Introduction to Communication Systems: Notes for Midterm

Important note: For your information, there will be six problems in the midterm. One (15%) will be on the spectrum drawing (e.g., $G(f)$, $G_+(f)$, $\tilde{G}(f)$, $\hat{G}(f)$, $\bar{S}_X(f)$, $\bar{S}_{\tilde{X}}(f)$, etc). Four (70%) will be from the sample problems. One (15%) will be from the slides (Up to 5-39).

Corrections

- Sample Problem 2(b) for Quiz 6: The last line of the solution should write, "$m'(t) = L_Q(f_m)\sin(2\pi f_m t) = \begin{cases} -\frac{f_m}{f_v}\sin(2\pi f_m t), & f_m \le f_v \\ -\sin(2\pi f_m t), & f_m > f_v \end{cases}$."
- Sample Problem 3(c) for Quiz 6: The Hilbert transform of a constant is zero. Hence, $\mathcal{H}\{1+\cos(2\pi f_0 t)+\cos^2(2\pi f_0 t)+\cos^3(2\pi f_0 t)\} =$ ~~$\frac{3}{2}+$~~ $\frac{7}{4}\sin(2\pi f_0 t)+\frac{1}{2}\sin(2\pi(2f_0)t)+\frac{1}{4}\sin(2\pi(3f_0)t)$.
- Slide 4-75: It should be "$J_n(\beta) \approx 0$ for $n \ge 2$."
- Slide 5-11: Remove "and coherent detection (observed at $x(t)$)"
- Slide 5-12: Please remove the left-most "[" in "$[E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)]$."
- Slide 5-16: The third "=" should be "≈." In other words,

$$\begin{aligned} y(t) &= \sqrt{(x^2(t))_{\text{LowPass}}} \\ &= \frac{1}{\sqrt{2}}\sqrt{[A_c(1+k_a m(t))+n_I(t)]^2+n_Q^2(t)} \\ &\approx \frac{1}{\sqrt{2}}[A_c(1+k_a m(t))+n_I(t)] \quad \text{if } A_c[1+k_a m(t)] \gg |\tilde{n}(t)| \end{aligned}$$

- Slide 5-21: It should be "$A_c[1+k_a m(t)] \gg |\tilde{n}(t)|$."
- Slide 5-26: In two places, please change to $\sigma_N^2 = E[|\tilde{n}(t)|^2]$
- Slide 5-40: "$\phi(t) - \phi(t)$" should be "$\psi(t) - \phi(t)$."

In what follows, the items leading with a ⋆ are more important, and the items leading with two ⋆ (i.e., ⋆⋆) are the most important.

Part I

- 1-13 ∼ 1-14 : Definition of stationarity and why we introduce stationarity

⋆ 1-16 : Definition of autocorrelation function

⋆ 1-17: Prove that the autocorrelation function becomes a function of time difference under stationarity.

⋆ 1-19: Definition of autocovariance (given in the first line) and its relation with autocorrelation and mean functions (established at the bottom)

Prove that the autocorrelation function of a random process is equal to the sum of its autocovariance function and the product of its mean functions at the two time instances, i.e., $R_X(t_1,t_2) = C_X(t_1,t_2) + \mu_X(t_1)\mu_X^*(t_2)$.

- ⋆⋆ 1-21: Definition of WSS (two conditions)
- 1-23: Definition of cyclostationarity
- ⋆⋆ 1-24 $\sim$ 1-25: Three important properties of WSS processes
- ⋆ 1-28 $\sim$ 1-42: Derivation of mean functions and autocorrelation functions (verification of WSS) for specific examples
- 1-43 and 1-44: Definition of cross-correlation function and correlation matrix
- 1-45 $\sim$ 1-48: Derivation of cross-correlation function
- ⋆⋆ 1-60 $\sim$ 1-62 : Definition and example of a stable LTI system
- ⋆ 1-63 and 1-65: Input-output relation of mean function and autocorrelation function for a stable LTI system
- ⋆⋆ 1-66: Prove that in a stable LTI system, WSS input induces WSS output.

