Sample Problems for the 6th Quiz (April 15, 2021)

Corrections to Sample Problems for Quiz 5:

- Problem 1(c): We provide more detail for its solution.

... Thus, no overmodulation dictates

$$0 \le 1 + k_a m(t) = \begin{cases} 1 - k_a t, & 0 \le t < 1; \\ 1 + 3k_a t - 4k_a, & 1 \le t < 2 \end{cases}$$

which implies

$$\min\left\{ \inf_{0\le t<1}(1-k_a t), \inf_{1\le t<2}(1+3k_a t - 4k_a) \right\} = \min\{1-k_a, 1-k_a\} \ge 0$$

Accordingly, when $1 - k_a \ge 0$, no overmodulation occurs.

- Problem 1(d): The problem statement does not coincide with the answer. I therefore change it to, "Will the answer be the same as (c) if one requires $|k_a m(t)| \le 1$ to ensure no overmodulation for DSB-C?

- Problem 2(e): The solution should be corrected as

$$\begin{aligned} S(f) &= \cdots \\ &= \frac{1}{2}\cancel{\tilde{M}}\, M_+(f - f_c) + \frac{1}{2}\cancel{\tilde{M}^*}\, M_+^*(-f - f_c) \end{aligned}$$

- Problem 2(f): The problem statement as well as the solution should be corrected as

$$\begin{aligned} S(f) &= \cdots \\ &= \begin{cases} M(f-f_c), & f > f_c \\ \frac{1}{2}M(0), & f = f_c \\ 0, & -f_c < f < f_c \quad \text{(This follows from (c).)} \\ \frac{1}{2}M^*(0), & f = -f_c \\ \cancel{\frac{1}{2}}M^*(-f-f_c), & f < -f_c \end{cases} \\ &= \begin{cases} M(f-f_c), & f > f_c \\ \frac{1}{2}M(0), & f = f_c \\ 0, & -f_c < f < f_c \\ \frac{1}{2}M(0), & f = -f_c \\ \cancel{\frac{1}{2}}M(f+f_c), & f < -f_c \end{cases} \end{aligned}$$

- Problem 3(b): For upper-sideband SSB,

$$\begin{aligned} \mathcal{F}\{X\} &= \cdots \\ \Bigg(&= \begin{cases} \frac{1}{2}M(f)e^{-j\phi}, & f > 0; \\ \frac{1}{2}M(f)e^{j\phi}, & f < 0 \end{cases} \Bigg) \end{aligned}$$

Similarly, for lower sideband SSB, we obtain

$$\frac{1}{2}(M(f) \quad = \quad \cdots$$
$$\left(= \begin{cases} \frac{1}{2}M(f)e^{j\phi}, & f > 0; \\ \frac{1}{2}M(f)e^{-j\phi}, & f < 0 \end{cases} \right)$$

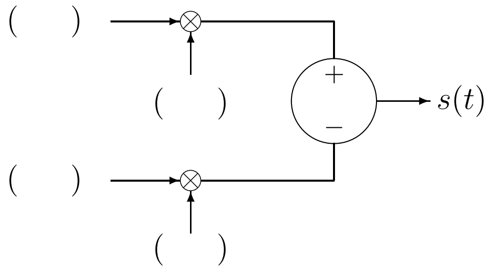
- Problem 6: Eq. (2) should be Denoting

$$\begin{cases} a_T(y) \triangleq \frac{1}{2T}\int_{T-y}^{T} E[X(x+y)X^*(x)]\mathrm{d}x \\ b_T(y) \triangleq \frac{1}{2T}\int_{-T}^{-T-y} E[X(x+y)X^*(y)]\mathrm{d}x \\ c_T(y) \triangleq \frac{1}{2T}\int_{-T}^{T} E[X(x+y)X^*(y)]\mathrm{d}x \end{cases} \tag{1}$$

