Sample Problems for Quiz 1 (2021 Spring)

1. (a) Prove that if a random process $X(t)$ is stationary, its autocorrelation function $R_X(t_1, t_2)$ is only a function of time difference $(t_1 - t_2)$.

   Hint: Here, we assume that the joint probability density function (pdf) $f_{X(t_1),X(t_2)}(x_1, x_2)$ exists.

   (b) Prove that autocorrelation function $R_X(t_1, t_2)$ equals autocovariance function $C_X(t_1, t_2)$ plus $\mu_X(t_1)\mu_X^*(t_2)$.

   Hint: Prove the equivalence: $C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2)$.

Solution.

(a)

$$
\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X^*(t_2)] \\
&\quad \text{(By definition of autocorrelation function)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^* f_{X(t_1),X(t_2)}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
&\quad \text{(By definition of expectation value)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^* f_{X(t_1-t_2),X(0)}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
&\quad \text{(Due to stationarity)} \\
&= E[X(t_1 - t_2)X^*(0)] \\
&= R_X(t_1 - t_2, 0) \\
\big(&= R_X(t_1 - t_2)\big)
\end{aligned}
$$

(b) See Slide 1-19.

2. (a) Give the conditions on the mean function $\mu_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$, which define a WSS process $X(t)$.

   (b) Give the conditions on the mean function $\mu_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$, which define the cyclostationary process $X(t)$.

Solution.

(a) $\mu_X(t)$ is a constant and $R_X(t_1, t_2) = R_X(t_1 - t_2)$ is only a function of time difference $(t_1 - t_2)$.

(b) $\mu_X(t+T) = \mu_X(t)$ and $R_X(t_1+T, t_2+T) = R_X(t_1, t_2)$ are both periodic function with the same period T.

Note: See Problem 5(a) for an example of cyclostationary process.

3. Name a reason why the following function cannot be the autocorrelation function $R_X(\tau)$ of some WSS process $X(t)$.

   Hint: The autocorrelation function $R_X(\tau)$ of a WSS process $X(t)$ must satisfy $R_X(0) = E[|X(t)|^2]$, $R_X(\tau) = R_X^*(-\tau)$ and $|\text{Re}\{R_X(\tau)\}| \leq R_X(0)$.

   (a) $g_1(\tau) = \cos(2\pi f_c \tau) + j\cos(2\pi f_c \tau)$

   (b) $g_2(\tau) = \tau^2$

   (c) $g_3(\tau) = 1 - |\tau - 1|$

   Solution.

   (a) $g_1(\tau)$ violates the property of conjugate symmetry: $g_1(\tau) \neq (g_1(-\tau))^*$.

   Note: It also violates the property that $g_1(0)$ must be a non-negative real number.

   (b) $g_2(\tau)$ violates the property that the real part peaks at zero: $|\text{Re}\{g_2(\tau)\}| = |g_2(\tau)| \not\leq g_2(0)$.

   (c) $g_3(\tau)$ violates the property of conjugate symmetry: $g_3(\tau) = (g_3(-\tau))^*$.

   Note: It also violates the property that the real part peaks at zero.

4. (a) Can the autocorrelation function at the origin $R_X(0)$ be a complex number with non-zero imaginary part? Justify your answer.

   (b) Prove that the real part of the autocorrelation function of a WSS random process $X(t)$ peaks at zero, i.e., $|\text{Re}\{R_X(\tau)\}| \leq R_X(0)$.

   Hint: Think of how to generate $R_X(\tau)$ and $R_X(0)$ simultaneously in an equation from $X(t+\tau)$ and $X(t)$.

   Solution.

   (a) The answer is No because by definition, $R_X(0) = E[X(t)X^*(t)] = E[|X(t)|^2]$ must be a (non-negative) real number.