<u>Part II</u>

- 2-2: Formula of Fourier transform pair
- 2-3: Sufficient condition for the existence of Fourier transform (and inverse Fourier transform)
- 2-5 to 2-8: Definition of Dirac delta function and its properties
- 2-11: Fourier series
- ⋆ 2-12 $\sim$ 2-14: The proof of Poisson sum formula, i.e., $g_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G\left(\frac{n}{T}\right) e^{j2\pi\frac{n}{T}t}$, where $g(t) = \begin{cases} g_T(t), & |t| < \frac{T}{2}; \\ 0, & \text{otherwise} \end{cases}$ is the generating function of a periodic function $g_T(t)$ with period T, and $G(f)$ is the (Fourier) spectrum of $g(t)$.

Proof: First, we note that

$$g_T(t) = \sum_{m=-\infty}^{\infty} g(t - mT).$$

Then, by Fourier series,

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} g_T(t)\exp\left(-j2\pi\frac{n}{T}t\right)\mathrm{d}t \\ &= \frac{1}{T}\int_{-T/2}^{T/2} g(t)\exp\left(-j2\pi\frac{n}{T}t\right)\mathrm{d}t \\ &= \frac{1}{T}G\left(\frac{n}{T}\right) \end{aligned}$$

and

$$g_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi\frac{n}{T}t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} G\left(\frac{n}{T}\right) e^{j2\pi\frac{n}{T}t}.$$

- ⋆ 2-19: Prove time-convolution ⇔ spectrum-multiplicaiton.
- • 2-26 and 2-29: Definitions of time-average autocorrelation and PSD, in particular

$$\bar{S}_X(f) = \lim_{T\to\infty} \frac{1}{2T} E[X(f)X_{2T}^*(f)]$$

- ⋆⋆ 2-31 and 2-32: Input-output relation of (time-average) autocorrelation functions and (time-average) PSDs for a stable LTI filter
- ⋆⋆ 2-38 to 2-43: Properties of PSD
- ⋆ 2-44 ∼ 2-49 and 2-58 ∼ 2-59: Derivations of PSDs for specific examples
- ⋆ 2-67 to 2-68: Definition and impracticability of white noise
- • 2-79 and 2-81 : Definitions of null-to-null and rms bandwidth.
- • 2-83 and 2-84 : Definition of noise equivalent bandwidth
- • 2-86: Notion of time-bandwidth product

Part III

- ⋆⋆ 3-5: Impulse response and transfer function of Hilbert transform
- ⋆ 3-8 ∼ 3-9: Prove that the Hilbert transform pairs are orthogonal to each other, where the inner product is defined as $\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)g^*(t)dt$.
- ⋆ 3-12 to 3-13: Prove $g(t) = \mathrm{Re}\{g_+(t)\} = \mathrm{Re}\{\tilde{g}(t)\exp(j2\pi f_c t)\} = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$ based on their spectrum relations
- • 3-15 to 3-16: Canonical transmitter and receiver
- • 3-22: Show that $Y(f) = H(f)X(f)$ implies $\tilde{Y}(f) = \frac{1}{2}\tilde{H}(f)\tilde{X}(f)$, where we define $H_+(f) = 2u(f)H(f)$ and $\tilde{H}(f) = H_+(f + f_c)$.
- ⋆⋆ 3-26 to 3-29: Proofs of Properties 1 and 2
- ⋆⋆ 3-33: Proof of Property 7
- ⋆⋆ 3-34: Proof of $R_{N_I,N_Q}(0) = E[N_I(t)N_Q^*(t)] = 0$, i.e., $N_I(t)$ and $N_Q(t)$ are orthogonal. Note that this is true even if $N_I(t)$ and $N_Q(t)$ do not have zero means.
- • 3-35 and 3-38: Good to know the two properties
- • 3-46 and 3-50: Reyleigh distribution and Rician distribution
- • 3-53 to 3-55: Properties 6 and 6′ of Bessel functions

Part IV

- ⋆⋆ 4-5: DSB-C and overmodulation
- ⋆⋆ 4-16 ∼ 4-18: Envelop detector

- 4-21: Linear modulation

⋆⋆ 4-22 and 4-23: Three kinds of linear modulations, which are DSB-C, SSB and VSB. In particular, you should know how to derive and draw their spectrums.