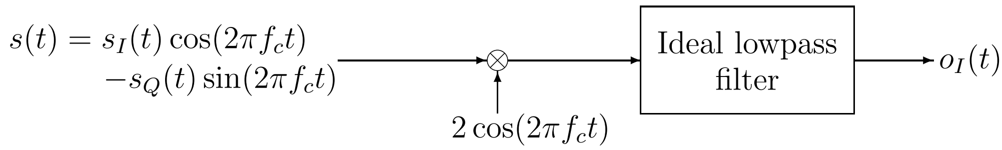
1. (a) The modulated signal to be transmitted via antenna is modeled as

$$s(t) = \mathrm{Re}\left\{[s_I(t) + js_Q(t)]e^{j2\pi f_c t}\right\}.$$

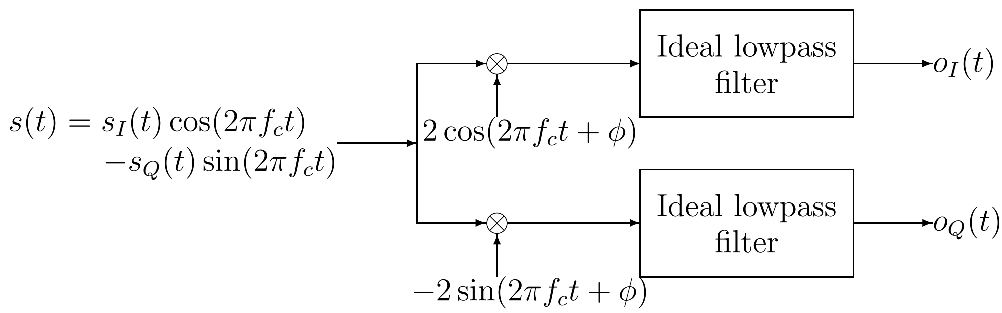
Place $s_I(t)$, $s_Q(t)$, $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ onto the four (   ) in the below figure such that $s(t)$ can be synthesized.



(b) Under *perfect synchronization*, determine the modulation output $o_I(t)$.



(c) Suppose the value of ϕ can be perfectly estimated via a separate low-power pilot tone, and hence is known to the receiver. Show that we can recover $s_I(t)$ from $o_I(t)$ and $o_Q(t)$.



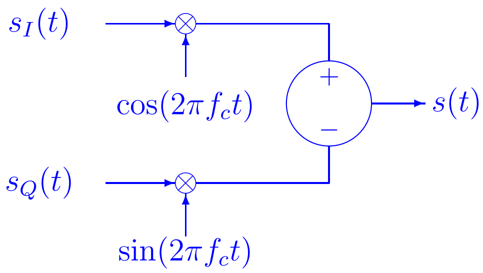
(d) The transmitter now adds a separate pilot tone to the transmission signal as

$$s(t) = \text{Re}\left\{(s_I(t) + js_Q(t) + A_p\cos(2\pi f_p t))e^{j2\pi f_c t}\right\}.$$

Note that we do not add $A_p e^{j2\pi f_p t}$ as in *Sample Problem 3(d) for Quiz 5* but $A_p\cos(2\pi f_p t)$. What will be $o_I(t)$ and $o_Q(t)$ based on the canonical receiver in (c)?

Solution.

(a)



(b) I provide two solutions. Solution 1 follows the receiver structure in a step-by-step manner, while Solution 2 is based on $s(t) = \text{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\}$.
(Solution 1)

$$\begin{aligned} s(t)\cdot 2\cos(2\pi f_c t) &= 2s_I(t)\cos(2\pi f_c t)\cos(2\pi f_c t) - 2s_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= s_I(t) + \underbrace{s_I(t)\cos(4\pi f_c t) - s_Q(t)\sin(4\pi f_c t)}_{\text{high freq. components}}. \end{aligned}$$

After filtering out the two high frequency components, we obtain $o_I(t) = s_I(t)$.

