Additional Sample Problems for the Final Exam

Corrections

- Solutions for Sample Problem 4(c) for Quiz 10:

  Derive

$$
\begin{aligned}
G(f) &= \Delta \cdot \mathcal{F}\{\mathbf{1}\{|t| \le 0.25\}\}\big(e^{j2\pi f(-0.25)} - e^{j2\pi f(0.25)}\big) \\
&= \Delta \cdot 0.5\,\text{sinc}(0.5f)\,\big(-j2\sin(0.5\pi f)\big) \\
&= -j\Delta\,\text{sinc}(0.5f)\,\sin(0.5\pi f)
\end{aligned}
$$

- Solutions for Sample Problem 4(a) for Quiz 11:

$$
h_{\text{ISI}}(iT_b) = \sum_{k=0}^{N-1} w_k a_{i-k} \left(= w_0 a_i + \underbrace{\sum_{k=1}^{N-1} w_k a_{i-k}}_{\text{ISI term}} \right)
$$

1. (Demonstration of how a pre-coder can be designed.) Design a pre-coder for the class III correlative level coding.

   Hint: For the class III correlative level coding, we have $c_k = 2a_k + a_{k-1} - a_{k-2}$.

   Solution. Using a similar derivation to Slides 8-64 and 8-72, we obtain

$$
\begin{aligned}
c_k &= 2a_k + a_{k-1} - a_{k-2} \\
&= 2(2\tilde{b}_k - 1) + (2\tilde{b}_{k-1} - 1) - (2\tilde{b}_{k-2} - 1) \\
&= 2(2\tilde{b}_k + \tilde{b}_{k-1} - \tilde{b}_{k-2}) - 2.
\end{aligned}
$$

   From the above equation, we need to represent b_k as a function of $\tilde{b}_k$, $\tilde{b}_{k-1}$ and $\tilde{b}_{k-2}$. Let $b_k = \alpha_0\tilde{b}_k \oplus \alpha_1\tilde{b}_{k-1} \oplus \alpha_2\tilde{b}_{k-2}$ with $\alpha_0, \alpha_1, \alpha_2 \in \{0, 1\}$. Then,

$\tilde{b}_k$	$\tilde{b}_{k-1}$	$\tilde{b}_{k-2}$	b_k	c_k
0	0	0	0	−2
0	0	1	α_2	−4
0	1	0	α_1	0
0	1	1	$\alpha_1 \oplus \alpha_2$	−2
1	0	0	α_0	2
1	0	1	$\alpha_0 \oplus \alpha_2$	0
1	1	0	$\alpha_0 \oplus \alpha_1$	4
1	1	1	$\alpha_0 \oplus \alpha_1 \oplus \alpha_2$	2

   From $c_k = -2$, we know the first row and the fourth row should give the same b_k-value, i.e., $\alpha_1 \oplus \alpha_2 = 0$; from $c_k = 0$, it requires $\alpha_1 = \alpha_0 \oplus \alpha_2$; and from $c_k = 2$, we must have $\alpha_0 = \alpha_0 \oplus \alpha_1 \oplus \alpha_2$. These summarize to:

$$
\begin{cases} \alpha_1 \oplus \alpha_2 = 0; \\ \alpha_1 = \alpha_0 \oplus \alpha_2; \\ \alpha_0 = \alpha_0 \oplus \alpha_1 \oplus \alpha_2 \end{cases}
\Rightarrow
\begin{cases} \alpha_1 \oplus \alpha_2 = 0; \\ \alpha_1 = \alpha_0 \oplus \alpha_2 \end{cases}
\Rightarrow (\alpha_0, \alpha_1, \alpha_2) = \begin{cases} (0,0,0) \\ (0,1,1) \end{cases}
$$

The solution of $(\alpha_0, \alpha_1, \alpha_2) = (0, 0, 0)$ must be excluded; hence, we choose $(\alpha_0, \alpha_1, \alpha_2) = (0, 1, 1)$. In summary, $b_k = \tilde{b}_{k-1} \oplus \tilde{b}_{k-2}$ (or $\tilde{b}_{k-1} = b_k \oplus \tilde{b}_{k-2}$), and the decision making at the receiver can be done as

$$b_k = \begin{cases} 0, & |c_k| = 2; \\ 1, & |c_k| = 0 \text{ or } 4, \end{cases}$$

as illustrated in the table below.

