Sample Problems for the 5th Quiz (April 8, 2021)

## Q&A

- Some students ask why

$$|\tilde{g}(t)| = |\tilde{g}_+(t)| = |g(t) + j\hat{g}(t)| = \sqrt{g_I^2(t) + g_Q^2(t)}$$

is called "envelope" of $g(t)$ in Slide 3-17 and whether we can "visualize" this "envelope" (as it is so named) from $g(t)$. The answer is yes if $f_c \gg W$ as in Slide 4-8, where $W$ is essentially the bandwidth of $\tilde{g}(t)$. We can visualize the trace of $|\tilde{g}(t)|$ from the curve of $g(t)$ because

$$\begin{aligned} g(t) &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\ &= \sqrt{g_I^2(t) + g_Q^2(t)}\cos\left(2\pi f_c t + \arctan\left(\frac{g_Q(t)}{g_I(t)}\right)\right) \\ &\approx \sqrt{g_I^2(t) + g_Q^2(t)}\cos(2\pi f_c t) \quad \text{if } f_c \gg W. \end{aligned}$$

Note that if $f_c \gg W$ does not hold, then the trace of $|\tilde{g}(t)|$ may be non-visualized due to the phase adjustment of $\arctan\left(\frac{g_Q(t)}{g_I(t)}\right)$.

1. (DSB-C or AM) Let the modulated signal for DSB-C be given by

$$s(t) = \sqrt{2}\,[1 + k_a\, m(t)]\cos(2\pi f_c t).$$

(a) Show that the Fourier transform of $s(t)$ is

$$S(f) = \frac{\sqrt{2}}{2}\left(\delta(f - f_c) + \delta(f + f_c)\right) + \frac{\sqrt{2}k_a}{2}\left(M(f - f_c) + M(f + f_c)\right),$$

where $M(f)$ is the Fourier transform of $m(t)$.

(b) Upon reception of $s(t)$, the receiver first performs squaring operation, i.e.,

$$s^2(t) \quad = \quad 2[1 + k_a m(t)]^2\cos^2(2\pi f_c t) = [1 + k_a m(t)]^2(1 + \cos(4\pi f_c t)),$$

followed by an ideal lowpass filter to obtain

$$[1 + k_a m(t)]^2.$$

Then, a simple square-rooter produces an output $r(t) = 1 + k_a m(t)$. The receiver can thus perfectly recover $m(t)$ by performing $\frac{1}{k_a}(r(t) - 1)$. Is there any additional technical requirement that needs to be mentioned here?

(c) Now suppose

$$m(t) = \begin{cases} -t, & 0 \le t < 1; \\ 3t - 4, & 1 \le t < 2. \end{cases}$$

What is the range of the modulation index $k_a$ over which no overmodulation for DSB-C occurs?

Note: $k_a$ must be a positive number.

(d) Will the answer be the same as (c) if one requires $|k_a m(t)| \leq 1$ to ensure no overmodulation for DSB-C?

**Solution.**

(a)

$$
\begin{aligned}
S(f) &= \mathcal{F}\left\{\sqrt{2}\cos(2\pi f_c t) + \sqrt{2}k_a\, m(t)\cos(2\pi f_c t)\right\} \\
&= \sqrt{2}\mathcal{F}\{\cos(2\pi f_c t)\} + \sqrt{2}k_a \mathcal{F}\{m(t)\} \star \mathcal{F}\{\cos(2\pi f_c t)\} \\
&= \frac{\sqrt{2}}{2}(\delta(f-f_c)+\delta(f+f_c)) + \sqrt{2}k_a M(f) \star \frac{1}{2}(\delta(f-f_c)+\delta(f+f_c)) \\
&= \frac{\sqrt{2}}{2}(\delta(f-f_c)+\delta(f+f_c)) + \frac{\sqrt{2}k_a}{2}(M(f-f_c)+M(f+f_c))
\end{aligned}
$$

(b) The square-rooter actually gives $\big|1 + k_a\, m(t)\big|$, instead of $\big(1 + k_a\, m(t)\big)$. Thus, if $1 + k_a\, m(t) < 0$, overmodulation occurs; as a result, such a simple and inexpensive envelop detector cannot recover $m(t)$ if $1 + k_a\, m(t) < 0$ for some $t$.

Note: Overmodulation in Slide 4-6 is "bad" for DSB-C mainly because one wishes to use simple "squarer+lowpass filter+square rooter" to recover $m(t)$. If a more sophisticated circuit, such as coherent detection, is realized, overmodulation will not be a problem for a receiver.

Also note that this envelop detector, ideally, will not have the non-visualized envelop problem as long as $f_c > 2W$. For detail, please see Problem 4.

(c) Overmodulation for DSB-C occurs when $1 + k_a m(t) < 0$. Thus, no overmodulation dictates

$$
0 \leq 1 + k_a m(t) = \begin{cases} 1 - k_a t, & 0 \leq t < 1; \\ 1 + 3k_a t - 4k_a, & 1 \leq t < 2 \end{cases}
$$

which implies

$$
\min\left\{\inf_{0\leq t<1}(1-k_a t), \inf_{1\leq t<2}(1+3k_a t - 4k_a)\right\} = \min\{1-k_a, 1-k_a\} \geq 0
$$

Accordingly, when $1 - k_a \geq 0$, no overmodulation occurs. The answer to this subproblem is therefore $0 < k_a \leq 1$.

(d) It is clear that

$$
1 \geq |k_a m(t)| = \begin{cases} |k_a \cdot (-t)|, & 0 \leq t < 1; \\ |k_a \cdot (3t-4)|, & 1 \leq t < 2 \end{cases}
$$

implies $|2k_a| \leq 1$. Hence, the answer is $0 < k_a \leq \frac{1}{2}$. The new condition is much stronger than the one in (c).

2. (Hilbert transform and SSB) Suppose $m(t)$ is a real-valued signal. Denote by $M(f)$ its Fourier transform (thus, $M(f) = M^*(-f)$).

   (a) Let $\hat{m}(t)$ be the Hilbert transform of $m(t)$. Give the relation of their Fourier transforms $M(f)$ and $\hat{M}(f)$? Is $|M(f)| = |\hat{M}(f)|$?
   Hint: $H_{\text{Hilbert}}(f) = -j\text{sgn}(f)$

(b) Give the range of frequency $f$ such that $j\hat{M}(f) = M(f)$, and also give the range of frequency $f$ such that $j\hat{M}(f) = -M(f)$.

(c) Let $m_+(t) = m(t) + j\hat{m}(t)$, and denote by $M_+(f)$ the Fourier transform of $m_+(t)$. Give the range of frequency $f$ such that $M_+(f) = 2M(f)$, and also give the range of frequency $f$ such that $M_+(f) = 0$.

(d) Let $m_-(t) = m(t) - j\hat{m}(t)$, and denote by $M_-(f)$ the Fourier transform of $m_-(t)$. Give the range of frequency $f$ such that $M_-(f) = 2M(f)$, and also give the range of frequency $f$ such that $M_-(f) = 0$.

