Sample Problems for Quiz 2 (March 15, 2021)

Correction

- In the solution of Problem 5(b) of Sample Problems for Quiz 1:

$$\begin{aligned} R_X(t_1, t_2) &= E[e^{j(2\pi f_c t_1 + \Theta)}(e^{j(2\pi f_c t_2 + \Theta)})^*] \\ &= E[e^{j(2\pi f_c t_1 + \Theta)} e^{-j(2\pi f_c t_2 + \Theta)}] \\ &= E[e^{j(2\pi f_c (t_1 - t_2) \cancel{+\Theta})} \cancel{e^{-j(2\pi f_c t_2 + \Theta)}}] \\ &= E[e^{j(2\pi f_c (t_1 - t_2))}] \\ &= e^{j2\pi f_c (t_1 - t_2)} \end{aligned}$$

1. In order to give you an intuition on stationarity and ergodicity of a random process, I cook up an example below.

   Let $\mathcal{A} = \{001001, 010010, 100100\}$. Define $r(\cdot)$ as a circularly right-shift operation. For example,

$$r(001001) = 100100 \text{ and } r(100100) = 010010.$$

   Note that the circularly right-shift operation can be regarded as a tick in time; thus, it can be considered as a circularly "time" shift, justifying the usage of the term of "stationary" below.

   A set $\mathcal{A}$ is said to be *ergodic* with respect to operation r if for every $\vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathcal{A}$, we have $r(\vec{x}) \in \mathcal{A}$.

   A random process[1] $X_1, X_2, X_3, X_4, X_5, X_6$ is said to be an *ergodic* process if for every ergodic set $\mathcal{A}$, $P_{\vec{X}}(\mathcal{A})$ is either 1 or 0.

   (a) Is $\mathcal{A}$ an ergodic set? Is $\mathcal{B} = \{001001, 110110\}$ an ergodic set? Justify your answer.

   (b) Suppose a random process $\vec{X}$ satisfy

$$P_{\vec{X}}(001001) = \frac{1}{2} \text{ and } P_{\vec{X}}(110110) = \frac{1}{2}.$$

   What is the relative frequency of X_3? What is the relative frequency of X_5? Are they identical? Justify your answer.

   (c) Is the random process $\vec{X}$ ergodic? Justify your answer.

[1]It is actually a 6-dimensional random vector because a random process usually has infinite number of components, e.g., $X_1, X_2, X_3, \ldots$; but we momentarily "regard" it as a "random process" in this problem for convenience.

(d) List all possible time averages one may obtain from observing $\vec{X}$.

(e) Suppose a random process $\vec{Y}$ satisfy

$$P_{\vec{Y}}(001001) = P_{\vec{Y}}(010010) = P_{\vec{Y}}(100100) = \frac{1}{3}.$$

Is $\vec{Y}$ ergodic? Is $\vec{Y}$ (firt-order) stationary in the sense of relative frequency? List all possible time averages one may obtain from observing $\vec{Y}$.

Solution.

(a) $\mathcal{A}$ is ergodic but $\mathcal{B}$ is not. You shall justify the answer by performing the r-operation on each of the elements in a set by yourself.

(b) Apparently, $P_{X_3}(0) = P_{X_3}(1) = \frac{1}{2}$ and $P_{X_5}(0) = P_{X_5}(1) = \frac{1}{2}$. So they are identical.

Note: We may say $X_1, X_2, X_3, X_4, X_5, X_6$ is first-order stationary in the sense of "relative frequency."

(c) Apparently

$$P_{\vec{X}}(\mathcal{A}) = P_{\vec{X}}(001001)+P_{\vec{X}}(010010)+P_{\vec{X}}(100100) = \frac{1}{2}+0+0 = \frac{1}{2}$$

is not equal to either 1 or 0 for ergodic set $\mathcal{A}$. So, $\vec{X}$ is not an ergodic process.

(d) When performing six consecutive observations (or measurements) on $\vec{X}$, one may observe 001001 with probability $\frac{1}{2}$ and may get 110110 with probability $\frac{1}{2}$. Thus, the time average can be 1/3 with probability $\frac{1}{2}$ and 2/3 with probability 1/2.

Note: In other words, if I ask 100 students to make six consecutive observation on $X_1, X_2, X_3, X_4, X_5, X_6$, half of them anticipatively get 001001, but the other half of students may result in 110110. Both groups of students faithfully perform their experiments and yet two different time averages are resulted. This example gives you the intuition why a (first-order) stationary process is not enough to guarantee the time average converges to the ensemble average.