$\tilde{b}_k$	$\tilde{b}_{k-1}$	$\tilde{b}_{k-2}$	b_k	c_k
0	0	0	0	-2
0	0	1	1	-4
0	1	0	1	0
0	1	1	0	-2
1	0	0	0	2
1	0	1	1	0
1	1	0	1	4
1	1	1	0	2

2. Give $\tilde{b}_k = b_k \oplus \tilde{b}_{k-1}$, where $b_k, \tilde{b}_k \in \{0, 1\}$ and "$\oplus$" is the exclusive-or operation.

   (a) Express the probability $\Pr[\tilde{b}_k = 0]$ as a function of the statistics of $\tilde{b}_{k-1}$ and b_k.

   (b) Let $\{b_k\}_{k=1}^{\infty}$ be i.i.d. with $\Pr(b_k = 1) = p$ for each k. Suppose we initialize $\tilde{b}_0 = 0$. Find $\Pr(\tilde{b}_1 = 0)$ and $\Pr(\tilde{b}_2 = 0)$. Is $\{\tilde{b}_k\}_{k=1}^{\infty}$ stationary?

   (c) Re-do (b) under $p = \frac{1}{2}$. Is $\{\tilde{b}_k\}_{k=1}^{\infty}$ stationary?

Solution.

(a) From Bayes rule,

$$\begin{aligned} \Pr(\tilde{b}_k = 0) &= \Pr(\tilde{b}_{k-1} = 0) \cdot \Pr(\tilde{b}_k = 0 | \tilde{b}_{k-1} = 0) + \Pr(\tilde{b}_{k-1} = 1) \cdot \Pr(\tilde{b}_k = 0 | \tilde{b}_{k-1} = 1) \\ &= \Pr(\tilde{b}_{k-1} = 0) \cdot \Pr(b_k = 0) + \Pr(\tilde{b}_{k-1} = 1) \cdot \Pr(b_k = 1) \end{aligned}$$

(b) From (a), we obtain

$$\Pr(\tilde{b}_1 = 0) = \underbrace{\Pr(\tilde{b}_0 = 0)}_{=1} \cdot (1-p) + \underbrace{\Pr(\tilde{b}_0 = 1)}_{=0} \cdot p = 1 - p$$

and

$$\begin{aligned} \Pr(\tilde{b}_2 = 0) &= \Pr(\tilde{b}_1 = 0) \cdot \Pr(b_1 = 0) + \Pr(\tilde{b}_1 = 1) \cdot \Pr(b_1 = 1) \\ &= (1-p) \cdot (1-p) + p \cdot p \\ &= 1 - 2p + 2p^2, \end{aligned}$$

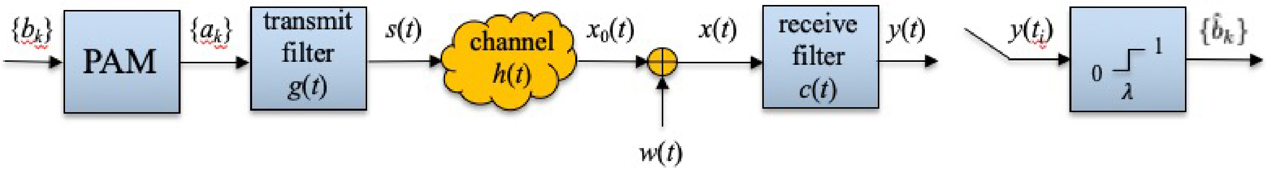
which implies $\{\tilde{b}_k\}$ is not necessarily stationary unless $p = \frac{1}{2}$.

(c) Since $p = \frac{1}{2}$, we have

$$\Pr(\tilde{b}_1 = 0) = \Pr(\tilde{b}_2 = 0) = \frac{1}{2}.$$

And we can follow Slides 8-65 and 8-66 to prove that $\{\tilde{b}_k\}_{k=1}^{\infty}$ is i.i.d. with $\Pr(\tilde{b}_k) = \frac{1}{2}$ for each k. Thus, $\{\tilde{b}_k\}_{k=1}^{\infty}$ is stationary.

3.



In our lectures, we derive

$$R_q(\tau_1, \tau_2; i) = \sum_{k=-\infty}^{\infty} q(iT_b - kT_b - \tau_1)q(iT_b - kT_b - \tau_2),$$

where $q(t) = g(t) \star h(t)$. Let $G(f)$ be the Fourier transform of $g(t)$.

