\psi(t)=\pi$,

$$2\pi n_d(t) = \lambda\psi'(t)\frac{\cos(\psi(t))+\lambda}{1+2\lambda\cos(\psi(t))+\lambda^2} = \lambda\psi'(t)\frac{-1+\lambda}{1-2\lambda+\lambda^2} = \frac{\lambda}{\lambda-1}\psi'(t).$$

Since

$$\lim_{\lambda\downarrow 1} = \frac{\lambda}{\lambda-1} = \infty \quad \text{and} \quad \lim_{\lambda\uparrow 1} = \frac{\lambda}{\lambda-1} = -\infty,$$

a discontinuity appears at $\lambda = 1$.

(c)

$$\begin{aligned}
&\Pr\left[\left|\frac{\psi(t)}{\pi}-1\right| \le \epsilon \text{ and } \left|\frac{r^2(t)}{A_c^2}-1\right| \le \epsilon\right] \\
&= \Pr\left[\left|\frac{\psi(t)}{\pi}-1\right| \le \epsilon\right]\cdot\Pr\left[\left|\frac{r^2(t)}{A_c^2}-1\right| \le \epsilon\right] \quad (r(t) \text{ and } \psi(t) \text{ are independent.}) \\
&= \Pr\left[(1-\epsilon)\pi \le \psi(t) \le (1+\epsilon)\pi\right]\cdot\Pr\left[A_c\sqrt{1-\epsilon} \le r(t) \le A_c\sqrt{1+\epsilon}\right] \\
&= \epsilon\cdot\left(\int_{A_c\sqrt{1-\epsilon}}^{A_c\sqrt{1+\epsilon}} \frac{x}{B_T N_0} e^{-\frac{x^2}{2B_T N_0}}\,dx\right) \\
&= \epsilon\left(-e^{-\frac{x^2}{2B_T N_0}}\Big|_{A_c\sqrt{1-\epsilon}}^{A_c\sqrt{1+\epsilon}}\right) \\
&= \epsilon\left(e^{-\frac{A_c^2(1-\epsilon)}{2B_T N_0}} - e^{-\frac{A_c^2(1+\epsilon)}{2B_T N_0}}\right) \\
&= 2\epsilon e^{-\frac{A_c^2}{2B_T N_0}}\sinh\left(\frac{A_c^2}{2B_T N_0}\epsilon\right) \quad \left(= 2\epsilon e^{-\rho}\sinh(\rho\epsilon)\right)
\end{aligned}$$

Note: This probability is therefor a function of carrier-to-noise power ratio $\rho = A_c^2/(2B_T N_0)$. When $\rho = 10$ and $\epsilon = 0.1$, this probability is equal to

$$2\times 0.1\times e^{-10}\sinh(10\cdot 0.1) \approx 10^{-5}.$$

When $\rho = 20$ and $\epsilon = 0.1$, it becomes

$$2\times 0.1\times e^{-20}\sinh(20\cdot 0.1) \approx 1.5\times 10^{-9}.$$

Experiments show that it requires $\rho \ge 20$ to guarantee that the clicking sound almost disappears.

(d)

$$\begin{aligned}
s(t)s_{\text{vco}}(t) &= 2\cos(2\pi f_c t + \phi(t))\cos(2\pi f_{\text{vco}} t + \phi_{\text{vco}}(t)) \\
&= \cos(2\pi(f_c - f_{\text{vco}})t + (1-\alpha)\phi(t)) + \cos(2\pi(f_c + f_{\text{vco}})t + (1+\alpha)\phi(t))
\end{aligned}$$

The frequency deviation of $s(t)s_{\text{vco}}(t)$ is given by

$$(\Delta f)_{\text{vco}} = \max_t \left|\frac{(1-\alpha)}{2\pi}\phi'(t)\right| = (1-\alpha)\cdot\Delta f,$$

where Δf is the frequency deviation of $s(t)$, given by

$$\Delta f = \max_t \frac{1}{2\pi}|\phi'(t)|.$$

As the frequency deviation of $s(t)s_{\text{vco}(t)}$ is $(1-\alpha)$ of the frequency derivation of $s(t)$, the effective bandwidth is reduced to $(1-\alpha)B_T$, where B_T is the transmission bandwidth of $s(t)$. Consequently, the bandpass filter $H_{\text{vco}}(f)$ is adequate to pass the first term of $s(t)s_{\text{vco}(t)}$, which is

$$\left(s(t)s_{\text{vco}(t)}\right)_{\text{bandpass}} = \cos(2\pi(f_c - f_{\text{vco}})t + (1-\alpha)\phi(t))$$

