Sample Problems for Quiz 4 (March 29, 2021)

Correction

- Slide 3-47:

$$\begin{aligned} x(t) &= A\cos(2\pi f_c t) + n(t) \\ &= A\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \end{aligned}$$

1. Prove (a), (b), (c) in Table A.6.3 using Properties 4 and 9 and $\mathcal{F}(\delta(t)) = 1$. You can see that for the Fourier transform of (real-valued) even symmetric or odd symmetric functions, the last tricky term in Property 9 of Table A6.2 is often ignored.

   (d) in Table A6.3 is not (real-valued) even symmetric, nor odd symmetric. So, rewrite it as the sum of an even symmetric function and an odd symmetric function and get its Fourier transform as the sum of the Fourier transforms of the even and odd symmetric functions.

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
8. Differentiation in the time domain	$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \underbrace{\frac{G(0)}{2}\delta(f)}_{\text{this term is tricky!}}$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$ (and additionally $g^*(-t) \rightleftharpoons G^*(f)$)
11. Multiplication in the time doman	$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda$
12. Convolution in the time doman	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)$

TABLE A6.3 Fourier-transform pairs

Time Function	Fourier Transform
(a) $\text{rect}\left(\frac{t}{T}\right)$	$T\text{sinc}(Tf)$
(b) $\text{sgn}(t)$	$\frac{1}{j\pi f}$
(c) $\Delta(t) := \begin{cases} 1 - \frac{\lvert t \rvert}{T}, & \lvert t \rvert < T \\ 0, & \lvert t \rvert \geq T \end{cases}$	$T\text{sinc}^2(Tf)$
(d) $u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

Notes: $u(t)$ = unit step function
$\delta(t)$ = delta function or unit impulse
$\text{rect}(t)$ = rectangular function of unit amplitude and unit duration centered on the origin
$\text{sgn}(t)$ = signum function
$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ sinc function

Solution.

(a)

$$\begin{aligned}
\mathcal{F}\left\{\text{rect}\left(\frac{t}{T}\right)\right\} &= \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{\left(\delta\left(\tau+\frac{T}{2}\right)-\delta\left(\tau-\frac{T}{2}\right)\right)}_{g(\tau)\text{in Property 9}}\mathrm{d}\tau\right\} \\
&= \frac{1}{j2\pi f}\underbrace{\left(e^{-j2\pi f(-\frac{T}{2})}-e^{-j2\pi f(\frac{T}{2})}\right)}_{G(f)} + \cancel{\underbrace{\left(e^{-j2\pi\cdot 0(-\frac{T}{2})}-e^{-j2\pi\cdot 0(\frac{T}{2})}\right)}_{G(0)}\frac{1}{2}\delta(f)} \\
&= \frac{1}{j2\pi f}(j2\sin(\pi f T)) = T\cdot\frac{\sin(\pi f T)}{\pi f T} = T\text{sinc}(Tf).
\end{aligned}$$

(b)

$$\mathcal{F}\{\text{sgn}\,(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{2\delta(\tau)}_{g(\tau)\text{ in Property 9}}\mathrm{d}\tau - 1\right\} = \frac{1}{j2\pi f}\underbrace{2}_{G(f)} + \frac{1}{2}\underbrace{2}_{G(0)}\delta(f) - \delta(f) = \frac{1}{j\pi f}.$$

(c)

$$\begin{aligned}
G_1(f) &= \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{\frac{1}{T}\left(\delta(\tau+T)-2\delta(\tau)+\delta(\tau-T)\right)}_{g(\tau)\text{ in Property 9}}\mathrm{d}\tau\right\} \\
&= \left(\frac{1}{j2\pi f}\right)\underbrace{\frac{1}{T}\left(e^{-j2\pi f(-T)}-2+e^{-j2\pi f(T)}\right)}_{G(f)} + \frac{1}{2}\underbrace{0}_{G(0)}\delta(f) \\
&= \left(\frac{1}{j2\pi f}\right)\frac{1}{T}\left(2\cos(2\pi f T)-2\right)
\end{aligned}$$

Note that

$$G_1(0) = \left.\frac{\frac{\mathrm{d}}{\mathrm{d}f}\left(2\cos(2\pi f T)-2\right)}{\frac{\mathrm{d}}{\mathrm{d}f}(j2\pi f T)}\right|_{f=0} = \left.\frac{-4\pi T\sin(2\pi f T)}{j2\pi T}\right|_{f=0} = 0.$$