(a) Argue that $R_q(\tau_1, \tau_2; i)$ has nothing to do with i. Hence, we can re-express it as $R_q(\tau_1, \tau_2)$.

(b) Is $R_q(\tau_1, \tau_2)$ only a function of $\tau_1 - \tau_2$? If your answer is affirmative, prove it; else, give a counterexample.

(c) Show that if the transmit filter $g(t)$ satisfies

$$\sum_{m=-\infty}^{\infty} G(-f_1)G(-f_2)\,\delta\left(f_1 + f_2 - \frac{m}{T_b}\right) = \underbrace{G(-f_1)G(-f_2)\delta\left(f_1 + f_2\right)}_{\text{the term for } m=0}, \tag{1}$$

then $R_q(\tau_1, \tau_2)$ is only a function of $\tau_1 - \tau_2$.
Hint: Perform the two-dimensional Fourier transform onto $R_q(\tau_1, \tau_2)$.

Solution.

(a) By setting $m = i - k$, $k$we obtain

$$R_q(\tau_1, \tau_2; i) = \sum_{m=-\infty}^{\infty} q(mT_b - \tau_1)q(mT_b - \tau_2)$$

and hence $R_q(\cdot, \cdot; \cdot)$ has nothing to do with i.

(b) Not necessarily. For example, suppose $q(t) = \left(1 - \frac{|t-T_b|}{T_b}\right) \mathbf{1}\{0 \le t < 2T_b\}$, as a result of $g(t) = h(t) = \frac{1}{\sqrt{T_b}}\mathbf{1}\{0 \le t < T_b\}$ (See Slide 8-11). Then, with $\tau_1 - \tau_2 = -\frac{T_b}{10}$, we obtain

$$R_q\left(0, \tfrac{T_b}{10}\right) = \sum_{m=-\infty}^{\infty} q(mT_b)q\left(mT_b - \tfrac{T_b}{10}\right) = q(T_b)q\left(T_b - \tfrac{T_b}{10}\right) = \tfrac{9}{10}$$

and

$$\begin{aligned} R_q\left(\tfrac{T_b}{10}, \tfrac{T_b}{5}\right) &= \sum_{m=-\infty}^{\infty} q\left(mT_b - \tfrac{T_b}{10}\right) q\left(mT_b - \tfrac{T_b}{5}\right) \\ &= q\left(T_b - \tfrac{T_b}{10}\right) q\left(T_b - \tfrac{T_b}{5}\right) + q\left(2T_b - \tfrac{T_b}{10}\right) q\left(2T_b - \tfrac{T_b}{5}\right) \\ &= \tfrac{9}{10} \cdot \tfrac{4}{5} + \tfrac{1}{10} \cdot \tfrac{1}{5} \\ &= \tfrac{37}{50}. \end{aligned}$$