⋆⋆ 4-25 and 4-26: Coherent receiver (Quadrature null effect of the coherent detector in 4-27)

- 4-28: Costas receiver

⋆⋆ 4-31: SSB (energy gap requirement)

⋆⋆ 4-35 and 4-37: VSB $M_{\text{VSB}}(f) = \frac{1}{2}M(f) - j\frac{1}{2}H_Q(f)M(f)$

- 4-39: Recover $M(f)$ from $M_{\text{VSB}}(f)$.

⋆⋆ 4-46: VSB $M_{\text{VSB}}(f) = \frac{1}{2}M(f) + j\frac{1}{2}M'(f)$, where $M'(f) = -H_Q(f)M(f)$

- 4-53: Frequency translation

- 4-56: Angle modulation (PM and FM)

⋆⋆ 4-60: Instantaneous frequency, frequency deviation Δf and modulation index β of a single-tone FM

⋆⋆ 4-66: Spectrum of Single-Tone FM and its relation with Bessel function (4-67 ∼ 4-70)

⋆ 4-79: Bandwidth of a General FM Wave based on Carson's rule

⋆ 4-80 ∼ 4-81 : Bandwidth of a General FM Wave based on universal curve

⋆⋆ 4-86 ∼ 4-93: Balanced frequency discriminator

- 4-95 ∼ 4-96: FM Stereo Multiplexing

- 4-102 ∼ 4-104: IF session and image interference

Part V

⋆⋆ 5-4 ∼ 5-6: Definitions of SNR_O, SNR_C and figure of merit

⋆⋆ 5-8 ∼ 5-13: Computations of SNR_O, SNR_C and figure of merit for DSB-C and coherent detector

⋆⋆ 5-14 ∼ 5-19: Computations of SNR_O, SNR_C and figure of merit for DSB-SC and envelop detector

⋆ 5-20 ∼ 5-23: Threshold effect of DSB-C and envelop detector (discussions in 5-34 and 5-35 are included)

- 5-24 ∼ 5-33: Outside the scope of the midterm

⋆⋆ 5-36 ∼ 5-39: Derivation of the equivalent additive noise of FM and balanced frequency discriminator

1. For a single-tone FM transmission with $m(t) = A_m \cos(2\pi f_m t)$, we have $\Delta f = \beta f_m = k_f A_m$, where Δf is the frequency deviation, β is the modulation index, and k_f is the frequency sensitivity.

   (a) Suppose $f_m = 15$ KHz. Give Δf if $\beta = 1.0$, 2.0 and 5.0.

   (b) For the computation of transmission bandwidth, Carson's rule gives $B_{T,\text{Carson}} = 2\Delta f(1 + \frac{1}{\beta})$ and the universal curve transmission bandwidth $B_{T,\text{UC}}$ follows the below table:

β	$B_{T,\text{UC}}/\Delta f$
0.1	20.0
0.3	13.3
0.5	8.0
1.0	6.0
2.0	4.0
5.0	3.2
10.0	2.8
20.0	2.5
30.0	2.3

   Is $B_{T,\text{UC}} \geq B_{T,\text{Carson}}$ always true for a given Δf?

Solution.

(a) $\Delta f = 15, 30, 75$ KHz respectively for $\beta = 1.0$, 2.0 and 5.0

(b) From $\frac{B_{T,\text{Carson}}}{\Delta f} = 2(1 + \frac{1}{\beta})$, we obtain

β	$B_{T,\text{UC}}/\Delta f$	$B_{T,\text{Carson}}/\Delta f$	$2/\beta$
0.1	20.0	22.0	20.00
0.3	13.3	8.67	6.67
0.5	8.0	6.00	4.00
1.0	6.0	4.00	2.00
2.0	4.0	3.00	1.00
5.0	3.2	2.40	0.40
10.0	2.8	2.20	0.20
20.0	2.5	2.10	0.10
30.0	2.3	2.07	0.067

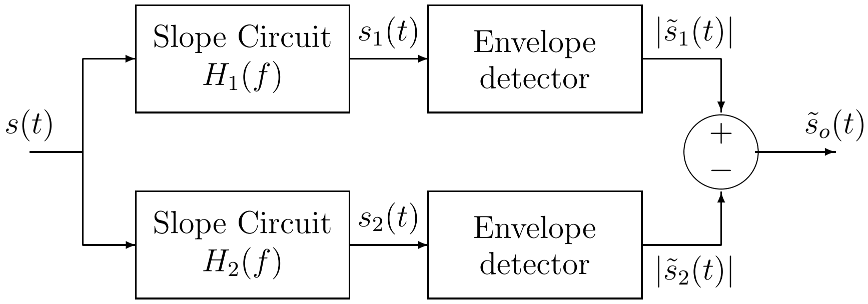
Therefore, when β is very small, we actually have $B_{T,\text{UC}} < B_{T,\text{Carson}}$.