(e) Show that the Fourier transform of

$$s(t) = \mathrm{Re}\{m_+(t)e^{j2\pi f_c t}\}$$

can be written as

$$S(f) = \frac{1}{2}M_+(f - f_c) + \frac{1}{2}M_+^*(-f - f_c).$$

(f) Use (c) and (e) to show that

$$S(f) = \begin{cases} M(f - f_c), & f > f_c \\ \frac{1}{2}M(0), & f = f_c \\ 0, & -f_c < f < f_c \\ \frac{1}{2}M(0), & f = -f_c \\ M(f + f_c), & f < -f_c \end{cases}$$

(g) Derive (f) directly from

$$s(t) = m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t).$$

Hint:

$$\mathcal{F}\{\cos(2\pi f_c t)\} = \frac{1}{2}(\delta(f - f_c) + \delta(f + c_c))$$

and

$$\mathcal{F}\{\sin(2\pi f_c t)\} = \frac{1}{2j}(\delta(f - f_c) - \delta(f + c_c))$$

**Solution.**

(a) $\hat{M}(f) = -j\mathrm{sgn}(f)\, M(f)$; hence, $|\hat{M}(f)| = |M(f)|$ (except $f = 0$).

(b)

$$j\hat{M}(f) = \mathrm{sgn}(f)\, M(f) = \begin{cases} M(f), & f > 0; \\ 0, & f = 0; \\ -M(f), & f < 0 \end{cases}$$

Thus, $j\hat{M}(f) = M(f)$ when $f > 0$ and $j\hat{M}(f) = -M(f)$ when $f < 0$.

(c)

$$\begin{aligned}
M_+(f) &= M(f) + j\,\hat{M}(f) \\
&= M(f) + j\,(-j\mathrm{sgn}(f))M(f) \\
&= M(f) + \mathrm{sgn}(f)\,M(f) \\
&= \begin{cases} 2M(f), & f > 0; \\ M(f), & f = 0; \\ 0, & f < 0 \end{cases}
\end{aligned}$$

Thus, $M_+(f) = 2M(f)$ when $f > 0$ and $M_+(f) = 0$ when $f < 0$.

(d)

$$\begin{aligned}
M_-(f) &= M(f) - j\,\hat{M}(f) \\
&= M(f) - j\,(-j\mathrm{sgn}(f))M(f) \\
&= M(f) - \mathrm{sgn}(f)\,M(f) \\
&= \begin{cases} 0, & f > 0; \\ M(f), & f = 0; \\ 2M(f), & f < 0 \end{cases}
\end{aligned}$$

Thus, $M_-(f) = 2M(f)$ when $f < 0$ and $M_-(f) = 0$ when $f > 0$.

(e)

$$\begin{aligned}
S(f) &= \int_{-\infty}^{\infty} \mathrm{Re}\{m_+(t)e^{j2\pi f_c t}\}e^{-j2\pi f t}dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(m_+(t)e^{j2\pi f_c t} + \left(m_+(t)e^{j2\pi f_c t}\right)^*\right) e^{-j2\pi f t}dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(m_+(t)e^{j2\pi f_c t} + m_+^*(t)e^{-j2\pi f_c t}\right) e^{-j2\pi f t}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} m_+(t)e^{-j2\pi(f-f_c)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} m_+^*(t)e^{-j2\pi(f+f_c)t}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} m_+(t)e^{-j2\pi(f-f_c)t}dt + \frac{1}{2}\left(\int_{-\infty}^{\infty} m_+(t)e^{-j2\pi(-f-f_c)t}dt\right)^* \\
&= \frac{1}{2}M_+(f-f_c) + \frac{1}{2}M_+^*(-f-f_c)
\end{aligned}$$

(f)

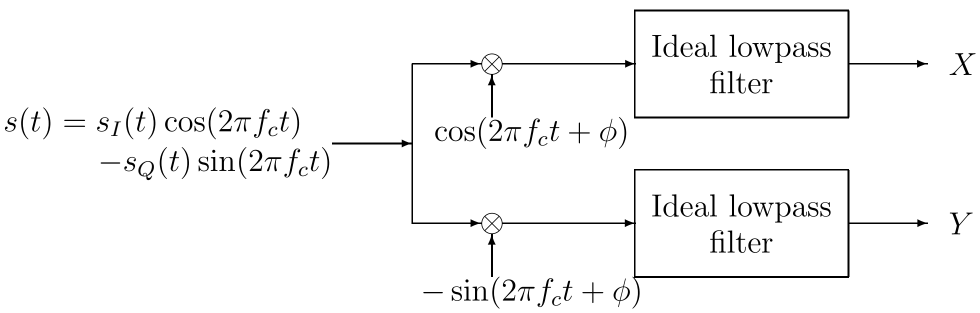
$$
\begin{aligned}
S(f) &= \frac{1}{2}M_+(f-f_c) + \frac{1}{2}M_+^*(-f-f_c) \\
&= \begin{cases} M(f-f_c), & f > f_c \\ \frac{1}{2}M(0), & f = f_c \\ 0, & -f_c < f < f_c \\ \frac{1}{2}M^*(0), & f = -f_c \\ M^*(-f-f_c), & f < -f_c \end{cases} \quad \text{(This follows from (c).)} \\
&= \begin{cases} M(f-f_c), & f > f_c \\ \frac{1}{2}M(0), & f = f_c \\ 0, & -f_c < f < f_c \\ \frac{1}{2}M(0), & f = -f_c \\ M(f+f_c), & f < -f_c \end{cases}
\end{aligned}
$$

where the last step follows from $M(f) = M^*(-f)$.

(g)

$$
\begin{aligned}
S(f) &= M(f) \star \mathcal{F}\{\cos(2\pi f_c t)\} - \hat{M}(f) \star \mathcal{F}\{\sin(2\pi f_c t)\} \\
&= M(f) \star \frac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right) \\
&\quad -\hat{M}(f) \star \frac{1}{2j}\left(\delta(f-f_c) - \delta(f+f_c)\right) \\
&= M(f) \star \frac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right) \\
&\quad +j\hat{M}(f) \star \frac{1}{2}\left(\delta(f-f_c) - \delta(f+f_c)\right) \\
&= \frac{M(f) + j\hat{M}(f)}{2} \star \delta(f-f_c) + \frac{M(f) - j\hat{M}(f)}{2} \star \delta(f+f_c) \\
&= \begin{cases} M(f-f_c), & f > f_c \\ \frac{1}{2}M(f-f_c), & f = f_c \\ 0 & -f_c < f < f_c \\ \frac{1}{2}M(f+f_c), & f = -f_c \\ M(f+f_c) & f < -f_c \end{cases}
\end{aligned}
$$

3. (Demodulation trick) A quick and simple way to obtain the demodulated output of a coherent receiver depicted below (instead of using the technical analysis in Slide 4-26) is to re-express the <u>transmitting signal $s(t)$</u> in terms of the <u>receiver carrier</u>.