(c) The two-dimensional Fourier transform of $R_q(\tau_1,\tau_2)$ is given by

$$
\begin{aligned}
S_q(f_1,f_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_q(\tau_1,\tau_2)e^{-\imath 2\pi f_1\tau_1}e^{-\imath 2\pi f_2\tau_2}d\tau_1 d\tau_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q(mT_b-\tau_1)q(mT_b-\tau_2)e^{-\imath 2\pi f_1\tau_1}e^{-\imath 2\pi f_2\tau_2}d\tau_1 d\tau_2\\
&= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} q(\underbrace{mT_b-\tau_1}_{=s_1})e^{-\imath 2\pi f_1\tau_1}d\tau_1\int_{-\infty}^{\infty} q(\underbrace{mT_b-\tau_2}_{=s_2})e^{-\imath 2\pi f_2\tau_2}d\tau_2\\
&= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} q(s_1)e^{-\imath 2\pi f_1(mT_b-s_1)}ds_1\int_{-\infty}^{\infty} q(s_2)e^{-\imath 2\pi f_2(mT_b-s_2)}ds_2\\
&= \sum_{m=-\infty}^{\infty} e^{-\imath 2\pi f_1 mT_b}e^{-\imath 2\pi f_2 mT_b}\underbrace{\int_{-\infty}^{\infty} q(s_1)e^{-\imath 2\pi(-f_1)s_1}ds_1}_{Q(-f_1)}\underbrace{\int_{-\infty}^{\infty} q(s_2)e^{-\imath 2\pi(-f_2)s_2}ds_2}_{Q(-f_2)}\\
&= Q(-f_1)Q(-f_2)\sum_{m=-\infty}^{\infty} e^{-\imath 2\pi(f_1+f_2)mT_b}\\
&= Q(-f_1)Q(-f_2)\frac{1}{T_b}\sum_{m=-\infty}^{\infty}\delta\left(f_1+f_2-\frac{m}{T_b}\right)\\
&= \frac{1}{T_b}H(-f_1)H(-f_2)\underbrace{\sum_{m=-\infty}^{\infty} G(-f_1)G(-f_2)\delta\left(f_1+f_2-\frac{m}{T_b}\right)}_{=G(-f_1)G(-f_2)\delta(f_1+f_2)\text{ by assumption}}\\
&\quad (\text{Apply } Q(f)=H(f)G(f))\\
&= \frac{1}{T_b}H(-f_1)H(-f_2)G(-f_1)G(-f_2)\delta(f_1+f_2)\\
&= \frac{1}{T_b}Q(-f_1)Q(-f_2)\delta(f_1+f_2).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
R_q(\tau_1,\tau_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{T_b}Q(-f_1)Q(-f_2)\delta(f_1+f_2)e^{\imath 2\pi f_1\tau_1}e^{\imath 2\pi f_2\tau_2}df_1 df_2\\
&= \frac{1}{T_b}\int_{-\infty}^{\infty} Q(-f_1)Q(f_1)e^{\imath 2\pi f_1(\tau_1-\tau_2)}df_1; \qquad (2)
\end{aligned}
$$

hence, $R_q(\tau_1,\tau_2)$ is only a function of $\tau_1-\tau_2$.

Note: An example of $G(f)$ that satisfies the condition in (1) (and hence satisfies that $R_q(\tau_1,\tau_2)$ is only a function of $\tau_1-\tau_2$) is

$$
G(f)=\begin{cases} T_b\,e^{-j2\pi f t_0}, & |f|<\frac{1}{2T_b};\\ 0, & \text{otherwise.}\end{cases}
$$

In such case, $g(t)=\text{sinc}((t-t_0)/T_b)$. In fact, any $G(f)$ that is zero outside $|f|<\frac{1}{2T_b}$ satisfies the condition in (1).

Taking $\tau = \tau_1 - \tau_2$ into (2), we obtain

$$R_q(\tau) \quad = \quad \frac{1}{T_b}\int_{-\infty}^{\infty} Q(-f)Q(f)e^{\imath 2\pi f\tau}df.$$

The Fourier transform of $R_q(\tau)$ is obviously

$$S_q(f) = \frac{1}{T_b}Q(-f)Q(f) = \frac{1}{T_b}|Q(f)|^2$$

provided $Q(-f) = Q^*(f)$ (equivalently, provided $q(t)$ is real).

3. This is a continuation from the previous problem.

From the previous block diagram, we have

$$y(iT_b) = \underbrace{\sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} c(\tau)q(iT_b - \tau - kT_b)d\tau}_{\xi_i} + \underbrace{\int_{-\infty}^{\infty} c(\tau)w(iT_b - \tau)d\tau}_{n_i},$$

where $q(t) = g(t) \star h(t)$.

Now, suppose $w(t)$ is a zero-mean WSS noise with PSD $S_w(f)$ (i.e., $w(t)$ is not necessarily a white noise), and suppose $\{a_k\}$ are real-valued zero-mean i.i.d. with variance 1 (i.e., each a_k not necessarily only takes values from $\{\pm 1\}$). Then, the minimum mean square error criterion (MMSE) minimizes

$$\begin{aligned} J_i &= E[e_i^2] = E[((\xi_i + n_i) - a_i)^2] \\ &= E[\xi^2] + E[n_i^2] + E[a_i^2] + 2E[\xi_i n_i] - 2E[n_i a_i] - 2E[\xi_i a_i]. \end{aligned}$$

It can be derived that $E[\xi_i^2] = \int_{-\infty}^{\infty} S_q(f)|C(f)|^2 df$.

(d) Show that $E[n_i^2] = \int_{-\infty}^{\infty} S_w(f)|C(f)|^2 df$.
Hint: $g(t)$, $h(t)$ and $c(t)$ are all real.