2. The balanced frequency discriminator for an FM modulated signal $s(t) = \cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau)$ is given below, where

$$H_1(f) = \begin{cases} j2\pi\left(f - f_c + \frac{B_T}{2}\right), & |f - f_c| \leq \frac{B_T}{2} \\ j2\pi\left(f + f_c - \frac{B_T}{2}\right), & |f + f_c| \leq \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

and

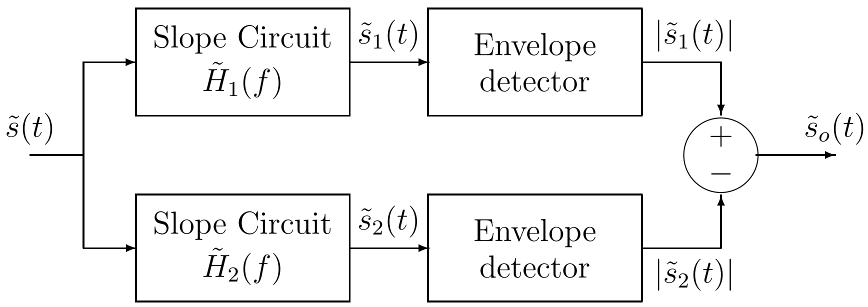
$$H_2(f) = \begin{cases} -j2\pi\left(f + f_c + \frac{B_T}{2}\right), & |f + f_c| \le \frac{B_T}{2} \\ -j2\pi\left(f - f_c - \frac{B_T}{2}\right), & |f - f_c| \le \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$



(a) The above block diagram can be transformed to its lowpass equivalent diagram with

$$\begin{cases} \tilde{s}(t) = \exp\left(j2\pi k_f \int_0^t m(\tau)d\tau\right) \\ \tilde{s}_1(t) = j\pi B_T \tilde{s}(t) + \dfrac{d\tilde{s}(t)}{dt} \\ \tilde{s}_2(t) = j\pi B_T \tilde{s}(t) - \dfrac{d\tilde{s}(t)}{dt} \end{cases}$$

as:



Show that

$$\tilde{s}_1(t) = j\pi B_T \left(1 + \frac{2k_f}{B_T} m(t)\right) \tilde{s}(t)$$

and

$$\tilde{s}_2(t) = j\pi B_T \left(1 - \frac{2k_f}{B_T} m(t)\right) \tilde{s}(t).$$

Note: We set $a = 1$ in the formula of $H_1(f)$ and $H_2(f)$ for analytical convenience.

(b) Further, represent $|\tilde{s}_1(t)|$ and $|\tilde{s}_2(t)|$ as a function of $m(t)$.

(c) Find the sufficient and necessary condition for the validity of $\tilde{s}_o(t) = 4\pi k_f m(t)$.

Solution.

(a) By noting that

$$\frac{d\tilde{s}(t)}{dt} = \frac{d\left(j2\pi k_f \int_0^t m(\tau)d\tau\right)}{dt} \cdot e^{j2\pi k_f \int_0^t m(\tau)d\tau} = j2\pi k_f m(t)\tilde{s}(t),$$

we derive

$$\begin{aligned}\tilde{s}_1(t) &= j\pi B_T \tilde{s}(t) + \frac{d\tilde{s}(t)}{dt} \\ &= j\pi B_T \tilde{s}(t) + j2\pi k_f m(t)\tilde{s}(t) \\ &= j\pi B_T \left(1 + \frac{2k_f}{B_T} m(t)\right) \tilde{s}(t)\end{aligned}$$

and

$$\begin{aligned}\tilde{s}_2(t) &= j\pi B_T \tilde{s}(t) - \frac{d\tilde{s}(t)}{dt} \\ &= j\pi B_T \tilde{s}(t) - j2\pi k_f m(t)\tilde{s}(t) \\ &= j\pi B_T \left(1 - \frac{2k_f}{B_T} m(t)\right) \tilde{s}(t).\end{aligned}$$