For example, according to the above figure, we derive:

$$\begin{cases} \text{Transmitter: } s(t) = \mathrm{Re}\{(s_I(t) + js_Q(t))e^{j2\pi f_c t}\} \\ \text{Receiver with carrier } 2e^{j(2\pi f_c t+\phi)}\text{:} \\ \qquad s(t) = \mathrm{Re}\Bigg\{ \underbrace{\left(\frac{1}{2}(s_I(t) + js_Q(t))e^{-j\phi}\right)}_{X+jY} 2e^{j(2\pi f_c t+\phi)} \Bigg\} \end{cases}$$

Hence, we obtain $X + jY = \frac{1}{2}(s_I(t) + js_Q(t))e^{-j\phi}$ after demodulation.

In Slides 4-25 and 4-26, we learn that $s_I(t) = m(t)$ and $s_Q(t) = 0$. We also learn that only $X$ is retained as an estimate of $m(t)$ at the output (because the receiver only implements the upper half of the Costas receiver), which gives

$$X = \mathrm{Re}\left\{\frac{1}{2}(\underbrace{s_I(t)}_{=m(t)} + j\underbrace{s_Q(t)}_{=0})e^{-j\phi}\right\} = \mathrm{Re}\left\{\frac{1}{2}m(t)e^{-j\phi}\right\} = \frac{1}{2}m(t)\cos(\phi)$$

Note that in Slides 4-25 and 4-26, the transmitter additionally amplifies the transmitted signal by multiplying it with $A_c$, and the receiver also amplifies the received signal with a multiplicative factor $A'_c$. This enlarges $X$ to $\frac{1}{2}A_cA'_cm(t)\cos(\phi)$ as exactly given in Slide 4-26. Here, we simply let $A_c = A'_c = 1$ for simplicity.

(a) In Slide 4-23 and 4-46, we produce $s_I(t)$ and $s_Q(t)$ as follows. For the two SSBs and VSB, give the output estimate $X$ as a function of $m(t)$, $\hat{m}(t)$, $m'(t)$ and $\phi$ using the analytical trick just introduced.

| Type of modulation | $s_I(t)$ | $s_Q(t)$ | Output estimate |
|---|---|---|---|
| DSB-SC | $m(t)$ | $0$ | $X$ |
| SSB | $m(t)$ | $\hat{m}(t)$ | $X$ |
| SSB | $m(t)$ | $-\hat{m}(t)$ | $X$ |
| VSB | $m(t)$ | $-m'(t)$ | $X$ |

(b) Find the Fourier transform of $X$ for the two SSBs in (a). You should express it as a function of $M(f)$ only.

(c) Re-do (a) and (b) for DSB-SC if the carrier frequency $f'_c$ at the receiver is not the same as that of the transmitter.

Hint:

$$\begin{cases} \text{Transmitter: } s(t) = \mathrm{Re}\{(s_I(t) + js_Q(t))e^{j2\pi f_c t}\} \\ \text{Receiver with carrier } 2e^{j(2\pi f'_c t+\phi)}\text{:} \\ \qquad s(t) = \mathrm{Re}\left\{(?)\, 2e^{j(2\pi f'_c t+\phi)}\right\} \end{cases}$$

(d) As indicated in Slide 4-33, a separated pilot tone is added to SSB such that

$$s(t) = \mathrm{Re}\left\{(m(t) + j\hat{m}(t) + A_p e^{j2\pi f_p t})e^{j2\pi f_c t}\right\}.$$

Give the output estimate $X$ as a function of $m(t)$, $\hat{m}(t)$, $f_p$ and $\phi$ using the analytical trick just introduced, if the receiver demodulates the signal based on local carrier $2e^{j(2\pi f_c t+\phi)}$.

**Solution.**

(a) $X = \text{Re}\left\{\frac{1}{2}(s_I(t) + js_Q(t))e^{-j\phi}\right\} = \frac{1}{2}(s_I(t)\cos(\phi) + s_Q(t)\sin(\phi))$. Thus we obtain

| Type of modulation | $s_I(t)$ | $s_Q(t)$ | Output estimate |
|---|---|---|---|
| SSB | $m(t)$ | $\hat{m}(t)$ | $\frac{1}{2}(m(t)\cos(\phi) + \hat{m}(t)\sin(\phi))$ |
| SSB | $m(t)$ | $-\hat{m}(t)$ | $\frac{1}{2}(m(t)\cos(\phi) - \hat{m}(t)\sin(\phi))$ |
| VSB | $m(t)$ | $-m'(t)$ | $\frac{1}{2}(m(t)\cos(\phi) - m'(t)\sin(\phi))$ |

(b) For upper-sideband SSB,

$$\begin{aligned}\mathcal{F}\{X\} &= \frac{1}{2}(M(f)\cos(\phi) + \hat{M}(f)\sin(\phi)) \\ &= \frac{1}{2}(M(f)\cos(\phi) + (-j\text{sgn}(f))M(f)\sin(\phi)) \\ &= \frac{1}{2}M(f)(\cos(\phi) - j\text{sgn}(f)\sin(\phi)) \\ \Bigg( &= \begin{cases}\frac{1}{2}M(f)e^{-j\phi}, & f > 0; \\ \frac{1}{2}M(f)e^{j\phi}, & f < 0\end{cases}\Bigg)\end{aligned}$$

Similarly, for lower sideband SSB, we obtain

$$\begin{aligned}&\frac{1}{2}(M(f)\cos(\phi) - \hat{M}(f)\sin(\phi)) \\ &= \frac{1}{2}(M(f)\cos(\phi) - (-j\text{sgn}(f))M(f)\sin(\phi)) \\ &= \frac{1}{2}M(f)(\cos(\phi) + j\text{sgn}(f)\sin(\phi)) \\ \Bigg( &= \begin{cases}\frac{1}{2}M(f)e^{j\phi}, & f > 0; \\ \frac{1}{2}M(f)e^{-j\phi}, & f < 0\end{cases}\Bigg)\end{aligned}$$

(c) Using the quick and simple analysis as follows. From

$$\begin{cases}\text{Transmitter: } s(t) = \text{Re}\{(s_I(t) + js_Q(t))e^{j2\pi f_c t}\} \\ \text{Receiver with carrier } 2e^{j2\pi f_c' t + \phi}\text{:} \\ \qquad s(t) = \text{Re}\left\{\underbrace{\left(\frac{1}{2}(s_I(t) + js_Q(t))e^{j(2\pi(f_c - f_c')t - \phi)}\right)}_{X+jY} 2e^{j(2\pi f_c' t + \phi)}\right\}\end{cases}$$

we obtain

$$X = \frac{1}{2}s_I(t)\cos(2\pi(f_c - f_c')t - \phi) - \frac{1}{2}s_Q(t)\sin(2\pi(f_c - f_c')t - \phi).$$

Hence, for DSB-SC (i.e., $s_I(t) = m(t)$ and $s_Q(t) = 0$),

$$X = \frac{1}{2}m(t)\cos(2\pi(f_c - f_c')t - \phi) = \frac{1}{2}m(t)\cos(2\pi(\Delta f)t - \phi),$$

where $\Delta f = f_c - f'_c$, and the Fourier transform of $X$ is

$$\begin{aligned}
&\frac{1}{2}M(f)\star\left(\cos(\phi)\frac{\delta(f-\Delta f)+\delta(f+\Delta f)}{2}+\sin(\phi)\frac{\delta(f-\Delta f)-\delta(f+\Delta f)}{2j}\right)\\
&= \frac{1}{4}M(f)\star\left(e^{-j\phi}\delta(f-\Delta f)+e^{j\phi}\delta(f+\Delta f)\right)\\
&= \frac{1}{4}\left[e^{-j\phi}M(f-\Delta f)+e^{j\phi}M(f+\Delta f)\right].
\end{aligned}$$

Thus, an additional shift in frequency occurs. As a result, $M(f)$ might be seriously distorted if $|f_c - f'_c| < W$, where $W$ is the bandwidth of $M(f)$.