Note: For your reference, we analyze the noise process due to feedback as follows. Rewrite $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$, where the PSDs of $n_I(t)$ and $n_Q(t)$ are equal to

$$\text{PSD}_{n_I}(f) = \text{PSD}_{n_Q}(f) = \begin{cases} N_0, & |f| < \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned} n(t)s_{\text{vco}}(t) &= 2n_I(t)\cos(2\pi f_c t)\cos(2\pi f_{\text{vco}}t + \phi_{\text{vco}}(t)) \\ &\quad -2n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_{\text{vco}}t + \phi_{\text{vco}}(t)) \\ &\xrightarrow{\text{bandpass } H(f)} n_{I,\text{vco}}(t)\cos(2\pi(f_c - f_{\text{vco}})t - \phi_{\text{vco}}(t)) \\ &\quad -n_{Q,\text{vco}}(t)\sin(2\pi(f_c - f_{\text{vco}})t - \phi_{\text{vco}}(t)) \end{aligned}$$

where the PSDs of $n_{I,\text{vco}}(t)$ and $n_{Q,\text{vco}}(t)$ are given by

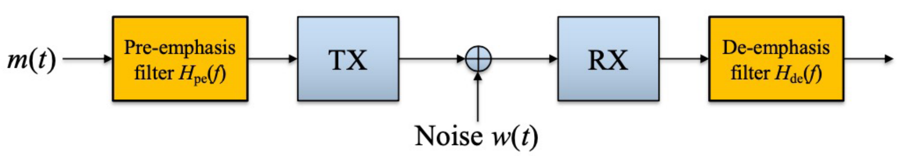
$$\begin{cases} N_0, & |f| < \frac{(1-\alpha)B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

This reduces to the threshold to:

$$\frac{A_c^2}{2(1-\alpha)B_T N_0} \geq 20 \Leftrightarrow \frac{A_c^2}{2B_T N_0} \geq (1-\alpha)20.$$

Accordingly, in principle, the feedback reduces the "noise-spike-free" threshold from 20 down to $(1-\alpha)20$.

3.



The introduction of pre-emphasis and de-emphasis filters can also help reduce the threshold considered in Problems 2(c) and 2(d). In order to un-distort the message $m(t)$, we dictate

$$H_{\text{pe}}(f)H_{\text{de}}(f) = 1 \text{ for } |f| < W.$$

Another condition that is not mentioned in our lectures is that we hope the message power is not altered by the pre-emphasis filter, i.e.,

$$P = \int_{-W}^{W} S_M(f)\mathrm{d}f = \int_{-W}^{W} S_M(f)|H_{\mathrm{pe}}(f)|^2\mathrm{d}f = \int_{-W}^{W} S_M(f)\frac{1}{|H_{\mathrm{de}}(f)|^2}\mathrm{d}f. \tag{2}$$

(a) Fix the PSDs of $m(t)$ and $n_o(t)$ as $S_M(f)$ and $S_{n_o}(f)$, respectively. Find the maximum improvement factor in Slide 5-63, i.e.,

$$I = \frac{\displaystyle\int_{-W}^{W} S_{n_o}(f)\mathrm{d}f}{\displaystyle\int_{-W}^{W} S_{n_o}(f)|H_{\mathrm{de}}(f)|^2\mathrm{d}f}$$

subject to the condition in (2).
Hint: Use Cauchy-Schwarz inequality, i.e.,

$$\left|\int f(x)g^*(x)\mathrm{d}x\right|^2 \le \left(\int |f(x)|^2\mathrm{d}x\right)\left(\int |g(x)|^2\mathrm{d}x\right)$$

with equality holding iff $f(x) = C \cdot g^*(x)$.

(b) Suppose $S_M(f) = M_0$ and $S_{n_o}(f) = N_0$ for $|f| < W$. Find the optimal improvement factor.

(c) Suppose $S_M(f) = M_0$ and $S_{n_o}(f) = \frac{N_0}{A_c^2}f^2$ for $|f| < W$. Find the optimal improvement factor.

Solution.

