Sample Problems for Quiz 3 (March 22, 2021)

Correction:

- On Slide 2-29, $X(t)$ should be complex conjugated, i.e.,

$$\begin{aligned}\bar{S}_X(f) &= \int_{-\infty}^{\infty}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[X(t+\tau)X^*(t)]\mathrm{d}t\right)e^{-j2\pi f\tau}\mathrm{d}\tau \\ &= \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}X(t+\tau)X_{2T}^*(t)\mathrm{d}t\right)e^{-j2\pi f\tau}\mathrm{d}\tau\right] \\ &= \lim_{T\to\infty}\frac{1}{2T}E[X(f)X_{2T}^*(f)],\ \text{where } X_{2T}\triangleq X(t)\cdot\mathbf{1}\{|t|\le T\}.\end{aligned}$$

1. Prove each of the seven properties in Table A6.2.

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
1. Linearity	$ag_1(t)+bg_2(t)\rightleftharpoons aG_1(f)+bG_2(f)$ where a and b are constants
2. Time scaling	$g(at)\rightleftharpoons \frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t)\rightleftharpoons G(f)$, then $G(t)\rightleftharpoons g(-f)$
4. Time shifting	$g(t-t_0)\rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_0 t)g(t)\rightleftharpoons G(f-f_0)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty}g(t)dt=G(0)$
7. Area under $G(f)$	$g(0)=\int_{-\infty}^{\infty}G(f)df$

Solution.

P1.

$$\begin{aligned}\int_{-\infty}^{\infty}(ag_1(t)+bg_2(t))e^{-j2\pi ft}\mathrm{d}t &= a\int_{-\infty}^{\infty}g_1(t)e^{-j2\pi ft}\mathrm{d}t+b\int_{-\infty}^{\infty}g_2(t)e^{-j2\pi ft}\mathrm{d}t \\ &= aG_1(f)+bG_2(f)\end{aligned}$$

P2. Letting $s=at$, we derive

$$\begin{aligned}\int_{-\infty}^{\infty}g(at)e^{-j2\pi ft}\mathrm{d}t &= \begin{cases}\int_{-\infty}^{\infty}g(s)e^{-j2\pi f\frac{s}{a}}\frac{1}{a}\mathrm{d}s, & a>0\\ \int_{\infty}^{-\infty}g(s)e^{-j2\pi f\frac{s}{a}}\frac{1}{a}\mathrm{d}s, & a<0\end{cases} \\ &= \int_{-\infty}^{\infty}g(s)e^{-j2\pi\frac{f}{a}s}\frac{1}{|a|}\mathrm{d}s \\ &= \frac{1}{|a|}G\left(\frac{f}{a}\right)\end{aligned}$$

P3.

$$\begin{aligned}\int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\mathrm{d}t &= \int_{-\infty}^{\infty} G(f)e^{-j2\pi tf}\mathrm{d}f \quad \text{(Exchange of } t \text{ and } f\text{)}\\ &= \int_{-\infty}^{\infty} G(f)e^{j2\pi(-t)f}\\ &= g(-t)\end{aligned}$$

P4. Letting $s = t - t_0$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi ft}\mathrm{d}t &= \int_{-\infty}^{\infty} g(s)e^{-j2\pi f(s+t_0)}\mathrm{d}s\\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} g(s)e^{-j2\pi fs}\mathrm{d}s\\ &= e^{-j2\pi ft_0}G(f)\end{aligned}$$

P5.

$$\int_{-\infty}^{\infty} e^{j2\pi f_0 t}g(t)e^{-j2\pi ft}\mathrm{d}t = \int_{-\infty}^{\infty} g(t)e^{-j2\pi(f-f_0)t}\mathrm{d}t = G(f-f_0)$$

P6.

$$\int_{-\infty}^{\infty} g(t)\mathrm{d}t = \int_{-\infty}^{\infty} g(t)e^{-j2\pi\,0\,t}\mathrm{d}t = G(0)$$

P7.

$$g(0) = \int_{-\infty}^{\infty} G(f)e^{j2\pi\,f\,0}\mathrm{d}f = \int_{-\infty}^{\infty} G(f)\mathrm{d}f$$

2. If $G(f)$ is the Fourier transform of $g(t)$, answer whether the following statement is correct or not, and which property in Table A6.2 your answer is based on.

(a) The Fourier transform of $2 \times g(t)$ is $2 \times G(f)$.

(b) The Fourier transform of $G(t)$ is $g^*(-f)$.