(b)

$$\begin{aligned}|\tilde{s}_1(t)| &= \left| j\pi B_T \left(1 + \frac{2k_f}{B_T} m(t)\right) \tilde{s}(t)\right| \\ &= |j\pi B_T| \left|1 + \frac{2k_f}{B_T} m(t)\right| |\tilde{s}(t)| \\ &= \pi B_T \left|1 + \frac{2k_f}{B_T} m(t)\right|\end{aligned}$$

and similarly

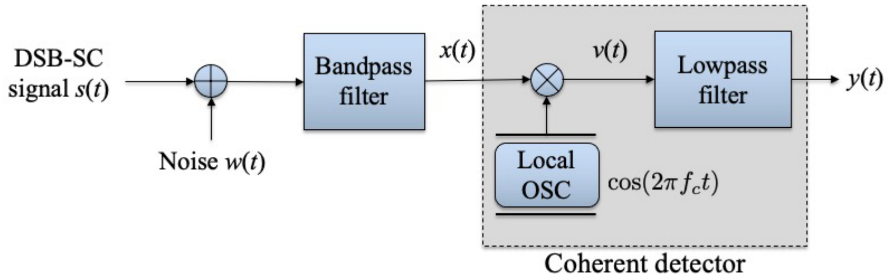
$$|\tilde{s}_2(t)| = \pi B_T \left|1 - \frac{2k_f}{B_T} m(t)\right|.$$

(c) $\left|\frac{2k_f}{B_T} m(t)\right| \le 1$, i.e., $|m(t)| \le \frac{B_T}{2k_f}$.

3. (a) Show that the SNR_C of the DSB-SC transmission (See the figure below) is equal to

$$\text{SNR}_\text{C} = \frac{P}{2WN_0},$$

where $s(t) = m(t)\cos(2\pi f_c t)$, $m(t)$ is a zero-mean WSS process with $E[m^2(t)] = P$ and with bandwidth W, and $w(t)$ is an additive white Gaussian noise with two-sided PSD $N_0/2$.

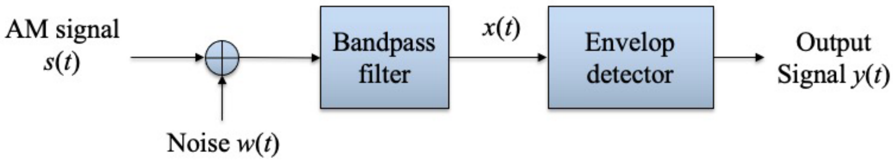


(b) Continue from (a). Find the SNR_O of the DSB-SC transmission.
Hint: You need to find the expression of the final output $y(t)$. Then, SNR_O is the average power of the *signal content* in $y(t)$ divided by the average power of the *noise content* in $y(t)$.

(c) Show that the SNR_C of the DSB-C transmission (See the figure below) is equal to

$$\text{SNR}_\text{C} = \frac{1 + k_a^2 P}{2WN_0},$$

where $s(t) = [1+k_a m(t)]\cos(2\pi f_c t)$, $m(t)$ is a zero-mean WSS process with $E[m^2(t)] = P$ and with bandwidth W, and $w(t)$ is an additive white Gaussian noise with two-sided PSD $N_0/2$.



(d) Show that under fixed N_0, the SNR_C of DSB-C is bounded due to the constraint $|k_a m(t)| \leq 1$, while the SNR_C of DSB-SC grows linearly with P.
Note: Hence, from the standpoint of "figure of merit," the reference target (i.e., SNR_C) of DSB-C is actually "worse" than the reference target of DSB-SC as P is moderately large.

Solution.