(e) Show that $E[\xi_i a_i] = \int_{-\infty}^{\infty} \text{Re}\{C(f)Q(f)\}df$.

(f) Since $E[a_i^2] = 1$ and $E[\xi_i n_i] = E[n_i a_i] = 0$, we have

$$J_i = \int_{-\infty}^{\infty} \underbrace{\left([S_q(f) + S_w(f)]\,|C(f)|^2 - 2 \cdot \text{Re}\{C(f)Q(f)\}\right)}_{=A(f)} df + 1.$$

Show that $C(f) = \frac{Q^*(f)}{S_q(f)+S_w(f)}$ minimizes $A(f)$.

(g) From (f), under what condition that the MMSE equalizer is a matched filter to $q(t)$?
Note: Since a_k is obtained from the sample at $t = kT_b$ (not at $t = (k+1)T_b$), the matched filter to $q(t)$ is $Q^*(f)$ (not $Q^*(f)e^{-\jmath 2\pi f T_b}$).

Solution.

(d)

$$
\begin{aligned}
E[n_i^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)E[w(iT_b-\tau_1)w(iT_b-\tau_2)]d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)R_w(\tau_1-\tau_2)d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)\left(\int_{-\infty}^{\infty} S_w(f)e^{j2\pi f(\tau_1-\tau_2)}df\right)d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty} S_w(f)\left(\int_{-\infty}^{\infty} c(\tau_1)e^{j2\pi f\tau_1}d\tau_1\right)\left(\int_{-\infty}^{\infty} c(\tau_2)e^{-j2\pi f\tau_2}d\tau_2\right)df \\
&= \int_{-\infty}^{\infty} S_w(f)C(-f)C(f)df \\
&= \int_{-\infty}^{\infty} S_w(f)|C(f)|^2 df \quad (\text{Because } C(-f)=C^*(f).)
\end{aligned}
$$

(e) See Slides 8-110∼111, and the fact that ξ_i and a_i are both real.

(f)

$$
\begin{aligned}
A(f) &= [S_q(f)+S_w(f)]\left(|C(f)|^2 - 2\mathrm{Re}\left\{\frac{Q(f)}{[S_q(f)+S_w(f)]}C(f)\right\}\right) \\
&= [S_q(f)+S_w(f)]\left|C(f)-\left(\frac{Q(f)}{S_q(f)+S_w(f)}\right)^*\right|^2 - \frac{|Q(f)|^2}{S_q(f)+S_w(f)}
\end{aligned}
$$

implies that the optimal design should satisfy

$$C(f) = \frac{Q^*(f)}{S_q(f)+S_w(f)}.$$

(g) It requires

$$\frac{Q^*(f)}{S_q(f)+S_w(f)} = \text{constant}\cdot Q^*(f),$$

based on which we know that the condition is $S_q(f)+S_w(f)$ is a constant over the range where $Q^*(f)$ is non-zero.

4. The derivation of the optimal linear receiver introduced in Slide 8-112 indicates that the transfer function of the MMSE equalizer is

$$C(f) = \frac{Q^*(f)}{S_q(f)+\frac{N_0}{2}}.$$

After approximating $S_q(f)$ by a periodic $\tilde{S}_q(f)$ defined in Slide 8-116, we notice that

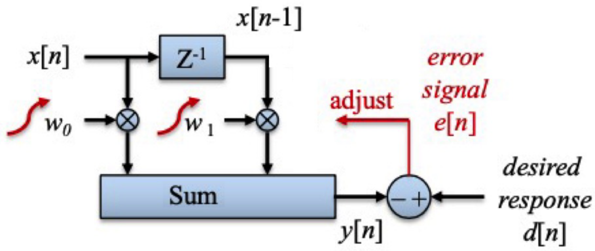
$$\Theta_q(f) = \frac{1}{\tilde{S}_q(f)+\frac{N_0}{2}}$$

becomes periodic with period $1/T_b$, and hence its inverse Fourier transform only exhibits pulses at $t = nT_b$ with integer n. Suppose the inverse Fourier transform of $\Theta_q(f)$ is given by

$$\theta_q(t) = \delta(t) - 2\delta(t-T_b) + \delta(t-2T_b).$$

(a) Draw the block diagram of the tapped-delay-line equalizer with impulse response $\theta_q(t)$.