(Solution 2) We can equivalently "imagine" that the receiver demodulates the modulated signal with receiver carrier $e^{j2\pi f_c t}$ (we do not need to use $2e^{j2\pi f_c t}$ as in Sample Problem 3 for Quiz 5 because we multiply $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ by 2), and then use the "reformulation trick" to obtain

$$s(t) = \text{Re}\left\{\underbrace{(s_I(t) + js_Q(t))}_{X+jY} e^{j2\pi f_c t}\right\}.$$

Hence, $X = s_I(t)$.

Note: Under perfect synchronization, the coherent receiver can recover $s_I(t)$ for all modulations listed below. Note that DSC-C and VSB-C can also be recovered by an envelop detector.

Modulations	$s_I(t)$	$s_Q(t)$	
DSC-C	$1 + k_a m(t)$	0	
DSB-SC	$m(t)$	0	
SSB	$m(t)$	$\hat{m}(t)$	Upper sideband transmission
SSB	$m(t)$	$-\hat{m}(t)$	Lower sideband transmission
VSB	$\frac{1}{2}m(t)$	$\frac{1}{2}m'(t)$	Slide 4-46
VSB-C	$\frac{1}{2}[1 + k_a m(t)]$	$\frac{1}{2}k_a m'(t)$	Slide 4-51

(c) (Solution 1) We derive

$$\begin{aligned} s(t) \cdot 2\cos(2\pi f_c t + \phi) &= 2s_I(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \phi) \\ &\quad -2s_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t + \phi) \\ &= s_I(t)\cos(\phi) + s_I(t)\cos(4\pi f_c t + \phi) \\ &\quad -s_Q(t)\sin(4\pi f_c t + \phi) + s_Q(t)\sin(\phi) \end{aligned}$$

and

$$\begin{aligned} s(t) \times \big(-2\sin(2\pi f_c t + \phi)\big) &= -2s_I(t)\cos(2\pi f_c t)\sin(2\pi f_c t + \phi) \\ &\quad +2s_Q(t)\sin(2\pi f_c t)\sin(2\pi f_c t + \phi) \\ &= -s_I(t)\sin(\phi) - s_I(t)\sin(4\pi f_c t + \phi) \\ &\quad +s_Q(t)\cos(\phi) - s_Q(t)\sin(4\pi f_c t + \phi), \end{aligned}$$

Passing the above two signals via a lowpass filter, we obtain

$$\begin{cases} o_I(t) = s_I(t)\cos(\phi) + s_Q(t)\sin(\phi) \\ o_Q(t) = -s_I(t)\sin(\phi) + s_Q(t)\cos(\phi) \end{cases}$$

Under the premise that ϕ can be perfectly estimated, we can recover $s_I(t)$ via

$$s_I(t) = o_I(t)\cos(\phi) - o_Q(t)\sin(\phi).$$

(Solution 2) Considering

$$s(t) = \mathrm{Re}\Big\{ \underbrace{(s_I(t) + js_Q(t))e^{-j\phi}}_{X+jY} e^{j(2\pi f_c t + \phi)} \Big\},$$

we obtain

$$o_I(t) = X = s_I(t)\cos(\phi) + s_Q(t)\sin(\phi)$$

and

$$o_Q(t) = Y = -s_I(t)\sin(\phi) + s_Q(t)\cos(\phi).$$

Under the premise that ϕ can be perfectly estimated, we can recover $s_I(t)$ via

$$s_I(t) = o_I(t)\cos(\phi) - o_Q(t)\sin(\phi).$$

(d) We directly adopt (Solution 2).

$$s(t) = \mathrm{Re}\Big\{ \underbrace{(s_I(t) + js_Q(t) + A_p\cos(2\pi f_p t))e^{-j\phi}}_{X+jY} e^{j(2\pi f_c t + \phi)} \Big\}.$$

Hence, we obtain

$$o_I(t) = X = s_I(t)\cos(\phi) + A_p\cos(2\pi f_p t)\cos(\phi) + s_Q(t)\sin(\phi)$$

and

$$o_Q(t) = Y = -s_I(t)\sin(\phi) - A_p\cos(2\pi f_p t)\sin(\phi) + s_Q(t)\cos(\phi).$$