(a) For the computation of SNR_C, we first obtain the **averaged signal power** (in the time-averaged form as $s(t)$ is not necessarily WSS):

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[s^2(t)]dt &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[m^2(t)\cos^2(2\pi f_c t)]dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[m^2(t)]\cos^2(2\pi f_c t)dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} P\cos^2(2\pi f_c t)dt \\
&\quad \text{Because } m(t) \text{ is WSS and } E[m^2(t)] = P \\
&= \lim_{T\to\infty} \frac{P}{4T}\int_{-T}^{T} [1+\cos(4\pi f_c t)]dt \\
&= \frac{P}{2},
\end{aligned}$$

where

$$\frac{1}{2T}\int_{-T}^{T}\cos(4\pi f_c t)dt = \frac{1}{8\pi f_c T}\sin(4\pi f_c t)\Big|_{-T}^{T} = \frac{\sin(4\pi f_c T)}{4\pi f_c T} \to 0 \text{ as } T \to \infty.$$

The **noise power in the message bandwidth** is given by

$$\int_{-W}^{W} S_w(f)df = \int_{-W}^{W}\frac{N_0}{2}df = WN_0.$$

Therefore,

$$\mathrm{SNR_C} = \frac{P/2}{WN_0} = \frac{P}{2WN_0}.$$

(b) The functional blocks give the following equations:

$$\begin{cases} s(t) = m(t)\cos(2\pi f_c t) \\ x(t) = s(t) + n(t) \\ \quad \text{where } n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \text{ is the filtered white noise} \\ v(t) = x(t)\cos(2\pi f_c t) \\ y(t) = v(t) \text{ passing through an ideal lowpass filter} \end{cases}$$

$$\Rightarrow \begin{cases} v(t) = [m(t)\cos(2\pi f_c t) + n(t)]\cos(2\pi f_c t) \\ y(t) = v(t) \text{ passing through an ideal lowpass filter} \end{cases}$$

$$\Rightarrow y(t) = \left[m(t)\cos^2(2\pi f_c t)\right]_{\text{Lowpass}} + \left[n_I(t)\cos^2(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t)\right]_{\text{Lowpass}}$$

$$\Rightarrow \boxed{y(t) = \tfrac{1}{2}m(t) + \tfrac{1}{2}n_I(t)}$$

$$\Rightarrow \mathrm{SNR_{O,DSB\text{-}SC}} = \frac{E[A_c^2 m^2(t)/4]}{E[n_I^2(t)/4]} = \frac{A_c^2 P}{E[n^2(t)]} = \frac{A_c^2 P}{2WN_0}$$

(c)

$$\begin{aligned} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[s^2(t)]dt &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[(1+k_a m(t))^2\cos^2(2\pi f_c t)]dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[(1+k_a m(t))^2]\cos^2(2\pi f_c t)dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(1+k_a^2 P)\cos^2(2\pi f_c t)dt \\ &\quad \text{Because } m(t) \text{ is zero-mean WSS and } E[m^2(t)] = P \\ &= \lim_{T\to\infty}\frac{(1+k_a^2 P)}{4T}\int_{-T}^{T}[1+\cos(4\pi f_c t)]dt \\ &= \frac{(1+k_a^2 P)}{2}, \end{aligned}$$

where

$$\frac{1}{2T}\int_{-T}^{T}\cos(4\pi f_c t)dt = \frac{1}{8\pi f_c T}\sin(4\pi f_c t)\Big|_{-T}^{T} = \frac{\sin(4\pi f_c T)}{4\pi f_c T} \to 0 \text{ as } T \to \infty.$$

The **noise power in the message bandwidth** is given by

$$\int_{-W}^{W} S_w(f)df = \int_{-W}^{W} \frac{N_0}{2} df = WN_0.$$

Therefore,

$$\text{SNR}_\text{C} = \frac{(1+k_a^2 P)/2}{WN_0} = \frac{1+k_a^2 P}{2WN_0}.$$

(d) $|k_a m(t)| \leq 1$ implies $k_a^2 m^2(t) \leq 1$, which in turn implies $E[k_a^2 m^2(t)] = k_a^2 P \leq 1$. Therefore,

$$\text{SNR}_\text{C,DSB-C} = \frac{1+k_a^2 P}{2WN_0} \leq \frac{1}{WN_0}.$$

This is in stark contrast to

$$\text{SNR}_\text{C, DSB-SC} = \frac{P}{2WN_0}$$

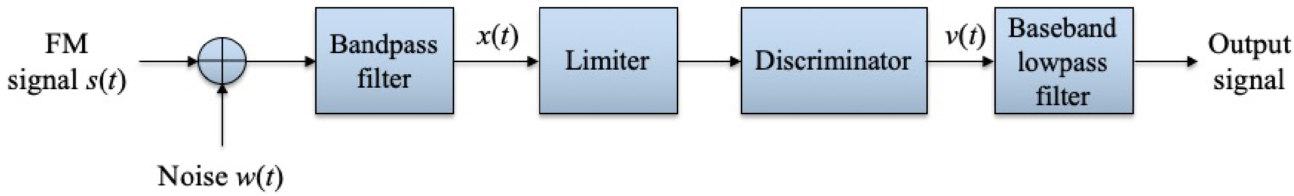
that grows linearly and unbounded with P.