Advanced Note: For those students who are interested in and hence self-study the subject of probability space, $\vec{X}$ is actually a function mapping from $\Omega = \{\omega_1, \omega_2\}$ to $\{0,1\}^6$ with $P(\omega_1) = P(\omega_2) = \frac{1}{2}$. This mapping can be divided into six sub-mappings:

$$\begin{cases} X_1(\omega_1) = 0 \\ X_1(\omega_2) = 1 \end{cases} \quad \begin{cases} X_2(\omega_1) = 0 \\ X_2(\omega_2) = 1 \end{cases} \quad \begin{cases} X_3(\omega_1) = 1 \\ X_3(\omega_2) = 0 \end{cases}$$

$$\begin{cases} X_4(\omega_1) = 0 \\ X_4(\omega_2) = 1 \end{cases} \quad \begin{cases} X_5(\omega_1) = 0 \\ X_5(\omega_2) = 1 \end{cases} \quad \begin{cases} X_6(\omega_1) = 1 \\ X_6(\omega_2) = 0 \end{cases}$$

Thus, these sub-mappings well-define a random process. For a discrete-time random process $\mathbf{X} = (X_1, X_2, \ldots)$, we may define the random process via a mapping $\mathbf{X}$ from Ω to $\{0,1\}^\infty$, or via each sub-mapping $X_i : \Omega \to \{0,1\}$. When $\mathbf{X} = \{X_t, t \in \Re\}$, we can also define this continuous-time random process via each sub-papping X_t from Ω to $\{0,1\}$. Hope this advanced note can help you understand the notion of defining a random process based on a probability space $(\Omega, \mathcal{F}, P)$.

(e) $\vec{Y}$ is a stationary ergodic process, and the time average is always equal to $1/3$ (with probability one). Your shall justify your answer by yourself by checking the corresponding properties.

Note: The textbook saves the effort of introducing the ergodic theory but directly defines the process under consideration to be *ergodic in the mean* and *ergodic in the autocorrelation*. This is a frequently used trick that is good from engineering standpoint but not well accepted from theoretical standpoint. Such a trick may obscure the understanding of the implication of ergodicity.

2. It is not easy to see why autocorrelation function is of significant importance to communication systems. In other words, why the knowledge of $R_X(\tau) = E[X(t+\tau)X^*(t)]$ is important cannot be seen from its definition. Thus, we use the scenario in Slide 1-61 as an example.

   (a) Suppose $\tau_1 = 16$ nanoseconds, $\tau_2 = 10$ natoseconds and $\tau_3 = 30$ nanoseconds. Let the input $X(t)$ be a WSS process. List all the time differences, corresponding to which the knowledge of $R_X(\tau) = E[X(t+\tau)X^*(t)]$ is necessary for the computation of the received power $E[|Y(t)|^2] = E[Y(t)Y^*(t)]$.

   (b) In (a), we see that $R_Y(0)$ is the weighted sum of $R_X(\tau)$ for many τ's. Since $R_X(\tau)$ is in general a complex number, how can we be certain that the received power $E[|Y(t)|^2] = E[Y(t)Y^*(t)]$ is real-valued?

Solution.

(a) The received power at time t is given by

$$\begin{aligned}
E[|Y(t)|^2] &= E[Y(t)Y^*(t)] \\
&= E\left[(\alpha_1 X(t-16) + \alpha_2 X(t-10) + \alpha_3(X(t-30))\right. \\
&\quad \left.(\alpha_1 X(t-16) + \alpha_2 X(t-10) + \alpha_3(X(t-30))^*\right] \\
&= E\left[\alpha_1\alpha_1^* X(t-16)X^*(t-16) + \alpha_1\alpha_2^* X(t-16)X^*(t-10)\right. \\
&\quad +\alpha_1\alpha_3^* X(t-16)X^*(t-30) + \alpha_2\alpha_1^* X(t-10)X^*(t-16) \\
&\quad +\alpha_2\alpha_2^* X(t-10)X^*(t-10) + \alpha_2\alpha_3^* X(t-10)X^*(t-30) \\
&\quad +\alpha_3\alpha_1^* X(t-30)X^*(t-16) + \alpha_3\alpha_2^* X(t-30)X^*(t-10) \\
&\quad \left.+\alpha_3\alpha_3^* X(t-30)X^*(t-30)\right] \\
&= \alpha_1\alpha_1^* R_X((t-16)-(t-16)) + \alpha_1\alpha_2^* R_X(t-16)-(t-10)) \\
&\quad +\alpha_1\alpha_3^* R_X((t-16)-(t-30)) + \alpha_2\alpha_1^* R_X((t-10)-(t-16)) \\
&\quad +\alpha_2\alpha_2^* R_X((t-10)-(t-10)) + \alpha_2\alpha_3^* R_X((t-10)-(t-30)) \\
&\quad +\alpha_3\alpha_1^* R_X((t-30)-(t-16)) + \alpha_3\alpha_2^* R_X((t-30)-(t-10)) \\
&\quad +\alpha_3\alpha_3^* R_X((t-30)X-(t-30)) \\
&= \alpha_1\alpha_1^* R_X(0) + \alpha_1\alpha_2^* R_X(-6) + \alpha_1\alpha_3^* R_X(14) + \alpha_2\alpha_1^* R_X(6) \\
&\quad +\alpha_2\alpha_2^* R_X(0) + \alpha_2\alpha_3^* R_X(20) + \alpha_3\alpha_1^* R_X(-14) + \alpha_3\alpha_2^* R_X(-20) \\
&\quad +\alpha_3\alpha_3^* R_X(0)
\end{aligned}$$

Hence, time differences, corresponding to which the knowledge of $R_X(\tau) = E[X(t+\tau)X^*(t)]$ is necessary for the computation of the received power $E[|Y(t)|^2] = E[Y(t)Y^*(t)]$, are -20, -14, -6, 0, 6, 14 and 20.