(b) Suppose an *adaptive* receiver is equipped with the ideal "desired response" that was obtained through, e.g., measurement of the output due to input $x(t)$ and filter $\theta_q(t)$. With the desired response $d[n] = x[n] - 2\,x[n-1] + x[n-2]$ and the adaptive equalizer below, represent $e[n]$ as a function of $x[n]$, $x[n-1]$, $x[n-2]$, w_0 and w_1.



(c) Continue from (b). Usually, $w_0 = 1$. By using the training sequence:

$$x[n] = \begin{cases} -1, & n \bmod 3 = 0; \\ +1, & n \bmod 3 = 1; \\ +1, & n \bmod 3 = 2, \end{cases}$$

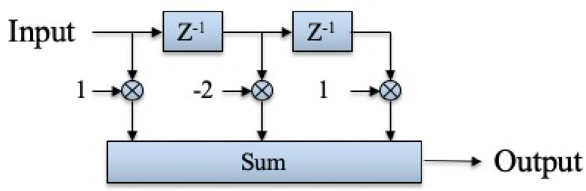
is it possible to adapt to a fixed w_1 such that $\lim_{n\to\infty} e[n] = 0$? Justify your answer. Hint: $y[n]$ is only a function of $x[n]$ and $x[n-1]$ but the desired response depends on $x[n-2]$.

(d) Re-do (c) for training sequence

$$x[n] = \begin{cases} -1, & n \text{ odd}; \\ 1, & n \text{ even}. \end{cases}$$

Solution.

(a)



(b) From the block diagram, we derive

$$\begin{aligned} e[n] &= d[n] - y[n] \\ &= d[n] - \sum_{k=0}^{1} w_k\, x[n-k] \\ &= (x[n] - 2\,x[n-1] + x[n-2]) - (w_0 x[n] + w_1 x[n-1]) \\ &= -(2 + w_1)\, x[n-1] + x[n-2] \end{aligned}$$

(c)

$$e[n] = -(2+w_1)\,x[n-1] + x[n-2] = \begin{cases} -1-w_1, & n \bmod 3 = 0 \\ 3+w_1, & n \bmod 3 = 1 \\ -(3+w_1), & n \bmod 3 = 2 \end{cases}$$

Because either $w_1 = -1$ or $w_1 = -3$ cannot make $\lim_{n\to\infty} e[n] = 0$, the answer to the question is negative

(d) We now have

$$e[n] = -(2+w_1)\,x[n-1] + x[n-2] = \begin{cases} -(3+w_1), & n \text{ odd} \\ 3+w_1, & n \text{ even} \end{cases}$$

Apparently, $w_1 = -3$ guarantees $e[n] = 0$; hence, it is surely possible that $e[n]$ converges to zero.

Note: Although this adaptive filter converges for the specifically designed training sequence, it actually "approximates" $\theta_q(\tau) = \delta(t) - 2\delta(t-T) + \delta(t-2T)$ by $\delta(t) - 3\delta(t-T)$. This approximation is "accurate" only for this specific input sequence, i.e., $\ldots, +1, -1, +1, -1, \ldots$ but for other data sequence, the two filters may generate very different output sequences.

Hence, even if the adaptive filter converges for a specific training sequence, it does not guarantee we obtain the "desired" adaptive equalizer.

5. An inner product satisfies that for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ and $\alpha \in \mathcal{F}$,

   1. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$, where equality holds iff $\boldsymbol{v} = \boldsymbol{0}$. (Here, $\boldsymbol{0}$ is the vector additive identity, i.e., $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$.)
   2. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (\langle \boldsymbol{v}, \boldsymbol{u} \rangle)^*$
   3. $\langle \boldsymbol{u} + \boldsymbol{w}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{w}, \boldsymbol{v} \rangle$
   4. $\langle \alpha \boldsymbol{u}, \boldsymbol{v} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{v} \rangle$

   (a) Let $\mathcal{V}$ be the set of all complex random variables and $\mathcal{F}$ be the set of complex numbers. Define $\langle X, Y \rangle = E[XY^*]$ for $X, Y \in \mathcal{V}$. Show that $\langle X, Y \rangle$ satisfies the above four properties.

   (b) Let $\mathcal{V}$ be the set of real-valued functions defined over $[0, T)$, and $\mathcal{F}$ be the set of real numbers. Define for $f(t), g(t) \in \mathcal{V}$,

   $$\langle f(t), g(t) \rangle = \int_0^T f^2(t) g^2(t) dt.$$

   does $\langle f(t), g(t) \rangle$ satisfy the above four properties? Justify your answer.