(d)

$$s(t) = \mathrm{Re}\left\{\underbrace{\frac{1}{2}\left(m(t)+j\hat{m}(t)+A_p e^{j2\pi f_p t}\right)e^{-j\phi}}_{X+jY}2e^{j(2\pi f_c t+\phi)}\right\}.$$

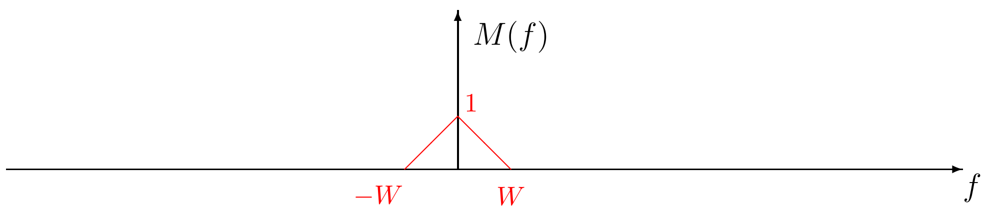
We then obtain

$$\begin{aligned}
X &= \mathrm{Re}\left\{\frac{1}{2}\left(m(t)+j\hat{m}(t)+A_p e^{j2\pi f_p t}\right)e^{-j\phi}\right\}\\
&= \frac{1}{2}m(t)\cos(\phi)+\frac{1}{2}\hat{m}(t)\sin(\phi)+\underbrace{\frac{1}{2}A_p\cos(2\pi f_p t-\phi)}_{\substack{\text{In principle, this part}\\ \text{can be technically separated from}\\ \frac{1}{2}m(t)\cos(\phi)+\frac{1}{2}\hat{m}(t)\sin(\phi).}}.
\end{aligned}$$

4. Here, $\hat{m}(t)$ is the Hilbert transform of $m(t)$. In other words, $\hat{M}(f) = H_{\text{Hilbert}}(f)M(f)$, where $H_{\text{Hilbert}}(f) = (-j\mathrm{sgn}(f))$. In addition, $M'(f) = H_Q(f)M(f)$, where

$$\frac{1}{j}H_Q(f)\begin{cases} = 1, & f \le -f_v \\ \in [0,1], & -f_v < f < 0 \\ 0, & f = 0 \end{cases} \quad \text{and} \quad H_Q(-f) = H_Q^*(f). \tag{1}$$

Suppose $M(f)$ is real-valued and is give by




(a) Does $H_{\text{Hilbert}}(f)$ satisfy (1) for any given $f_v < W$? Justify your answer.

(b) Plot $j\hat{M}(f)$.

(c) Plot $M(f) + j\hat{M}(f)$ for upper sideband SSB.

(d) Plot the spectrum of $\mathrm{Re}\{(m(t) + j\hat{m}(t))e^{j2\pi f_c t}\} = m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)$.

(e) Plot $M(f) - j\hat{M}(f)$ for lower sideband SSB.

(f) Plot the spectrum of $\mathrm{Re}\{(m(t) - j\hat{m}(t))e^{j2\pi f_c t}\} = m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)$.

(g) Further assume that

$$\frac{1}{j}H_Q(f) = \begin{cases} 1, & f \le -\frac{W}{2} \\ \frac{1}{2}, & -\frac{W}{2} < f < 0 \\ 0, & f = 0 \end{cases}$$

Plot $jM'(f)$.

(h) Continue from (g). Plot $M(f) - jM'(f)$.

**Solution.**

(a)

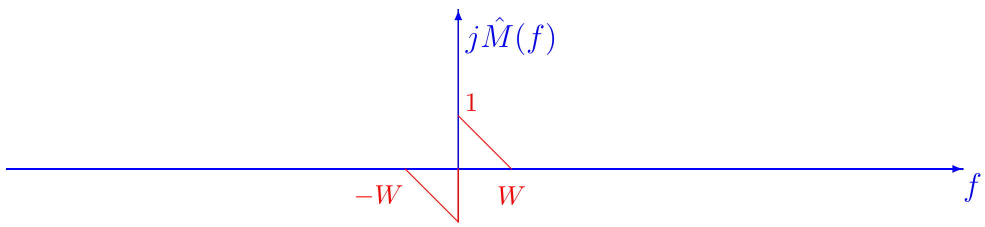
$$\frac{1}{j}H_{\text{Hilbert}}(f) = -\mathrm{sgn}(f) \begin{cases} = 1, & f \le -f_v \\ \in [0,1], & -f_v < f < 0 \\ 0, & f = 0 \end{cases}$$

and

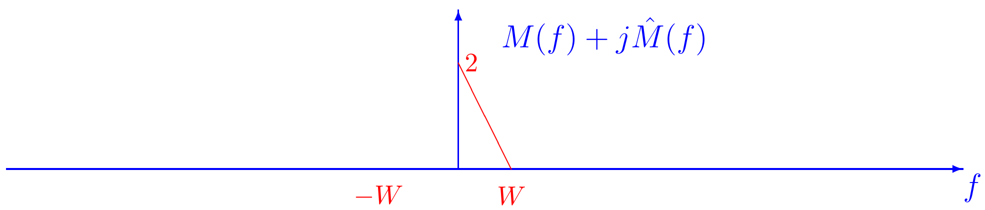
$$H_{\text{Hilbert}}(-f) = -j\mathrm{sgn}(-f) = j\mathrm{sgn}(f) = (-j\mathrm{sgn}(f))^* = H^*_{\text{Hilbert}}(f).$$

Thus, Hilbert transformer is a special (impractical) choice of $H_Q(f)$.

(b) $j\hat{M}(f) = j(-j\mathrm{sgn(f)})M(f) = \mathrm{sgn}(f)M(f)$.