(a) We shall minimize

$$\int_{-W}^{W} S_{n_o}(f)|H_{\mathrm{de}}(f)|^2\mathrm{d}f$$

subject to

$$P = \int_{-W}^{W} S_M(f)\frac{1}{|H_{\mathrm{de}}(f)|^2}\mathrm{d}f.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left(\int_{-W}^{W} S_{n_o}(f)|H_{\mathrm{de}}(f)|^2\mathrm{d}f\right)\underbrace{\left(\int_{-W}^{W} S_M(f)\frac{1}{|H_{\mathrm{de}}(f)|^2}\mathrm{d}f\right)}_{=P}\\
= &\left(\int_{-W}^{W}\left|\sqrt{S_{n_o}(f)}H_{\mathrm{de}}(f)\right|^2\mathrm{d}f\right)\left(\int_{-W}^{W}\left|\frac{\sqrt{S_M(f)}}{H^*_{\mathrm{de}}(f)}\right|^2\mathrm{d}f\right)\\
\ge &\left|\int_{-W}^{W}\sqrt{S_{n_o}(f)}H_{\mathrm{de}}(f)\left(\frac{\sqrt{S_M(f)}}{H^*_{\mathrm{de}}(f)}\right)^*\mathrm{d}f\right|^2\\
= &\left|\int_{-W}^{W}\sqrt{S_{n_o}(f)S_M(f)}\mathrm{d}f\right|^2
\end{aligned}$$

with equality holding iff

$$\sqrt{S_{n_o}(f)}H_{\text{de,opt}}(f) = C \cdot \frac{\sqrt{S_M(f)}}{H^*_{\text{de,opt}}(f)} \quad \left(\text{equivalently, } |H_{\text{de,opt}}(f)|^2 = C \cdot \sqrt{\frac{S_M(f)}{S_{n_o}(f)}}\right).$$

Consequently,

$$I_{\max} = \frac{P \int_{-W}^{W} S_{n_o}(f)\mathrm{d}f}{\left(\int_{-W}^{W} \sqrt{S_{n_o}(f)S_M(f)}\mathrm{d}f\right)^2}.$$

Note:

$$P = \int_{-W}^{W} S_M(f)\frac{1}{|H_{\text{de}}(f)|^2}\mathrm{d}f = \frac{1}{C}\int_{-W}^{W}\sqrt{S_M(f)S_{n_o}(f)}\mathrm{d}f$$

$$\Rightarrow C = \frac{1}{P}\int_{-W}^{W}\sqrt{S_M(f)S_{n_o}(f)}\mathrm{d}f$$

$$\Rightarrow |H_{\text{de}}(f)|^2 = \left(\frac{1}{P}\int_{-W}^{W}\sqrt{S_M(f)S_{n_o}(f)}\mathrm{d}f\right)\sqrt{\frac{S_M(f)}{S_{n_o}(f)}}$$

(b) $P = \int_{-W}^{W} S_M(f)\mathrm{d}f = 2WM_0$ and $I_{\max} = \frac{(2WM_0)(2WN_0)}{(2W\sqrt{M_0N_0})^2} = 1$.
Note: In such case, no improvement can be obtained by pre- and de-emphasis filters and

$$|H_{\text{de,opt}}(f)|^2 = \left(\frac{2W}{P}\sqrt{M_0N_0}\right)\sqrt{\frac{M_0}{N_0}} = 1.$$

(c) $P = \int_{-W}^{W} S_M(f)\mathrm{d}f = 2WM_0$ and

$$I_{\max} = \frac{(2WM_0)\int_{-W}^{W}\frac{N_0}{A_c^2}f^2\mathrm{d}f}{\left(\frac{\sqrt{M_0N_0}}{A_c}\int_{-W}^{W}|f|\mathrm{d}f\right)^2} = \frac{(2W)(\frac{2}{3}W^3)}{(W^2)^2} = \frac{4}{3} \quad (\approx 1.25 \text{ dB})$$

Note: In such case,

$$|H_{\text{de,opt}}(f)|^2 = \left(\frac{1}{2WM_0}\int_{-W}^{W}\sqrt{M_0\frac{N_0}{A_c^2}f^2}\mathrm{d}f\right)\sqrt{\frac{M_0}{\frac{N_0}{A_c^2}f^2}} = \left(\frac{1}{2W}\int_{-W}^{W}|f|\mathrm{d}f\right)\frac{1}{|f|} = \frac{W}{2|f|}.$$

We can double-check:

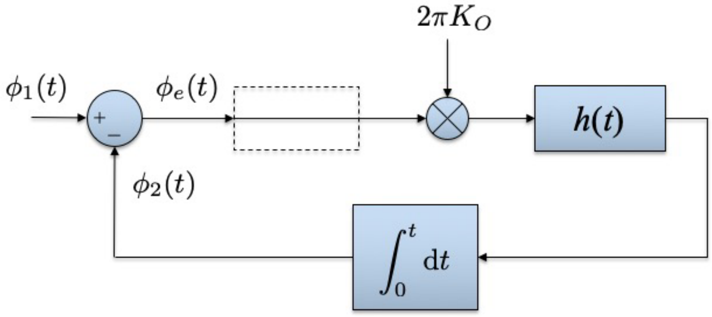
$$\int_{-W}^{W} S_M(f)\frac{1}{|H_{\text{de}}(f)|^2}\mathrm{d}f = \int_{-W}^{W} M_0\frac{2|f|}{W}\mathrm{d}f = 2WM_0 = P.$$