(b)

$$
\begin{aligned}
0 &\le E\left[|X(t+\tau)+X(t)|^2\right] \\
&= E\left[(X(t+\tau)+X(t))\,(X(t+\tau)+X(t))^*\right] \\
&\qquad \text{(Since } |A|^2 = AA^* \text{ for a complex number } A\text{)} \\
&= E\left[(X(t+\tau)+X(t))\,(X^*(t+\tau)+X^*(t))\right] \\
&\qquad \text{(Since } (A+B)^* = A^*+B^* \text{ for two complex numbers } A \text{ and } B\text{)} \\
&= E[X(t+\tau)X^*(t+\tau)+X(t+\tau)X^*(t)+X(t)X^*(t+\tau)+X(t)X^*(t)] \\
&= E[X(t+\tau)X^*(t+\tau)]+E[X(t+\tau)X^*(t)]+E[X(t)X^*(t+\tau)]+E[X(t)X^*(t)] \\
&\qquad \text{(Since } E[Y+Z]=E[Y]+E[Z] \text{ for two random variables } Y \text{ and } Z\text{)} \\
&= R_X(0)+R_X(\tau)+R_X(-\tau)+R_X(0) \\
&\qquad \text{(By definition of autocorrelation function)} \\
&= R_X(0)+R_X(\tau)+R_X^*(\tau)+R_X(0) \\
&\qquad \text{(From } R_X(\tau)=R_X^*(-\tau)\text{)} \\
&= 2R_X(0)+2\mathrm{Re}\{R_X(\tau)\},
\end{aligned}
$$

which implies $\mathrm{Re}\{R_X(\tau)\} \ge -R_X(0)$. Similarly, we derive

$$
\begin{aligned}
0 &\le E\left[|X(t+\tau)-X(t)|^2\right] \\
&= E\left[(X(t+\tau)-X(t))\,(X(t+\tau)-X(t))^*\right] \\
&= E\left[(X(t+\tau)-X(t))\,(X^*(t+\tau)-X^*(t))\right] \\
&= E[X(t+\tau)X^*(t+\tau)-X(t+\tau)X^*(t)-X(t)X^*(t+\tau)+X(t)X^*(t)] \\
&= E[X(t+\tau)X^*(t+\tau)]-E[X(t+\tau)X^*(t)]-E[X(t)X^*(t+\tau)]+E[X(t)X^*(t)] \\
&= R_X(0)-R_X(\tau)-R_X(-\tau)+R_X(0) \\
&= R_X(0)-R_X(\tau)-R_X^*(\tau)+R_X(0) \\
&= 2R_X(0)-2\mathrm{Re}\{R_X(\tau)\},
\end{aligned}
$$

which implies $\mathrm{Re}\{R_X(\tau)\} \le R_X(0)$. The above two inequalities jointly imply $|\mathrm{Re}\{R_X(\tau)\}| \le R_X(0)$.

Note: From the Hint, one way to generate $R_X(\tau)$ and $R_X(0)$ simultaneously in an equation from $X(t+\tau)$ and $X(t)$ is to examine

$$E[(X(t+\tau)+X(t))(X(t+\tau)+X(t))^*]$$

and

$$E[(X(t+\tau)-X(t))(X(t+\tau)-X(t))^*].$$

5. (a) Give $X(t)=\cos(2\pi f_c t+\Theta)$, where Θ is uniformly distributed over $[-\frac{\pi}{2},\frac{\pi}{2})$. Find the mean function $\mu_X(t)$ and the autocorrelation function $R_X(t_1,t_2)$. Is $X(t)$ WSS? Is $X(t)$ cyclostationary?

(b) Give $X(t) = \cos(2\pi f_c t + \Theta) + j\sin(2\pi f_c t + \Theta)$, where is uniformly distributed over $[-\pi, \pi)$. Find the mean function $\mu_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$. Is $X(t)$ WSS? Is $X(t)$ cyclostationary?

Hint: It is sometimes convenient to denote

$$\cos(2\pi f_c t + \Theta) + j\sin(2\pi f_c t + \Theta) = e^{j(2\pi f_c t + \Theta)}.$$

Solution.