Note: By a bandpass filter, we can further obtain:

$$\begin{cases} s_I(t)\cos(\phi) + s_Q(t)\sin(\phi) \\ -s_I(t)\sin(\phi) + s_Q(t)\cos(\phi) \\ A_p\cos(2\pi f_p t)\cos(\phi) \\ -A_p\cos(2\pi f_p t)\sin(\phi) \end{cases}$$

provided f_p is larger than W, where W is the bandwidth of $s_I(t)$ and $s_Q(t)$. Then we can use the latter two to have an estimate of (or to control) the phase.

2. (VSB with Carrier) Suppose $s(t) = \text{Re}\left\{(\sqrt{2}[1 + k_a m(t)] + j\sqrt{2}k_a m'(t))e^{j2\pi f_c t}\right\}$.

   (a) Find the output of an envelope detector (that is structured with a squarer, an ideal lowpass filter and a square rooter) due to input $s(t)$.

   (b) Find $m'(t)$, provided $m(t) = \cos(2\pi f_m t)$ and $L_Q(f) = \begin{cases} 1, & f < -f_v; \\ -\frac{f}{f_v}, & |f| \le |f_v|; \\ -1, & f > f_v. \end{cases}$

   Note: From Slide 4-46, $M'(f) = -jM(f)L_Q(f)$, where $M'(f) = \mathcal{F}\{m'(t)\}$ and $M(f) = \mathcal{F}\{m(t)\}$.

   (c) Slide 4-51 states that the distortion of VSB with Carrier can be compensated by reducing the amplitude sensitivity k_a or increasing the width of the vestigial sideband $W + f_v$. Show that the distortion term $k_a^2(m'(t))^2$ in (b) is proportional to k_a^2/f_v^2, provided $f_v \ge f_m$.

Solution.

(a)

$$\begin{aligned} s^2(t) &= \left(\sqrt{2}(1 + k_a m(t))\cos(2\pi f_c t) - \sqrt{2}k_a m'(t)\sin(2\pi f_c t)\right)^2 \\ &= 2(1 + k_a m(t))^2\cos^2(2\pi f_c t) + 2k_a^2(m'(t))^2\sin^2(2\pi f_c t) \\ &\quad -4(1 + k_a m(t))\cos(2\pi f_c t) \cdot k_a m'(t)\sin(2\pi f_c t) \end{aligned}$$

After passing $s^2(t)$ through an ideal lowpass filter, we obtain

$$(1 + k_a m(t))^2 + k_a^2(m'(t))^2.$$

The last stage of square rooter then gives

$$\sqrt{(1 + k_a m(t))^2 + k_a^2(m'(t))^2}.$$

(b)

$$\begin{aligned} M'(f) &= -jM(f)L_Q(f) \quad \text{(Slide 4-46)} \\ &= -j\frac{\delta(f - f_m) + \delta(f + f_m)}{2}L_Q(f) \\ &= \frac{L_Q(f_m)\delta(f - f_m) + L_Q(-f_m)\delta(f + f_m)}{2j} \quad (L_Q(f) = -L_Q(-f)) \\ &= L_Q(f_m)\frac{\delta(f - f_m) - \delta(f + f_m)}{2j} \end{aligned}$$

Hence, $m'(t) = L_Q(f_m)\sin(2\pi f_m t) = \begin{cases} -\frac{f_m}{f_v}\sin(2\pi f_m t), & f_m \le f_v \\ -\sin(2\pi f_m t), & f_m > f_v \end{cases}$

Note: We can combine the two cases and write $m'(t) = -\min\left\{\frac{f_m}{f_v}, 1\right\}\sin(2\pi f_m t)$. Also, in the (Fourier) frequency domain, we need to specify the value of, e.g., $L_Q(f)$ not just for $f \ge 0$ but also for $f < 0$. However, in the time domain, f_m and f_v can never be negative! In fact, this is another difference between the instantaneous frequency and the Fourier frequency. The former can never be negative while the latter ranges over the entire real line.