Solution.

(a)
- $E[X \times X^*] = E[|X|^2] \geq 0$, where equality holds iff $X = 0$ (with probability 1).
- $E[XY^*] = (E[YX^*])^*$
- $E[(X+Y)Z^*] = E[XZ^*] + E[YZ^*]$

- $E[\alpha XY^*] = \alpha E[XY^*]$

(b) The first two axioms hold but the last two axioms do not.

- $\int_0^T f^2(t)f^2(t)dt \geq 0$, where equality holds iff $f(t) = 0$.
- $\int_0^T f^2(t)g^2(t)dt = (\int_{-\infty}^{\infty} g^2(t)f^2(t)dt)^* = \int_{-\infty}^{\infty} g^2(t)f^2(t)dt$
- $\int_0^T (f(t)+g(t))^2 h^2(t)dt = \int_0^T f^2(t)h^2(t)dt + \int_0^T g^2(t)h^2(t)dt + 2\int_0^T f(t)g(t)h^2(t)dt$ is not necessarily equal to $\int_0^T f^2(t)h^2(t)dt + \int_0^T g^2(t)h^2(t)dt$
- $\int_0^T (\alpha f(t))^2 g^2(t)dt = \alpha^2 \int_0^T f^2(t)g^2(t)dt$ is not necessarily equal to $\alpha \int_0^T f^2(t)g^2(t)dt$.

6. Define the inner product of two complex-valued signals $x(t)$ and $y(t)$ as

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt.$$

Answer the following questions.

(a) Suppose $\boldsymbol{n}(t)$ is a white random process with PSD $N_0/2$, and Show that $\{\boldsymbol{a}_i\}_{i=1}^{\infty}$ satisfies

$$E[\boldsymbol{a}_i \boldsymbol{a}_j^*] = \begin{cases} \frac{N_0}{2}, & i = j \\ 0, & i \neq j \end{cases}$$

provided that $\{f_i(t)\}_{i=1}^{\infty}$ are orthonormal.

(b) Prove that $\langle x(t), y(t)\rangle = \langle X(f), Y(f)\rangle$, where $X(f)$ and $Y(f)$ are Fourier transforms of $x(t)$ and $y(t)$, respectively, and

$$\langle X(f), Y(f)\rangle = \int_{-\infty}^{\infty} X(f)Y^*(f)df.$$

Solution.

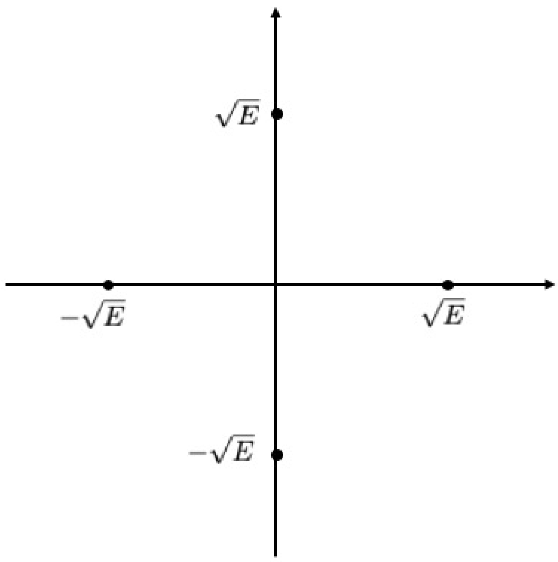
(a)

$$\begin{aligned} E[\boldsymbol{a}_i \boldsymbol{a}_j^*] &= E\left[\left(\int_{-\infty}^{\infty} \boldsymbol{n}(t) f_i^*(t)dt\right)\left(\int_{-\infty}^{\infty} \boldsymbol{n}(s) f_j^*(s)ds\right)^*\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{n}(t)\boldsymbol{n}^*(s)] f_i^*(t) f_j(s)dtds \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-s) f_i^*(t) f_j(s)dtds \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} f_i^*(t) f_j(t)dt \\ &= \begin{cases} \frac{N_0}{2}, & i = j \\ 0, & i \neq j \end{cases} \end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)y^*(t)dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(f_1)e^{\imath 2\pi f_1 t}df_1\right)\left(\int_{-\infty}^{\infty} Y(f_2)e^{\imath 2\pi f_2 t}df_2\right)^* dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(f_1)e^{\imath 2\pi f_1 t}df_1\right)\left(\int_{-\infty}^{\infty} Y^*(f_2)e^{-\imath 2\pi f_2 t}df_2\right) dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f_1)Y^*(f_2)\left(\int_{-\infty}^{\infty} e^{-\imath 2\pi (f_2-f_1)t}dt\right) df_1 df_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f_1)Y^*(f_2)\delta(f_2-f_1)df_1 df_2 \\
&= \int_{-\infty}^{\infty} X(f_1)Y^*(f_1)df_1.
\end{aligned}$$