4. (a) For the linearization approximation of phase-locked loop below, show that

$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{jf}{jf + k_0 H(f)}$$

where $H(f)$, $\Phi_e(f)$ and $\Phi_1(f)$ are Fourier transforms of $h(t)$, $\phi_e(t)$ and $\phi_1(t)$, respectively.

Linearization approximation model for PLL.



(b) If $H(f) = 1$ and

$$\phi_1(t) = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

determine $\phi_e(t)$ and $\phi_2(t)$.

Hint:

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
8. Differentiation in the time domain	$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$

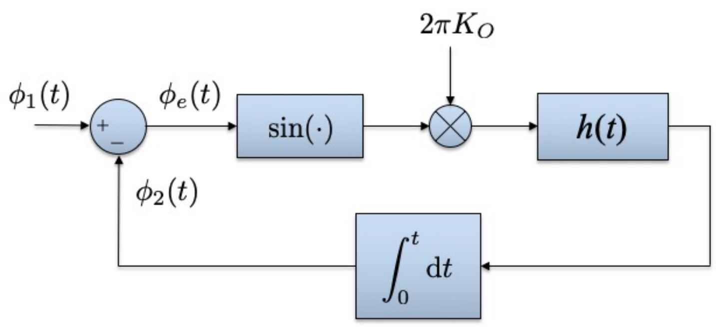
TABLE A6.3 Fourier-transform pairs

Time Function	Fourier Transform
$\exp(-at)u(t)$, $a > 0$	$\frac{1}{a+j2\pi f}$
$\exp(-a\|t\|)$, $a > 0$	$\frac{2a}{a^2+(2\pi f)^2}$

Notes: $u(t)$ = unit step function

(c) How long it takes to have $|\phi_e(t)| = |\phi_1(t) - \phi_2(t)| \le e^{-3} \approx 0.05$? Will a larger k_0 increase the time to reach $|\phi_e(t)| \le e^{-3}$?

(d) Continue from (b) and (c). In the system, we actually have $\phi_1(t) = \phi_2(t) = \phi_e(t) = 0$ for $t < 0$ and $\phi_1(t) = 1$ when $t > 0$, and therefore $\phi_e(t)$ can be larger than 0.5 at some t. It is thus inaccurate to simplify $\sin(\phi_e(t))$ as $\phi_e(t)$. Since $0 < \sin(\theta) < \theta$ for $0 < \theta < \pi$, will the actual time to achieve $|\phi_e(t)| = |\phi_1(t) - \phi_2(t)| \le e^{-3} \approx 0.05$ longer than the time obtained from linearization approximation, or shorter than the time obtained from linearization approximation? Justify your answer.



Solution.

(a) See Slide 5-71.

(b)

$$\Phi_e(f) = \frac{j2\pi f}{j2\pi f + 2\pi k_0}\Phi_1(f) = (j2\pi f\,\Phi_1(f))\cdot\frac{1}{j2\pi f + 2\pi k_0}$$

implies

$$\begin{aligned}\phi_e(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t)\right) \star \mathcal{F}^{-1}\left\{\frac{1}{j2\pi f + 2\pi k_0}\right\}\\ &= \delta(t) \star e^{-2\pi k_0 t}u(t)\\ &= \underline{e^{-2\pi k_0 t}u(t)}\end{aligned}$$

Hence,

$$\phi_2(t) = \phi_1(t) - \phi_e(t) = \underline{(1 - e^{-2\pi k_0 t})u(t)}.$$

(c) $e^{-2\pi k_0 t} \leq e^{-3}$ implies $t \geq \frac{3}{2\pi k_0}$; hence, a larger k_0 will reduce the time to achieve $e^{-2\pi k_0 t} \leq e^{-3}$.

(d) Since $\sin(\phi_e(t)) < \phi_e(t)$,

$$\phi_2(t) = \int_0^t 2\pi k_0 \sin(\phi_e(s))\mathrm{d}s$$

should be smaller than $\int_0^t 2\pi k_0 \phi_e(s)ds$. As a result, $\phi_e(t) = \phi_1(t) - \phi_2(t)$ shall take longer time to achieve e^{-3} (in comparison with the one obtained from the linearization approximation).