Then,

$$\begin{aligned}
\mathcal{F}\{\Delta(t)\} &= \mathcal{F}\left\{\int_{-\infty}^{t}\underbrace{\int_{-\infty}^{\tau}\frac{1}{T}\left(\delta(s+T)-2\delta(s)+\delta(s-T)\right)\mathrm{d}s}_{g_1(\tau)\text{ in Property 9}}\,\mathrm{d}\tau\right\} \\
&= \frac{1}{j2\pi f}G_1(f) + \frac{1}{2}\underbrace{0}_{G_1(0)}\delta(f) \\
&= -\frac{1}{4\pi^2 f^2 T}\left(2\cos(2\pi f T)-2\right) = \frac{1}{4\pi^2 f^2 T}\left(2-2\cos(2\pi f T)\right) \\
&= T\frac{\sin^2(\pi f T)}{\pi^2 f^2 T^2} = T\text{sinc}^2(Tf)
\end{aligned}$$

(d) $u(t) = \frac{1}{2}(1 + \text{sgn}(t))$; hence,

$$\mathcal{F}\{u(t)\} = \frac{1}{2}\mathcal{F}\{1 + \text{sgn}(t)\} = \frac{1}{2}\left(\delta(f) + \frac{1}{j\pi f}\right).$$

2. (a) Prove the following statement using Property 10 of Table A6.2: $g(t)$ is real if and only if $G(f) = G^*(-f)$ (conjugate symmetric).

   (b) Prove the following statement using Property 10 of Table A6.2: $g(t)$ is pure imaginary if and only if $G(f) = -G^*(-f)$ (conjugate anti-symmetric).

   (c) Justify that all functions can be represented as the sum of a conjugate symmetric function and a conjugate antisymmetric function.

   (d) If $G(f) = \mathcal{F}\{g(t)\}$, express $\text{Re}\{G(f)\}$ as a function $g(t)$.

   Solution.

   (a) $g(t)$ real iff $g(t) = g^*(t)$ iff $G(f) = G^*(-f)$ from Property 10 of Table A6.2.

   (b) $g(t)$ imaginary iff $g(t) = -g^*(t)$ iff $G(f) = -G^*(-f)$ from Property 10 of Table A6.2.

   (c)
   $$g(t) = \underbrace{\frac{g(t) + g^*(-t)}{2}}_{\text{conjugate symmetric}} + \underbrace{\frac{g(t) - g^*(-t)}{2}}_{\text{conjugate antisymmetric}}$$

   (d) $\text{Re}\{G(f)\} = \mathcal{F}\left\{\frac{g(t)+g^*(-t)}{2}\right\}$

3. Find the Hilbert transform $\hat{g}(t)$ of the following function $g(t)$.

   Hint:
   $$\hat{G}(f) = H_{\text{hilbert}}(f)\, G(f) = (-j\text{sgn}(f))\, G(f)$$

   and

   TABLE A6.3 Fourier-transform pairs

Time Function	Fourier Transform
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

Notes: $u(t)$ = unit step function
$\delta(t)$ = delta function or unit impulse
and unit duration centered on the origin
$\text{sgn}(t)$ = signum function
$\text{sinc}(t)$ = sinc function

(a) $g(t) = \delta(t)$.

(b) $g(t) = \cos(2\pi f_c t)$ with $f_c > 0$.

Solution.

(a) It is clear that $\hat{g}(t) = \delta(t) \star \frac{1}{\pi t} = \frac{1}{\pi t}$.

(b)

$$\begin{aligned}
\hat{G}(f) &= H_{\text{hilbert}}(f)G(f) \\
&= (-j\text{sgn}(f))\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)] \\
&= \frac{1}{2}[\delta(f-f_c)(-j\text{sgn}(f))+\delta(f+f_c)(-j\text{sgn}(f))] \\
&= \frac{1}{2}[\delta(f-f_c)(-j\text{sgn}(f_c))+\delta(f+f_c)(-j\text{sgn}(-f_c))] \\
&\quad \text{(Note that } \lambda(f)\delta(f-f_c)=\lambda(f_c)\delta(f-f_c).\text{)} \\
&= \frac{1}{2}[\delta(f-f_c)(-j)+\delta(f+f_c)(j)] \\
&= \frac{1}{2}(-j)[\delta(f-f_c)-\delta(f+f_c)] \\
&= \frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)].
\end{aligned}$$

As a result, $\hat{g}(t) = \sin(2\pi f_c t)$.