Note: $\text{SNR}_\text{C, DSB-C}$ cannot be improved by growing P, not to mention its figure of merit is often much less than 1.

4. For an FM receiver below, we have $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$. Denote the passband noise process (i.e., the noise process observed at the output of the bandpass filter) as

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) = r(t)\cos(2\pi f_c t + \Psi(t)),$$

where $n_I(t) = r(t)\cos[\Psi(t)]$ and $n_Q(t) = r(t)\sin[\Psi(t)]$.



Due to the limiter, the amplitude of $x(t)$ is of no influence on its output, and only the phase remains. Hence, we can assume the amplitude at the output of the limiter is 1 for simplicity. Show that the output of the limiter $x_\text{limiter}(t)$ is given by

$$x_\text{limiter}(t) = \cos[2\pi f_c t + \theta(t)]$$

where

$$\theta(t) = \phi(t) + \tan^{-1}\left(\frac{r(t)\sin[\Psi(t) - \phi(t)]}{A_c + r(t)\cos[\Psi(t) - \phi(t)]}\right).$$

Hint: For simplicity during derivation, you can denote $\mathtt{A} = 2\pi f_c t + \phi(t)$ and $\mathtt{B} = \Psi(t) - \phi(t)$ and complete the following derivation:

$$
\begin{aligned}
x(t) &= A_c \cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \Psi(t)] \\
&= A_c \cos[\underbrace{2\pi f_c t + \phi(t)}_{=\mathtt{A}}] + r(t)\cos[\underbrace{2\pi f_c t + \phi(t)}_{=\mathtt{A}} + \underbrace{\Psi(t) - \phi(t)}_{=\mathtt{B}}] \\
&= A_c \cos(\mathtt{A}) + r(t)\cos(\mathtt{A}+\mathtt{B}) \\
&= \cdots
\end{aligned}
$$

Solution. Denoting $\mathtt{A} = 2\pi f_c t + \phi(t)$ and $\mathtt{B} = \Psi(t) - \phi(t)$, we obtain

$$
\begin{aligned}
x(t) &= A_c \cos(\mathtt{A}) + r(t)\cos(\mathtt{A}+\mathtt{B}) \\
&= A_c \cos(\mathtt{A}) + r(t)\cos(\mathtt{A})\cos(\mathtt{B}) - r(t)\sin(\mathtt{A})\sin(\mathtt{B}) \\
&= [A_c + r(t)\cos(\mathtt{B})]\cos(\mathtt{A}) - r(t)\sin(\mathtt{B})\sin(\mathtt{A}) \\
&= \sqrt{[A_c + r(t)\cos(\mathtt{B})]^2 + [r(t)\sin(\mathtt{B})]^2} \\
&\quad \times \left(\frac{[A_c + r(t)\cos(\mathtt{B})]}{\sqrt{[A_c + r(t)\cos(B)]^2 + [r(t)\sin(B)]^2}} \cos(\mathtt{A}) \right. \\
&\qquad \left. - \frac{r(t)\sin(\mathtt{B})}{\sqrt{[A_c + r(t)\cos(\mathtt{B})]^2 + [r(t)\sin(\mathtt{B})]^2}} \sin(\mathtt{A}) \right) \\
&= \sqrt{[A_c + r(t)\cos(\mathtt{B})]^2 + [r(t)\sin(\mathtt{B})]^2}\,(\cos(\phi_{\text{noise}}(t))\cos(\mathtt{A}) - \sin(\phi_{\text{noise}}(t))\sin(\mathtt{A})) \\
&\overset{\text{limiter}}{\to} \cos(\mathtt{A} + \phi_{\text{noise}}(t)),
\end{aligned}
$$

where

$$
\phi_{\text{noise}}(t) = \tan^{-1}\left(\frac{r(t)\sin(B)}{A_c + r(t)\cos(B)} \right).
$$