(c) The Fourier transform of $g(t-a)$ is $G(f)e^{-j2\pi af}$.

(d) The Fourier transform of $g(2t)$ is $2G(\frac{f}{2})$.

(e) $G(0) = \int_{-\infty}^{\infty} g(t)dt$

Solution.

(a) Based on the linearity property of the Fourier transform, the statement is correct.

(b) Based on the duality property of the Fourier transform, the statement is incorrect since $g(-f)$ may not equal $g^*(-f)$.

(c) Based on the time shifting property of the Fourier transform, the statement is correct.

(d) Based on the time scaling property of the Fourier transform, the statement is incorrect. The correct answer should be $(1/2)G(f/2)$.

(e) Based on the "Area under $g(t)$" property of the Fourier transform, the statement is correct.

3. The time-average autocorrelation function $\bar{R}_X(\tau)$ of complex-valued random process $X(t)$ is given by

$$\bar{R}_X(\tau) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau)X^*(t)]dt,$$

and the time-average power spectra density (PSD) is defined as

$$\bar{S}_X(f) := \int_{-\infty}^{\infty} \bar{R}_X(\tau) e^{-j2\pi f\tau} d\tau.$$

(a) Show that

$$\bar{S}_X(f) = \lim_{T\to\infty} \frac{1}{2T} E[X(f)X_{2T}^*(f)],$$

where

$$X_{2T}(t) := \begin{cases} X(t), & |t| \le T; \\ 0, & \text{otherwise} \end{cases} = X(t) \cdot \mathbf{1}\{|t| \le T\}.$$

(b) Find the time-averaged PSD of $X(t)$, given that $X(t) = 1$ is a constant (i.e., non-random) function.

Hint: A quote from Table A6.3: Fourier transform pairs.

Time Function	Fourier Transform
$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(Tf) = T\,\frac{\sin(\pi Tf)}{\pi Tf}$

Notes: $\text{rect}(t)$ = rectangular function of unit amplitude and unit duration centered on the origin

Hence, $\text{rect}(\frac{t}{T}) = \begin{cases} 1, & |f| < \frac{T}{2}; \\ 0, & \text{otherwise.} \end{cases}$

(c) In reality, we could not know or measure $X(t)$ for $-\infty < t < \infty$. Instead we shall use the so-called "periodogram" to estimate $\bar{S}_X(f)$:

$$\bar{S}_X(f) \approx \frac{1}{2T} E[X_{2T}(f)X_{2T}^*(f)],$$

which can be computed based on a window period of $X(t)$. Find the periodogram approximation of $\bar{S}_X(f)$ in (b).

Solution.

(a)

$$
\begin{aligned}
\bar{S}_X(f) &= \int_{-\infty}^{\infty} \bar{R}_X(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau)X^*(t)]dt \right) e^{-j2\pi f\tau} d\tau \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} X^*(t) \left(\int_{-\infty}^{\infty} X(t+\tau) e^{-j2\pi f\tau} d\tau \right) dt \right] \quad \text{(Let } s = t+\tau.\text{)} \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} X^*(t) \left(\int_{-\infty}^{\infty} X(s) e^{-j2\pi f\cdot(s-t)} ds \right) dt \right] \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} X^*(t) \left(\int_{-\infty}^{\infty} X(s) e^{-j2\pi fs} ds \right) e^{j2\pi ft} dt \right] \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} X^*(t) X(f) e^{j2\pi ft} dt \right] \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[X(f) \left(\int_{-T}^{T} X(t) e^{-j2\pi ft} dt \right)^* \right] \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[X(f) X_{2T}^*(f) \right]
\end{aligned}
$$

(b) There are two ways to derive the PSD. A formal way is perhaps to derive the time-average autocorrelation function and find its Fourier transform as follows:

$$
\begin{aligned}
\bar{R}_X(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau)X^*(t)]dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \quad \text{(Because } X(t) = 1.\text{)} \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{S}_X(f) &= \int_{-\infty}^{\infty} \bar{R}_X(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} e^{-j2\pi f\tau} d\tau \quad \text{(Because } \bar{R}_X(\tau) = 1.\text{)} \\
&= \delta(-f),
\end{aligned}
$$

where the last step holds because the Fourier transform of $\delta(t)$ is 1 (using the replication property of the delta function) and therefore by duality property of the Fourier transform, the Fourier transform of 1 is $\delta(-f)$. Note that $\delta(-f) = \delta(f)$ since δ-function is symmetric.