(a) For wide-sense stationarity, we need to check whether the mean function $\mu_X(t)$ is a constant, (functionally) independent of t, and whether the autocorrelation function $R_X(t_1, t_2)$ is only a function of time difference $(t_1 - t_2)$. So we derive

$$\begin{aligned}
\mu_X(t) &= E[\cos(2\pi f_c t + \Theta)] \\
&= \int_{-\pi/2}^{\pi/2} \cos(2\pi f_c t + \theta)\frac{1}{\pi}d\theta \\
&= \frac{1}{\pi}\sin(2\pi f_c t + \theta)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{1}{\pi}\sin\left(2\pi f_c t + \frac{\pi}{2}\right) - \frac{1}{\pi}\sin\left(2\pi f_c t - \frac{\pi}{2}\right) \\
&= \frac{1}{\pi}\cos\left(2\pi f_c t\right) + \frac{1}{\pi}\cos\left(2\pi f_c t\right) \\
&= \frac{2}{\pi}\cos(2\pi f_c t).
\end{aligned}$$

and

$$\begin{aligned}
R_X(t_1, t_2) &= E[\cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)] \\
&= \int_{-\pi/2}^{\pi/2} \cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)\frac{1}{\pi}d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{2\pi}\left[\cos(2\pi f_c(t_1 + t_2) + 2\theta) + \cos(2\pi f_c(t_1 - t_2))\right] d\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{4\pi}\left[\cos(2\pi f_c(t_1 + t_2) + \theta') + \cos(2\pi f_c(t_1 - t_2))\right] d\theta' \\
&= \frac{1}{2}\cos(2\pi f_c(t_1 - t_2)).
\end{aligned}$$

Consequently, $X(t)$ is not WSS because $\mu_X(t)$ is not a constant. However, $X(t)$ is cyclostationary since $\mu_X(t + T) = \mu_X(t)$ and $R_X(t_1 + T, t_2 + T) = R_X(t_1, t_2)$ with $T = \frac{1}{f_c}$.

Note: With a better calibration mechanism, one may limit the phase difference between the transmitter and the receiver to be within $[-\frac{\pi}{2}, \frac{\pi}{2})$; however, the received signal $X(t) = \cos(2\pi f_c t + \Theta)$ is no longer WSS but cyclostationary.

(b) Again, for wide-sense stationarity, we need to check whether the mean function $\mu_X(t)$ is a constant, (functionally) independent of t, and whether the autocorrelation function $R_X(t_1, t_2)$ is only a function of time difference $(t_1 - t_2)$. So we derive

$$\begin{aligned}
\mu_X(t) &= E[\cos(2\pi f_c t + \Theta) + j\sin(2\pi f_c t + \Theta)] \\
&= E[\cos(2\pi f_c t + \Theta)] + jE[\sin(2\pi f_c t + \Theta)] \\
&= \int_{-\pi}^{\pi} \cos(2\pi f_c t + \theta)\frac{1}{2\pi}d\theta + j\int_{-\pi}^{\pi} \sin(2\pi f_c t + \theta)\frac{1}{2\pi}d\theta \\
&= \frac{1}{2\pi}\sin(2\pi f_c t + \theta)\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\cos(2\pi f_c t + \theta)\Big|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
R_X(t_1, t_2) &= E[e^{j(2\pi f_c t_1 + \Theta)}(e^{j(2\pi f_c t_2 + \Theta)})^*] \\
&= E[e^{j(2\pi f_c t_1 + \Theta)}e^{-j(2\pi f_c t_2 + \Theta)}] \\
&= E[e^{j2\pi f_c (t_1 - t_2)}] \\
&= e^{j2\pi f_c (t_1 - t_2)}
\end{aligned}$$

Since $\mu_X(t)$ is a constant, and $R_X(t_1, t_2)$ is only a function of $(t_1 - t_2)$, $X(t)$ is a WSS process.

As a WSS process must be a cyclostationary process (with arbitrary period T), $X(t)$ is also a cyclostationary process.