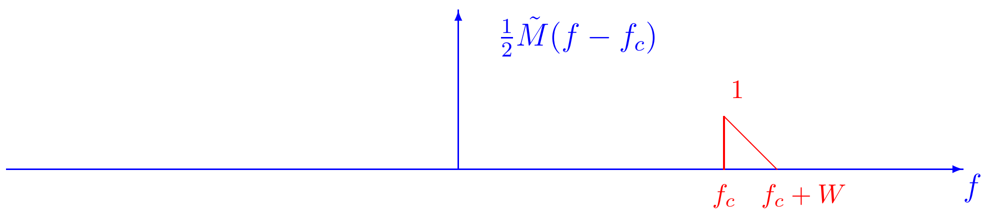

(c)

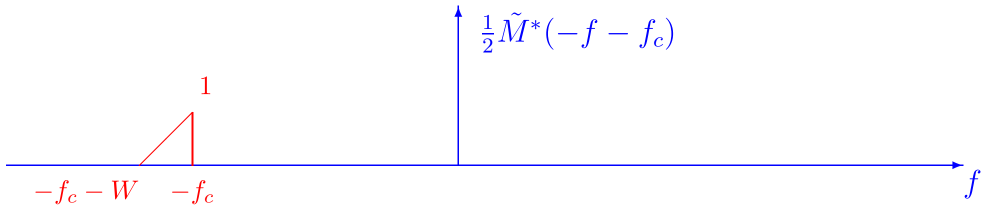


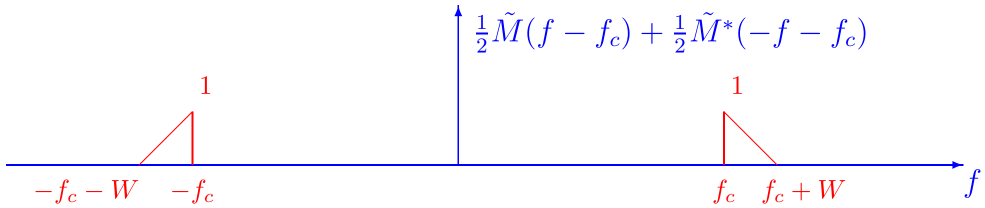

(d) From Problem 1(f), we know that the spectrum of $\mathrm{Re}\{m_+(t)e^{j2\pi f_c t}\} = m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)$, where $m_+(t) = m(t) + j\hat{m}(t)$, is equal to

$$\frac{1}{2}\left[\tilde{M}(f - f_c) + \tilde{M}^*(-f - f_c)\right].$$

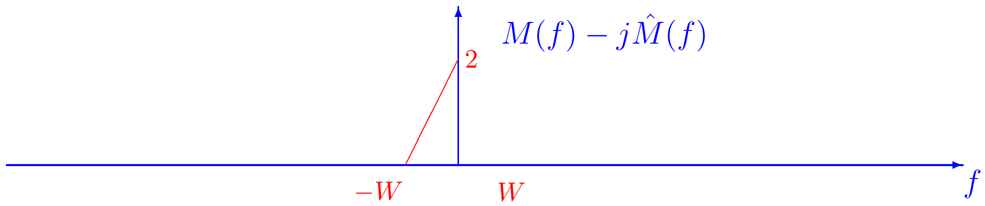
Since $\tilde{M}(f) = M(f) + j\hat{M}(f)$, we plot:



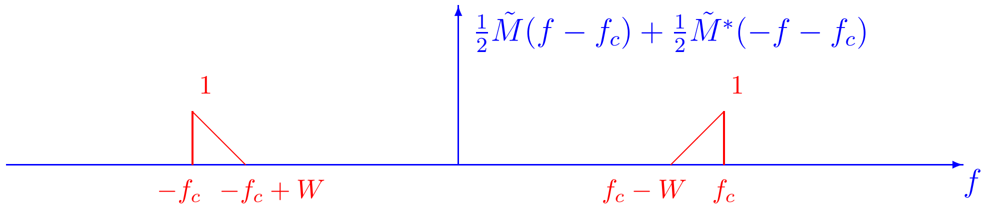


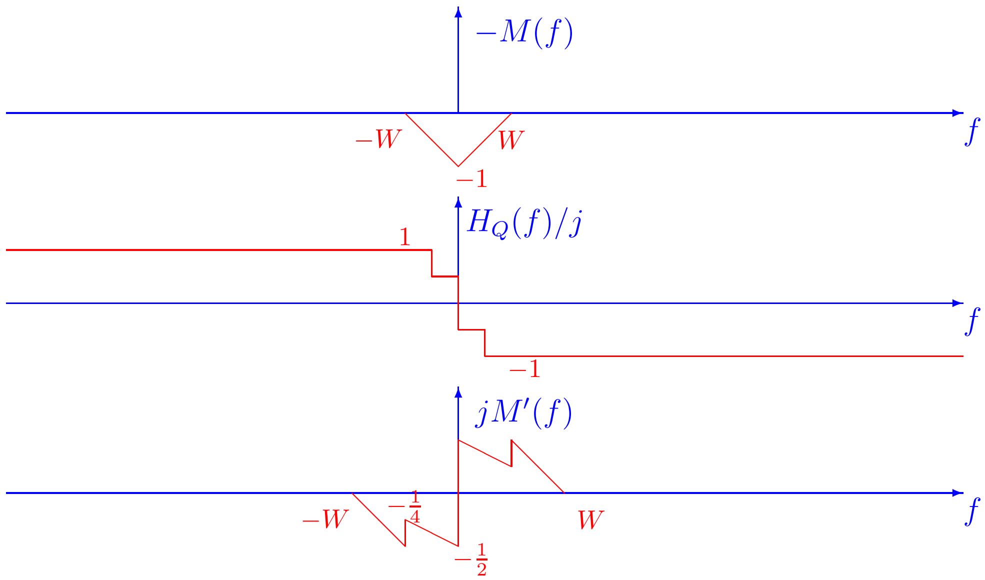

(e)



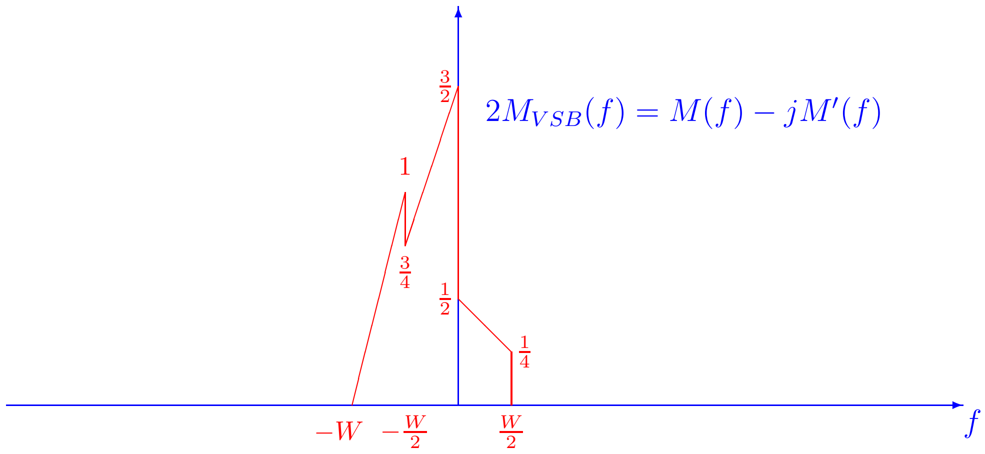

(f) With $\tilde{M}(f)=M(f)-j\hat{M}(f)$, we have:




(g) $jM'(f)=jH_Q(f)M(f)=\frac{H_Q(f)}{j}(-M(f))$




(h)




5. The passband filter $H(f)$ for VSB modulation is required to satisfy two conditions:

$$\begin{cases} |H(f_c - f)| + |H(f_c + f)| = 1 & \text{for } |f| < f_v \\ H(f - f_c) + H(f + f_c) = 1 & \text{for } |f| < W \end{cases}$$

Thus, given $f = \pm 0.01$ MHz, $f_c = 1$ MHz, $W = 0.04$ MHz and $f_v = 0.02$ MHz, we have

$$\begin{cases} |H(0.99)| + |H(1.01)| = 1 \\ H(-0.99) + H(1.01) = 1 \\ H(-1.01) + H(0.99) = 1 \end{cases}$$

(a) If $H(1.01) = \frac{1+j}{2}$, determine the value of $H(-0.99)$.

(b) Can we determine the value of $H(0.99)$ and $H(-1.01)$? Justify your answer.

(c) Can a third condition that $H(f)$ is conjugate symmetric, i.e., $H(-f) = H^*(f)$, be satisfied under $H(1.01) = \frac{1+j}{2}$? Justify your answer.

(d) Show that

$$\begin{cases} |H(f_c - f)| + |H(f_c + f)| = 1 & \text{for } |f| < f_v \\ H(f - f_c) + H(f + f_c) = 1 & \text{for } |f| < W \\ H(f) = H^*(-f) \end{cases}$$

imply $H(f)$ is real for $|f - f_c| < f_v$.

**Solution.**

(a) $H(-0.99) = 1 - \frac{1+j}{2} = \frac{1-j}{2}$.

(b) $|H(0.99)| = 1 - \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{2-\sqrt{2}}{2}$. Hence,

$$H(0.99) = \left(\frac{2-\sqrt{2}}{2}\right) e^{j\theta}.$$

Also,

$$H(-1.01) = 1 - H(0.99) = 1 - \left(\frac{2-\sqrt{2}}{2}\right) e^{j\theta}.$$

We cannot determine $H(0.99)$ and $H(-1.01)$ because $\theta$ is unknown.

(c) For conjugate symmetric $H(f)$, we only need to specify $H(f)$ for $f > 0$. Thus the second condition becomes

$$H^*(f_c - f) + H(f_c + f) = 1 \text{ for } |f| < W$$

which implies

$$H(0.99) = (1 - H(1.01))^* = \frac{1-j}{2}.$$

However, this violates the first condition that requires $|H(0.99)| + |H(1.01)| = 1$ because

$$\left|\frac{1-j}{2}\right| + \left|\frac{1+j}{2}\right| = \sqrt{2} \neq 1.$$

Note: This concludes that $H(1.01)$ must not have an imaginary part.

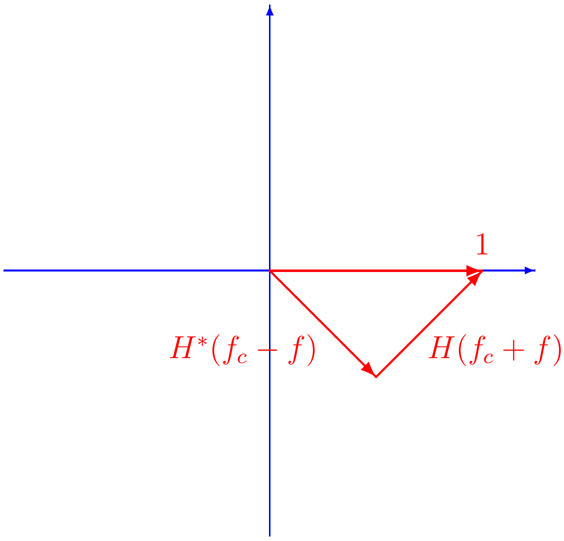
(d) For conjugate symmetric $H(f)$, the second condition becomes

$$H^*(f_c - f) + H(f_c + f) = 1 \text{ for } |f| < W.$$

We thus have

$$\begin{cases} |H^*(f_c - f)| + |H(f_c + f)| = 1 \text{ for } |f| < f_v; \\ H^*(f_c - f) + H(f_c + f) = 1 \text{ for } |f| < f_v. \end{cases}$$

We plot $H^*(f_c - f) + H(f_c + f) = 1$ for $|f| < f_v$ in the complex domain.




Then, $|H^*(f_c - f)| + |H(f_c + f)| = 1$ for $|f| < f_v$ indicates the three vectors in the above figure cannot form a triangle and hence $H^*(f_c - f)$ and $H(f_c + f)$ must be both real-valued.

**(The next two problems are only for your reference. They will not be a part of the exam or quizzes.)**

6. In this problem, we demonstrate the idea behind the Wiener-Khintchine theorem in response to queries from some students. As given in our lecture, the time-average autocorrelation function is defined as

$$\bar{R}_X(\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau)X^*(t)]\mathrm{d}t.$$

Denoting

$$\begin{cases} a_T(y) \triangleq \dfrac{1}{2T}\displaystyle\int_{T-y}^{T} E[X(x+y)X^*(x)]\mathrm{d}x \\ b_T(y) \triangleq \dfrac{1}{2T}\displaystyle\int_{-T}^{-T-y} E[X(x+y)X^*(y)]\mathrm{d}x \\ c_T(y) \triangleq \dfrac{1}{2T}\displaystyle\int_{-T}^{T} E[X(x+y)X^*(y)]\mathrm{d}x \end{cases} \tag{2}$$

we assume that there exists a function $\epsilon(T)$ such that

$$\max_{0\le y<2T} \max\{|a_T(y)|, |b_T(y)|\} \le \epsilon(T) \quad \text{and} \quad \lim_{T\to\infty} T\cdot\epsilon(T) = 0 \tag{3}$$

and also assume that

$$\lim_{T\to\infty}\int_{|y|>2T} |c_T(y)|\mathrm{d}y = 0. \tag{4}$$

Prove that

$$\overline{\text{PSD}}(f) \triangleq \lim_{T\to\infty}\frac{1}{2T}E[|X_{2T}(f)|^2]$$

is the Fourier transform of the time-average autocorrelation function $\bar{R}_X(\tau)$, where $X_{2T}(f)$ is the Fourier transform of

$$X_{2T}(t) \triangleq \begin{cases} X(t), & |t| < T; \\ 0, & \text{otherwise.} \end{cases}$$