4. Check if the following statements are true.

(a) A signal with pulse shape $p(t)$ that has a smaller time-bandwidth product in general demands less bandwidth.

(b) A time-limited signal can be also band-limited.

(c) The noise equivalent bandwidth is defined as the rms bandwidth of the noise process.

Solution.

(a) Yes, since the bandwidth demand for a given time duration is proportional to the time-bandwidth product.

(b) No, since a time-limited signal must be band-unlimited.

(c) No. The noise equivalent bandwidth of a filter (or a system) is the bandwidth of an ideal low-pass filter through which the same noise power at the filter output (or the system output) is resulted.

5. From Slide 3-12, we know that

$$g(t) = \mathrm{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}.$$

Prove that

$$G(f) = \frac{1}{2}(\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c)).$$

Note: Here, $\tilde{G}(f)$ may be complex-valued.

Solution.

$$\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \mathrm{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}\, e^{-j2\pi f t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\big(\tilde{g}(t)e^{j2\pi f_c t} + (\tilde{g}(t)e^{j2\pi f_c t})^*\big)e^{-j2\pi f t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\big(\tilde{g}(t)e^{j2\pi f_c t} + \tilde{g}^*(t)e^{-j2\pi f_c t}\big)e^{-j2\pi f t}\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \tilde{g}(t)e^{-j2\pi(f - f_c)t}\mathrm{d}t + \frac{1}{2}\left(\int_{-\infty}^{\infty} \tilde{g}(t)e^{-j2\pi(-f-f_c)\tau}\mathrm{d}t\right)^* \\
&= \frac{1}{2}\big(\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c)\big)
\end{aligned}$$

6. Suppose

$$N(t) = \mathrm{Re}\left\{(N_I(t) + jN_Q(t))e^{j2\pi f_c t}\right\} = N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t).$$

(a) Prove that if *i)* $N(t)$ is WSS and *ii)* $N_I(t)$ and $N_Q(t)$ are jointly WSS, then

$$\text{Property 1:} \quad \begin{cases} R_{N_I}(\tau) = R_{N_Q}(\tau) \\ R_{N_I,N_Q}(\tau) = -R_{N_Q,N_I}(\tau) \end{cases}$$

(b) Based on (a), further show that $R_N(\tau) = R_{N_I}(\tau)\cos(2\pi f_c t) - R_{N_Q,N_I=Q}\sin(2\pi f_c t)$.

(c) Show that $R_{\tilde{N}}(\tau) = 2R_{N_I}(\tau) + j2R_{N_Q,N_I}(\tau)$.

(d) Argue based on (b) and (c) that $R_N(\tau) = \frac{1}{2}\mathrm{Re}\{R_{\tilde{N}}(\tau)e^{j2\pi f_c \tau}\}$.

(e) Based on (d), show that

$$S_N(f) = \frac{1}{4}\left(R_{\tilde{N}}(f - f_c) + R_{\tilde{N}}(-f - f_c)\right).$$

Solution.

(a) Slides 3-28 and 3-29

(b) Take (a) into the derivation in Slide 3-29.

(c) Slide 3-30

(d) It can be easily verified that

$$\begin{aligned}
\frac{1}{2}\text{Re}\{R_{\tilde{N}}(\tau)e^{j2\pi f_c\tau}\} &= \frac{1}{2}\text{Re}\{[2R_{N_I}(\tau)+j2R_{N_Q,N_I}(\tau)]e^{j2\pi f_c\tau}\} \quad \text{(From (c))}\\
&= \text{Re}\{[R_{N_I}(\tau)+jR_{N_Q,N_I}(\tau)]e^{j2\pi f_c\tau}\}\\
&= R_{N_I}(\tau)\cos(2\pi f_c t) - R_{N_Q,N_I}(\tau)\sin(2\pi f_c\tau)\\
&= R_N(\tau) \quad \text{(From (b))}
\end{aligned}$$

(e) Slide 3-33