(b) Since $R_X(\tau)$ and $R_X(-\tau)$ will simultaneously appear in the weighted sum for the computation of $R_Y(0)$ and the sum of $R_X(\tau)$ and $R_X(-\tau) = R_X^*(\tau)$ (thus, this is an important property) is real-valued, we can be certain that $R_Y(0)$ must be real-valued.

Note: By this example, you can see that the real part of $R_X(\tau)$ is conceptually more important than the imaginary part as the knowledge of $\mathrm{Re}\{R_X(\tau)\}$ fully decides $R_Y(0)$.

3. (a) If we admit (by the Sifting property of the Dirac delta function) that the Fourier transform of the Dirac delta function $\delta(t)$ is 1, show that the Fourier transform of 1 is $\delta(f)$.

(b) Suppose $G(f)$ is continuous. Then, we obtain from the replication property that

$$\frac{1}{T}G\left(\frac{n}{T}\right)e^{j2\pi\frac{n}{T}t} = \int_{-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)\frac{1}{T}G\left(f\right)e^{j2\pi ft}\mathrm{d}f,$$

show that the (extended) Fourier transform of the periodic function

$$g_T(t) = \sum_{n=-\infty}^{\infty} g(t-nT)$$

is equal to

$$\frac{1}{T}G\left(f\right)\left(\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)\right).$$

Solution.

(a) Admitting

$$\int_{-\infty}^{\infty}\delta(t)e^{-j2\pi ft}\mathrm{d}t = \int_{-\infty}^{\infty}\delta(t-0)e^{-j2\pi ft}\mathrm{d}t = e^{j2\pi f\cdot 0} = 1,$$

we know from the Fourier transform pair that the inverse Fourier transform of 1 must be $\delta(t)$, which can be written as

$$\delta(t) = \int_{-\infty}^{\infty} 1\cdot e^{j2\pi ft}\mathrm{d}f. \tag{1}$$

In the above formula, t and f are just arguments and can be exchanged. Hence, we can rewrite (1) as

$$\delta(f) = \int_{-\infty}^{\infty} 1\cdot e^{j2\pi tf}\mathrm{d}t = \int_{-\infty}^{\infty} 1\cdot e^{-j2\pi(-f)t}\mathrm{d}t,$$

implying that the Fourier transform of function 1 is $\delta(-f)$. As $\delta(-f) = \delta(f)$, we conclude that the Fourier transform of 1 is $\delta(f)$. Note: Since constant function 1 is not absolutely integrable, its Fourier transform does not exist (as pointed out by Dirichlet). However, for convenience, engineers often admit this "fact" and use it widely in their derivations. Such an "admission" facilitates the general establishment of communications theory (as can be seen from the next subproblem). Due to the conflict of the two viewpoints, some books may say the Fourier transform of 1 does not exist but others may admit it. Both are correct in their own viewpoints. Now you shall know how tricky it is to introduce the Dirac delta function.

In order to unify the two viewpoints, a few researchers prefer to use *the extended Fourier transform* and say that the extended Fourier transform of $\delta(t)$ is 1. But still, some articles do not bother to add "extended" and simply use Fourier transform in their description.

(b) From

$$\begin{aligned} g_T(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T} G\left(\frac{n}{T}\right) e^{j2\pi \frac{n}{T} t} \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \frac{1}{T} G(f) e^{j2\pi f t} \mathrm{d}f \\ &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{T} G(f) \left(\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \right)}_{G_T(f)} e^{j2\pi f t} \mathrm{d}f \end{aligned}$$

we admit that the Fourier transform of $g_T(t)$ is

$$G_T(f) = \frac{1}{T} G(f) \left(\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \right).$$

Note: Dirichlet would say the Fourier transform of the periodic function $g_T(t)$ does not exist (as a period function is not absolutely integrable). So, he may prefer to see the statement that $G_T(f)$ is the *extended Fourier transform* of $g_T(t)$.

4. (a) Prove that for a stable LTI filter, a WSS input induces a WSS output.

(b) Prove that the convolution operation in time domain $\equiv$ the multiplication operation in frequency domain.

Solution.

(a) See Slide 1-63 and 1-65. Simply show that $\mu_Y(t)$ is a constant and $R_Y(t,u)$ is only a function of $(t-u)$.

Note: It is suggested to memorize the general input-output relation of mean function and autocorrelation function for a stable LTI filter in Slide 1-65, as well as the simplified input-output relation when assuming the input process is WSS.

(b) See Slide 2-18.

5. Please memorize the Fourier transform pair in Slide 2-2. You shall carefully read the three examples in Slides 1-28, 1-33 and 1-45. Variations of the three may appear in the midterm exam.