(c) Just take the result in (b) into the distortion term:

$$k_a^2(m'(t))^2 = k_a^2\frac{f_m^2}{f_v^2}\sin^2(2\pi f_m t)$$

Note: In general, $k_a^2(m'(t))^2 = k_a^2\min\{\frac{f_m^2}{f_v^2}, 1\}\sin^2(2\pi f_m t)$. The result demonstrates that the amount of distortion varies with f_m. As a result, the higher frequency part of a TV signal experiences more distortion than its lower frequency part.

3. (Instantaneous frequency and Fourier frequency) The instantaneous frequency and the Fourier frequency are two *different* notions.

The instantaneous frequency is the operational frequency usually obtained from the spacing between adjacent crossings with respect to the mid-value of a signal (because this is what we observe instantaneously from the curve). For a signal of the shape $\cos(2\pi\phi(t))$, the instantaneous frequence can be obtained from $\frac{d}{dt}\phi(t)$.

The Fourier frequency, however, is an "imaginary" quantity for convenience of decomposing a signal into pieces that it is made of. In 1822, Joseph Fourier found that some nice function $g(t)$ could be expressed as a linear combination of harmonics $\{\cos(2\pi\, n f_0 t)\}_{n\,\text{integer}}$, i.e.,

$$g(t) = \sum_{n=0}^{\infty} c_n\cos(2\pi\, n f_0 t) = \sum_{n=0}^{\infty} c_n\frac{(e^{-j2\pi\, n f_0 t} + e^{j2\pi\, n f_0 t})}{2} = \sum_{n=-\infty}^{\infty} G(n\, f_0)e^{-j2\pi\, n f_0},$$

where $G(nf_0) + G(-nf_0) = G(nf_0) + G^*(nf_0) = \frac{c_n}{2}$. An extension of the nice expression is the Fourier transform:

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{-j2\pi f t}dt.$$

It is useful in the analysis of communication systems because we can transform the convolution operation of a signal and an impulse response (which occurs often in practice) into a multiplication of their Fourier pieces.

We now demonstrate how nonlinearity induces (Fourier) harmonics.

(a) How many (Fourier) harmonics can be obtained at the functional output $y(t) = f(x(t))$ when a single tone $x(t) = \cos(2\pi f_0 t)$ is fed into it, if $f(x(t)) = x^2(t)$? List the harmonics of the functional output. What is the Hilbert transform of the functional output? What is the instantaneous frequency of the functional output?

Hint:[1] $\mathcal{H}\{\cos(2\pi k f_0 t)\} = \sin(2\pi k f_0 t)$

(b) Answer the same questions as in (a) except the one about the instantaneous frequency if $f(x(t)) = x^3(t)$.

(c) Answer the same questions as in (b) if $f(x(t)) = 1 + x(t) + x^2(t) + x^3(t)$.

(d) How many (Fourier) harmonics can be obtained at the functional output

$$f(x(t)) = \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(\tau)\mathrm{d}\tau\right)$$

when a single tone $\cos(2\pi f_0 t)$ is fed into it? What is the Hilbert transform of the functional output? What is the instantaneous frequency of the functional output?
Hint: $e^{j\beta\sin(\phi)} = \sum_{n=-\infty}^{\infty} J_n(\beta)e^{jn\phi}$ and $J_{-n}(\beta) = (-1)^n J_n(\beta)$.

Solutions.

(a) We must decompose $\cos^2(2\pi f_0 t)$ into a linear combination of harmonics as did by Joseph Fourier. As a result of

$$\cos^2(2\pi f_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi(2f_0)t),$$

two harmonics, i.e., 1 and $\cos(4\pi f_0 t)$, are induced by nonlinear functional operation $f(x(t)) = x^2(t)$. With the Fourier decomposition, the Hilbert transform of $\cos^2(2\pi f_0 t)$ is straightforwardly

$$\mathcal{H}\{\cos^2(2\pi f_0 t)\} = 0 + \frac{1}{2}\sin(2\pi(2f_0)t) = \sin(4\pi f_0 t).$$

By the constant spacing of 1/2-crossings, the instantaneous frequency of $\cos^2(2\pi f_0 t)$ is a constant equal to $2f_0$.