**Solution.**

$$\begin{aligned}
\overline{\text{PSD}}(f) &= \lim_{T\to\infty}\frac{1}{2T}E\left[\left(\int_{-T}^{T} X(t)e^{-j2\pi ft}\mathrm{d}t\right)\left(\int_{-T}^{T} X(t')e^{-j2\pi ft'}\mathrm{d}t'\right)^*\right] \\
&= \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{-T}^{T}\int_{-T}^{T} X(t)X^*(t')e^{-j2\pi ft}e^{j2\pi ft'}\mathrm{d}t\mathrm{d}t'\right] \quad (x = t' \text{ and } y = t-t') \\
&= \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{0}^{2T}\left(\int_{-T}^{T-y} X(x+y)X^*(x)\mathrm{d}x\right)e^{-j2\pi fy}\mathrm{d}y\right. \\
&\qquad\left. + \int_{-2T}^{0}\left(\int_{-T-y}^{T} X(x+y)X^*(x)e^{-j2\pi fy}\mathrm{d}x\right)\mathrm{d}y\right] \quad (t' = x \text{ and } t = x+y) \\
&= \lim_{T\to\infty}\left(\int_0^{2T}[c_T(y)-a_T(y)]e^{-j2\pi fy}\mathrm{d}y + \int_{-2T}^{0}[c_T(y)-b_T(y)]e^{-j2\pi fy}\mathrm{d}y\right) \\
&= \lim_{T\to\infty}\left(\int_{-\infty}^{\infty} c_T(y)e^{-j2\pi fy}\mathrm{d}y - \int_0^{2T} a_T(y)e^{-j2\pi fy}\mathrm{d}y - \int_{-2T}^{0} b_T(y)e^{-j2\pi fy}\mathrm{d}y\right. \\
&\qquad\left. - \int_{|y|>2T} c_T(y)e^{-j2\pi fy}\mathrm{d}y\right).
\end{aligned}$$

Observing that

$$\left|\int_0^{2T} a_T(y)e^{-j2\pi fy}\mathrm{d}y\right| \le \int_0^{2T}|a_T(y)e^{-j2\pi fy}|\mathrm{d}y \le \int_0^{2T}\epsilon(T)\mathrm{d}y \to 0 \quad \text{as } T\to\infty$$

$$\left|\int_0^{2T} b_T(y)e^{-j2\pi fy}\mathrm{d}y\right| \le \int_0^{2T} |b_T(y)e^{-j2\pi fy}|\mathrm{d}y \le \int_0^{2T} \epsilon(T)\mathrm{d}y \to 0 \quad \text{as } T\to\infty$$

$$\left|\int_{|y|>2T} c_T(y)e^{-j2\pi fy}\mathrm{d}y\right| \le \int_{|y|>2T} |c_T(y)|\mathrm{d}y \to 0 \quad \text{as } T\to\infty$$

we obtain

$$\begin{aligned}\overline{\mathrm{PSD}}(f) &= \lim_{T\to\infty}\int_{-\infty}^{\infty} c_T(y)e^{-j2\pi fy}\mathrm{d}y\\ &= \int_{-\infty}^{\infty}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{-T} E[X(x+y)X^*(y)]\mathrm{d}x\right)e^{-j2\pi fy}\mathrm{d}y\\ &= \int_{-\infty}^{\infty}\bar{R}_X(y)e^{-j2\pi fy}\mathrm{d}y.\end{aligned}$$

Note: Some books directly adopt $\overline{\mathrm{PSD}}(f) \triangleq \lim_{T\to\infty}\frac{1}{2T}E[|X_{2T}(f)|^2]$ as the definition of the time-average PSD and claim that it is the Fourier transform of the time-average autocorrelation function $\bar{R}_X(\tau)$. From the above proof, you shall realize some assumptions must be made in order to validate it.

Thus, I adopt a different approach in my lescture. Note that from Sample Problem 3(a) for Quiz 3, it is always valid that

$$\bar{S}_X(f) = \lim_{T\to\infty}\frac{1}{2T}E[X(f)X^*_{2T}(f)]$$

is the Fourier transform of

$$\bar{R}_X(\tau) \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X(t+\tau)X^*(t)]\mathrm{d}t.$$

In fact, if the time-average autocorrelation function is no longer functionally independent of the location of the integration window, then we can show that

$$\lim_{T\to\infty}\frac{1}{2T}E[X(f)X^*_{(L-T,L+T)}(f)]$$

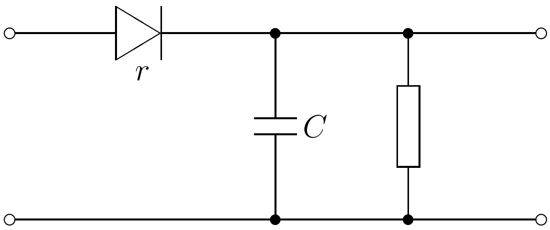
is the Fourier transform of

$$\lim_{T\to\infty}\frac{1}{2T}\int_{L-T}^{L+T} E[X(t+\tau)X^*(t)]\mathrm{d}t$$

where

$$X_{(L-T,L+T)}(t) \triangleq \begin{cases} X(t), & |t-L|<T;\\ 0, & \text{otherwise.}\end{cases}$$

7. The "squarer + lowpass filter + square rooter" concatenation is just one kind of envelope detectors. In fact, the term *envelope detector* can be used to refer to any electronic circuit that takes in a high-frequency amplitude modulated signal and provides an output that is the *envelope* of the original signal. As an example, one can use a diode and a capacitor to form a simple envelope detector that is even much cheaper than the "squarer + lowpass filter + square rooter" concatenation.




(a) The input-output relation of the simple envelope detector above can actually be characterized by

$$\frac{v_{\text{out}}(f)}{v_{\text{in}}(f)} = \frac{1}{1 + jf/f_0}$$

where $f_0$ is determined by the capacitor $C$. Find $v_{\text{out}}(t)$ if $v_{\text{in}}(t) = \delta(t)$.
Hint: $v_{\text{in}}(t) = \delta(t)$ implies $v_{\text{in}}(f) = 1$ and use the below table.

**TABLE A6.3** Fourier-transform pairs

| Time Function | Fourier Transform |
|---|---|
| $\exp(-at)u(t)$, $a > 0$ | $\frac{1}{a+j2\pi f}$ |
| $\exp(-a\|t\|)$, $a > 0$ | $\frac{2a}{a^2+(2\pi f)^2}$ |
| $\delta(t)$ | 1 |
| 1 | $\delta(f)$ |

Notes: $u(t)$ = unit step function
$\delta(t)$ = delta function or unit impulse

(b) Can the concatenation of squarer, ideal lowpass filter, square rooter and dc term remover perfectly recover $m(t)$ from the DSB-C signal $s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$ if $f_c > 2W$? Justify your answer.
Hint: Here, we assume that $1 + k_a m(t)$ is always non-negative.

**Solution.**

(a) Apparently, $v_{\text{out}}(f) = \frac{1}{1+jf/f_0} = \frac{2\pi f_0}{2\pi f_0 + j2\pi f}$. Hence, according to the table,

$$v_{\text{out}}(t) = 2\pi f_0 e^{-2\pi f_0 t} u(t).$$

Note: If $f_0$ is small (i.e., $C$ is large), then this envelop detector is simply to connect the adjacent peaks of the amplitude modulated waves by a straight line that well approximates the $e^{-2\pi f_0 t}$ curve, as the black dashed line in Slide 4-7. Hence, non-visualized envelope occurs when $f_c$ is not much larger than $W$.