Alternatively, we use the formula in (a). We know that $X(t) = 1$ and $X_{2T}(t) = \text{rect}(\frac{t}{2T})$. Hence, their Fourier transforms are respectively $X(f) = \delta(f)$ and $X_{2T}(f) = 2T\,\text{sinc}(2Tf)$.

Thus,

$$\begin{aligned}
\bar{S}_X(f) &= \lim_{T\to\infty} \frac{1}{2T} E\left[X(f)X_{2T}^*(f)\right] \\
&= \lim_{T\to\infty} \frac{1}{2T} E\left[\delta(f)\cdot 2T\,\mathrm{sinc}(2Tf)\right] \\
&= \lim_{T\to\infty} \delta(f)\,\mathrm{sinc}(2Tf) \\
&= \lim_{T\to\infty} \delta(f)\,\mathrm{sinc}(2T\cdot 0) \\
&= \lim_{T\to\infty} \delta(f) \\
&= \delta(f),
\end{aligned}$$

where we use the property that $g(t)\delta(t) = g(0)\delta(t)$ (i.e., only the value of $g(t)$ at $t = 0$ matters when it multiplies $\delta(t)$) and $\mathrm{sinc}(0) = 1$.

Note: I favor the alternative way because it is much more "thoughtful."

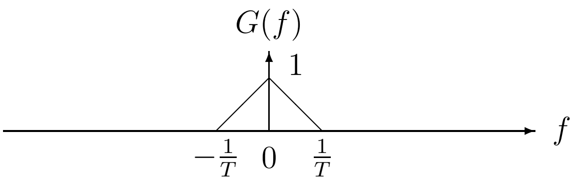
(c) (Let's use the thoughtful approach.) Since $X_{2T}(f) = 2T\,\mathrm{sinc}(2Tf)$, we have

$$\begin{aligned}
\frac{1}{2T} E[X_{2T}(f)X_{2T}^*(f)] &= \frac{1}{2T} E[|X_{2T}(f)|^2] \\
&= \frac{1}{2T}(4T^2\,\mathrm{sinc}^2(2Tf)) \\
&= 2T\,\mathrm{sinc}^2(2Tf).
\end{aligned}$$

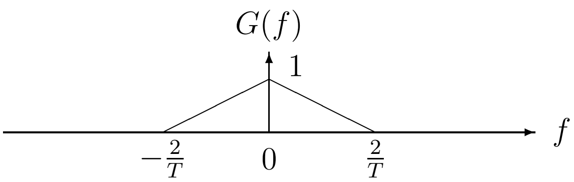
4. Suppose $f(t) = \sum_{n=-\infty}^{\infty} g(t - nT)$, which is periodic with period T. Plot the Fourier transform $F(f)$ of $f(t)$, if $G(f)$ is given as follows.

Hint: See Slide 2-14. The Fourier transform of a periodic function must be in the form of a pulse train.

(a)

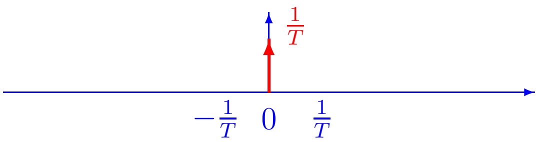


(b)

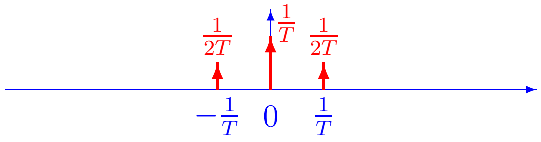


Solution.

(a)



(b)



5. (a) Use $R_X^*(\tau) = R_X(-\tau)$ to prove that PSD $S_X(f)$ must be real-valued.
   (b) Use *i*) $\underline{R_X(\tau) = R_X(-\tau)}$ and *ii*) $\underline{R_X(\tau)\text{ real}}$ to prove that PSD $S_X(f)$ is real and symmetric.
   (c) The autocorrelation function of a signal with random phase is periodic, and is given by $R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$. What is the problem encountered when using the technique in Slide 2-43 to prove that $S_X(f) \geq 0$.
   Hint: The Fourier transform of $\cos(2\pi f_c \tau)$ is $\frac{1}{2}(\delta(f - f_c) + \delta(f + f_c))$.

Solution.

(a) See Slide 2-40.

(b) Use (a) and the derivation in Slide 2-41.

(c) The PSD $S_X(f) = \frac{A^2}{4}(\delta(f - f_c) + \delta(f + f_c))$ is discontinuous at $f = f_c$ and $f = -f_c$. Hence, by replication property of the δ-function, $S_X(f_c)$ is actually undertermined.