(b) As a result of

$$\cos^3(2\pi f_0 t) = \frac{3}{4}\cos(2\pi f_0 t) + \frac{1}{4}\cos(2\pi(3f_0)t),$$

two harmonics, i.e., $\cos(2\pi f_0 t)$ and $\cos(2\pi(3f_0)t)$, are induced by nonlinear functional operation $f(x(t)) = x^3(t)$. With the Fourier decomposition, the Hilbert transform of $\cos^3(2\pi f_0 t)$ is straightforwardly $\mathcal{H}\{\cos^3(2\pi f_0 t)\} = \frac{3}{4}\sin(2\pi f_0 t) + \frac{1}{4}\sin(2\pi(3f_0)t)$.
Note: $\cos^3(2\pi f_0 t)$ actually have the same constant zero-crossing spacing as $\cos(2\pi f_0 t)$. But its instantaneous frequency, e.g., at $t = 0$ is apparently faster than f_0. No widely acceptable definition of instantaneous frequency for a signal of different shape from $\cos(2\pi\phi(t))$ has been proposed in the literature.

[1]

$$\begin{aligned}\mathcal{H}\{\cos(2\pi k f_0 t)\} &= (-j\mathrm{sgn}(f))\,\mathcal{F}\{\cos(2\pi k f_0 t)\} = (-j\mathrm{sgn}(f))\,\frac{\delta(f-kf_0)+\delta(f+kf_0)}{2}\\ &= \frac{\mathrm{sgn}(f)\,\delta(f-kf_0)+\mathrm{sgn}(f)\,\delta(f+kf_0)}{2j} = \frac{\delta(f-kf_0)-\delta(f+kf_0)}{2j} = \sin(2\pi k f_0)\end{aligned}$$

(c) As a result of

$$
\begin{aligned}
&1+\cos(2\pi f_0 t)+\cos^2(2\pi f_0 t)+\cos^3(2\pi f_0 t)\\
&= \frac{3}{2}+\frac{7}{4}\cos(2\pi f_0 t)+\frac{1}{2}\cos(2\pi(2f_0)t)+\frac{1}{4}\cos(2\pi(3f_0)t)
\end{aligned}
$$

four harmonics, i.e., 1, $\cos(2\pi f_0 t)$, $\cos(2\pi(2f_0)t)$ and $\cos(2\pi(3f_0)t)$, are induced by nonlinear functional operation $f(x(t)) = 1 + x(t) + x^2(t) + x^3(t)$. The Hilbert transform of this functional output is straightforwardly $\mathcal{H}\{1+\cos(2\pi f_0 t)+\cos^2(2\pi f_0 t)+\cos^3(2\pi f_0 t)\} =$ ~~$\frac{3}{2}+$~~ $\frac{7}{4}\sin(2\pi f_0 t)+\frac{1}{2}\sin(2\pi(2f_0)t)+\frac{1}{4}\sin(2\pi(3f_0)t)$.

Note: From (a), (b) and (c), you shall realize how nonlinearity induces (Fourier) harmonics. For a functional $f(x(t)) = a_0 + a_1 x(t) + \cdots + a_n x^n(t) + \cdots$, infinite number of harmonics $\{\cos(2\pi n f_0 t)\}_{n=0}^{\infty}$ may be induced at its output.