(b) From the below procedure,

$$\begin{aligned} s(t) \quad &\xrightarrow{\text{squarer}} \quad s^2(t) = A_c^2(1+k_a m(t))^2 \cos^2(2\pi f_c t) \\ &\qquad\qquad = \frac{A_c^2}{2}(1+k_a m(t))^2[1+\cos(4\pi f_c t)] \\ &\xrightarrow{\text{ideal lowpass}} \quad \frac{A_c^2}{2}(1+k_a m(t))^2 \quad (\text{Hold when } f_c > 2W) \\ &\xrightarrow{\text{square rooter}} \quad \frac{A_c}{\sqrt{2}}(1+k_a m(t)) \\ &\xrightarrow{\text{dc term remover}} \quad \frac{A_c}{\sqrt{2}} k_a m(t) \end{aligned}$$

it can be seen that as long as $f_c > 2W$, the system can perfectly recover $m(t)$. Hence, the answer to the question is YES.

8. Below I repeat Sample Problem 1 for Quiz 4 with new proofs. Prove (a), (b), (c) in Table A.6.3 using Properties 4 and 9 and $\mathcal{F}(\delta(t)) = 1$. You can see that for the Fourier transform of (real-valued) even symmetric or odd symmetric functions, the last tricky term in Property 9 of Table A6.2 is often ignored.

   (d) in Table A6.3 is not (real-valued) even symmetric, nor odd symmetric. So, rewrite it as the sum of an even symmetric function and an odd symmetric function and get its Fourier transform as the sum of the Fourier transforms of the even and odd symmetric functions.

**TABLE A6.2** Summary of properties of the Fourier transform

| Property | Mathematical Description |
|---|---|
| 4. Time shifting | $g(t-t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$ |
| 8. Differentiation in the time domain | $\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$ |
| 9. Integration in the time domain | $\int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \underbrace{\frac{G(0)}{2}\delta(f)}_{\text{this term is tricky!}}$ |
| 10. Conjugate functions | If $g(t) \rightleftharpoons G(f)$,<br>then $g^*(t) \rightleftharpoons G^*(-f)$ (and additionally $g^*(-t) \rightleftharpoons G^*(f)$) |
| 11. Multiplication in the time doman | $g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda$ |
| 12. Convolution in the time doman | $\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)$ |

**TABLE A6.3** Fourier-transform pairs

| Time Function | Fourier Transform |
|---|---|
| (a) $\mathrm{rect}\left(\frac{t}{T}\right)$ | $T\mathrm{sinc}(Tf)$ |
| (b) $\mathrm{sgn}(t)$ | $\frac{1}{j\pi f}$ |
| (c) $\Delta(t) := \begin{cases} 1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \geq T \end{cases}$ | $T\mathrm{sinc}^2(Tf)$ |
| (d) $u(t)$ | $\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$ |

Notes: $u(t)$ = unit step function
$\delta(t)$ = delta function or unit impulse
$\text{rect}(t)$ = rectangular function of unit amplitude and unit duration centered on the origin
$\text{sgn}(t)$ = signum function
$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ sinc function

**Solution.**

(a)

$$\begin{aligned}
\mathcal{F}\left\{\text{rect}\left(\frac{t}{T}\right)\right\} &= \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{\left(\delta\left(\tau+\frac{T}{2}\right)-\delta\left(\tau-\frac{T}{2}\right)\right)}_{g(\tau)\text{in Property 9}}\mathrm{d}\tau\right\} \\
&= \frac{1}{j2\pi f}\underbrace{\left(e^{-j2\pi f(-\frac{T}{2})}-e^{-j2\pi f(\frac{T}{2})}\right)}_{G(f)}+\cancel{\underbrace{\left(e^{-j2\pi\cdot 0(-\frac{T}{2})}-e^{-j2\pi\cdot 0(\frac{T}{2})}\right)}_{G(0)}\frac{1}{2}\delta(f)} \\
&= \frac{1}{j2\pi f}(j2\sin(\pi f T)) = T\cdot\frac{\sin(\pi f T)}{\pi f T} = T\text{sinc}(Tf).
\end{aligned}$$

(b)

$$\mathcal{F}\{\text{sgn}\,(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{2\delta(\tau)}_{g(\tau)\text{ in Property 9}}\mathrm{d}\tau - 1\right\} = \frac{1}{j2\pi f}\underbrace{2}_{G(f)}+\frac{1}{2}\underbrace{2}_{G(0)}\delta(f)-\delta(f) = \frac{1}{j\pi f}.$$

(c)

$$\begin{aligned}
G_1(f) &= \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{\frac{1}{T}\left(\delta(\tau+T)-2\delta(\tau)+\delta(\tau-T)\right)}_{g(\tau)\text{ in Property 9}}\mathrm{d}\tau\right\} \\
&= \left(\frac{1}{j2\pi f}\right)\underbrace{\frac{1}{T}\left(e^{-j2\pi f(-T)}-2+e^{-j2\pi f(T)}\right)}_{G(f)}+\frac{1}{2}\underbrace{0}_{G(0)}\delta(f) \\
&= \left(\frac{1}{j2\pi f}\right)\frac{1}{T}\left(2\cos(2\pi f T)-2\right)
\end{aligned}$$

Note that

$$G_1(0) = \left.\frac{\frac{\mathrm{d}}{\mathrm{d}f}\left(2\cos(2\pi f T)-2\right)}{\frac{\mathrm{d}}{\mathrm{d}f}(j2\pi f T)}\right|_{f=0} = \left.\frac{-4\pi T\sin(2\pi f T)}{j2\pi T}\right|_{f=0} = 0.$$

Then,

$$\begin{aligned}
\mathcal{F}\{\Delta(t)\} &= \mathcal{F}\Bigg\{\int_{-\infty}^{t}\underbrace{\int_{-\infty}^{\tau}\frac{1}{T}\big(\delta(s+T)-2\delta(s)+\delta(s-T)\big)\mathrm{d}s}_{g_1(\tau)\text{ in Property 9}}\,\mathrm{d}\tau\Bigg\} \\
&= \frac{1}{j2\pi f}G_1(f)+\frac{1}{2}\underbrace{0}_{G_1(0)}\delta(f) \\
&= -\frac{1}{4\pi^2 f^2 T}\left(2\cos(2\pi fT)-2\right) = \frac{1}{4\pi^2 f^2 T}\left(2-2\cos(2\pi fT)\right) \\
&= T\frac{\sin^2(\pi fT)}{\pi^2 f^2 T^2} = T\mathrm{sinc}^2(Tf)
\end{aligned}$$

(d) $u(t)=\frac{1}{2}(1+\mathrm{sgn}(t))$; hence,

$$\mathcal{F}\{u(t)\}=\frac{1}{2}\mathcal{F}\{1+\mathrm{sgn}(t)\}=\frac{1}{2}\left(\delta(f)+\frac{1}{j\pi f}\right).$$