(d) First, we derive $2\pi k_f \int_0^t x(\tau)\mathrm{d}\tau = \frac{k_f}{f_0}\sin(2\pi f_0 t)$. Then, with $\beta = \frac{k_f}{f_0}$, we have

$$
\begin{aligned}
&\cos(2\pi f_c t + \beta\sin(2\pi f_0 t))\\
&= \frac{1}{2}\left(e^{j2\pi f_c t}e^{j\beta\sin(2\pi f_0 t)} + e^{-j2\pi f_c t}e^{j\beta\sin(-2\pi f_0 t)}\right)\\
&= \frac{1}{2}\left(e^{j2\pi f_c t}\sum_{n=-\infty}^{\infty} J_n(\beta)e^{jn(2\pi f_0 t)} + e^{-j2\pi f_c t}\sum_{n'=-\infty}^{\infty} J_{n'}(\beta)e^{jn'(-2\pi f_0 t)}\right)\\
&= \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi f_c t + 2\pi n f_0 t)
\end{aligned}
$$

Thus, those Fourier harmonics in $\{\cos(2\pi(f_c + nf_0)f)\}_{n=-\infty}^{\infty}$ are induced at the functional output. With the Fourier decomposition, the Hilbert transform of $\cos(2\pi f_c t + \beta\sin(2\pi f_0 t))$ is straightforwardly

$$
\mathcal{H}\{\cos(2\pi f_c t+\beta\sin(2\pi f_0 t))\} = \sum_{n=-\infty}^{\infty} J_n(\beta)\sin(2\pi f_c t+2\pi n f_0 t) = \sin(2\pi f_c t+\beta\sin(2\pi f_0 t)),
$$

where

$$
\begin{aligned}
&\sin(2\pi f_c t + \beta\sin(2\pi f_0 t))\\
&= \frac{1}{2j}\left(e^{j2\pi f_c t}e^{j\beta\sin(2\pi f_0 t)} - e^{-j2\pi f_c t}e^{j\beta\sin(-2\pi f_0 t)}\right)\\
&= \frac{1}{2j}\left(e^{j2\pi f_c t}\sum_{n=-\infty}^{\infty} J_n(\beta)e^{jn(2\pi f_0 t)} - e^{-j2\pi f_c t}\sum_{n'=-\infty}^{\infty} J_{n'}(\beta)e^{jn'(-2\pi f_0 t)}\right)\\
&= \sum_{n=-\infty}^{\infty} J_n(\beta)\sin(2\pi f_c t + 2\pi n f_0 t).
\end{aligned}
$$

The instantaneous frequency of $\cos(2\pi\phi(t)) = \cos(2\pi f_c t + \beta\sin(2\pi f_0 t))$ is given by

$$
f_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = f_c + \frac{\beta}{2\pi}2\pi f_0\cos(2\pi f_0 t) = f_c + k_f\cos(2\pi f_0 t).
$$

4. The time-averaged PSD of a deterministic signal $x(t)$ can be obtained via

$$\bar{S}_X(f) = \lim_{T\to} \frac{1}{2T} X(f) X_{2T}^*(f)$$

where $X(f)$ and $X_{2T}(f)$ are the Fourier transforms of $x(t)$ and $x(t)\cdot\mathbf{1}\{|t| < T\}$, respectively.

(a) Show that if

$$X(f) = \sum_{n=-\infty}^{\infty} J_n \cdot [\delta(f - \underbrace{mW}_{f_c} - nW) + \delta(f + \underbrace{mW}_{f_c} + nW)] \quad \text{for } m \text{ fixed},$$

where J_n is a real number satisfying $J_{-n} = (-1)^n J_n$, then

$$\bar{S}_X(f) = \sum_{k=-\infty}^{\infty} J_k \left(J_k + (-1)^k J_{2m+k}\right) [\delta(f - mW - kW) + \delta(f + mW + kW)]$$

Hint: $X_{2T}(f) = X(f) \star 2T\text{sinc}(2Tf)$, where "$\star$" is the convolution operation.

(b) Determine $\int_{-\infty}^{\infty} \bar{S}_X(f) df$, provided $\sum_{n=-\infty}^{\infty} J_n^2 = 1$.

Solution.

(a)

$$\begin{aligned}
X_{2T}(f) &= X(f) \star 2T\text{sinc}(2Tf) \\
&= \left(\sum_{n=-\infty}^{\infty} J_n \cdot [\delta(f - mW - nW) + \delta(f + mW + nW)]\right) \star 2T\text{sinc}(2Tf) \\
&= \sum_{n=-\infty}^{\infty} J_n \cdot \big[2T\text{sinc}(2T(f - mW - nW)) + 2T\text{sinc}(2T(f + mW + nW))\big].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\bar{S}_X(f) &= \lim_{T\to\infty} \frac{1}{2T} X(f) X_{2T}^*(f) \\
&= \lim_{T\to\infty} \left[\sum_{k=-\infty}^{\infty} J_k \cdot [\delta(f - mW - kW) + \delta(f + mW + kW)] \right] \\
&\quad \left[\sum_{n=-\infty}^{\infty} J_n \cdot \left[\mathrm{sinc}(2T(f - mW - nW)) + \mathrm{sinc}(2T(f + mW + nW)) \right] \right] \\
&= \lim_{T\to\infty} \left[\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f - mW - kW) \mathrm{sinc}(2T(f - mW - nW)) \right. \\
&\qquad + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f - mW - kW) \mathrm{sinc}(2T(f + mW + nW)) \\
&\qquad + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f + mW + kW) \mathrm{sinc}(2T(f - mW - nW)) \\
&\qquad \left. + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f + mW + kW) \mathrm{sinc}(2T(f + mW + nW)) \right] \\
&= \lim_{T\to\infty} \left[\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f - mW - kW) \mathrm{sinc}(2T(mW + kW - mW - nW)) \right. \\
&\qquad + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f - mW - kW) \mathrm{sinc}(2T(mW + kW + mW + nW)) \\
&\qquad + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f + mW + kW) \mathrm{sinc}(2T(-mW - kW - mW - nW)) \\
&\qquad \left. + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f + mW + kW) \mathrm{sinc}(2T(-mW - kW + mW + nW)) \right]
\end{aligned}
$$

$$= \lim_{T\to\infty}\left[\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f - mW - kW)\underbrace{\mathrm{sinc}(2TW(k-n))}_{n=k}\right.$$

$$+ \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f - mW - kW)\underbrace{\mathrm{sinc}(2TW(2m+k+n))}_{n=-2m-k}$$

$$+ \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f + mW + kW)\underbrace{\mathrm{sinc}(2TW(-2m-k-n))}_{n=-2m-k}$$

$$\left.+ \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_k J_n \cdot \delta(f + mW + kW)\underbrace{\mathrm{sinc}(2TW(-k+n))}_{n=k}\right]$$

(Note that $\mathrm{sinc}(x) = 0$ for all non-zero integers x.)

$$= \sum_{k=-\infty}^{\infty} J_k^2 \cdot \delta(f - mW - kW) + \sum_{k=-\infty}^{\infty} J_k J_{-2m-k} \cdot \delta(f - mW - kW)$$

$$+ \sum_{k=-\infty}^{\infty} J_k J_{-2m-k} \cdot \delta(f + mW + kW) + \sum_{k=-\infty}^{\infty} J_k^2 \cdot \delta(f + mW + kW)$$

$$= \sum_{k=-\infty}^{\infty} J_k \left(J_k + (-1)^k J_{2m+k}\right)\left[\delta(f - mW - kW) + \delta(f + mW + kW)\right]$$

(b)

$$\int_{-\infty}^{\infty} \bar{S}_X(f) df = \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} J_k \left(J_k + (-1)^k J_{2m+k}\right)\right.$$

$$\left.\left[\delta(f - mW - kW) + \delta(f + mW + kW)\right]\right) \mathrm{d}f$$

$$= \sum_{k=-\infty}^{\infty} J_k \left(J_k + (-1)^k J_{2m+k}\right)$$

$$= 2 + 2\sum_{k=-\infty}^{\infty} (-1)^k J_k J_{2m+k}$$