7. (a) Suppose $\boldsymbol{x} = \boldsymbol{s}_m + \boldsymbol{n}$, where $\boldsymbol{s}_1 = (\sqrt{E}, 0)$, $\boldsymbol{s}_2 = (0, \sqrt{E})$, $\boldsymbol{s}_3 = (-\sqrt{E}, 0)$ and $\boldsymbol{s}_4 = (0, -\sqrt{E})$. Assume the prior probability for each $\boldsymbol{s}_m$ is $1/4$. The noise $\boldsymbol{n}$, however, is Gaussian distributed with mean $\boldsymbol{\mu} = (\sqrt{E}, 0)$ and covariance matrix $\sigma^2\mathbb{I}$, where $\mathbb{I}$ is the identity matrix. Find the ML decision rule.



(b) Draw the best decision region for $m = 1, 2, 3, 4$.

(c) Suppose the receiver mistakenly treats the noise as zero-mean, and use the best partitions for zero-mean additive Gaussian noise. What is the symbol error rate obtained by such a careless receiver if $\sigma^2 = 0$ and $\boldsymbol{\mu} = (2\sqrt{E}, 0)$ (i.e., if $\Pr[\boldsymbol{n} = (2\sqrt{E}, 0)] = 1$)?

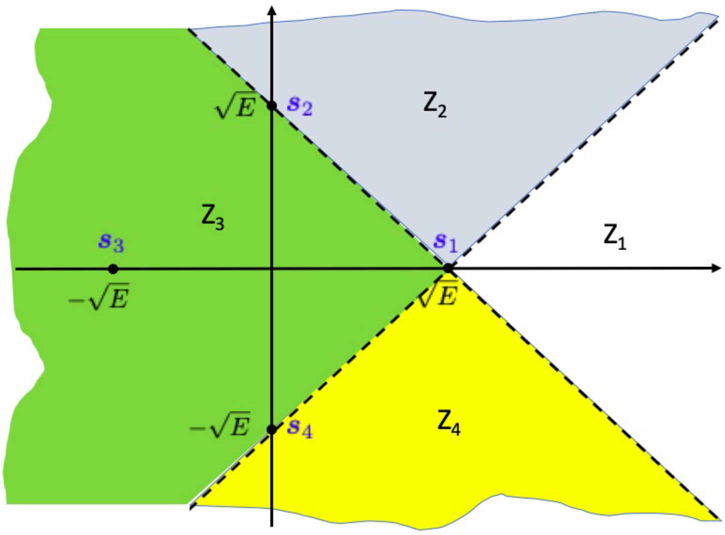
Solution.

(a)

$$\begin{aligned} d_{\mathrm{ML}}(\boldsymbol{x}) &= \arg\max_{1\le m\le 4} \log f(\boldsymbol{x}|\boldsymbol{s}_m) \\ &= \arg\max_{1\le m\le 4} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1-(s_{m,1}+\sqrt{E}))^2-\frac{1}{2\sigma^2}(x_2-s_{m,2})^2} \\ &= \arg\min_{1\le m\le 4} \left[(x_1-(s_{m,1}+\sqrt{E}))^2+(x_2-s_{m,2})^2\right] \\ &= \arg\min_{1\le m\le 4} \|\boldsymbol{x}-\boldsymbol{s}_m-\boldsymbol{\mu}\|^2 \quad \left(=\arg\min_{1\le m\le 4}\|\boldsymbol{x}-(\boldsymbol{s}_m+\boldsymbol{\mu})\|^2\right) \end{aligned}$$

Note: If the mean of the additive noise is not equal to zero, then a compensation subtraction of this mean should be added in the ML decision.

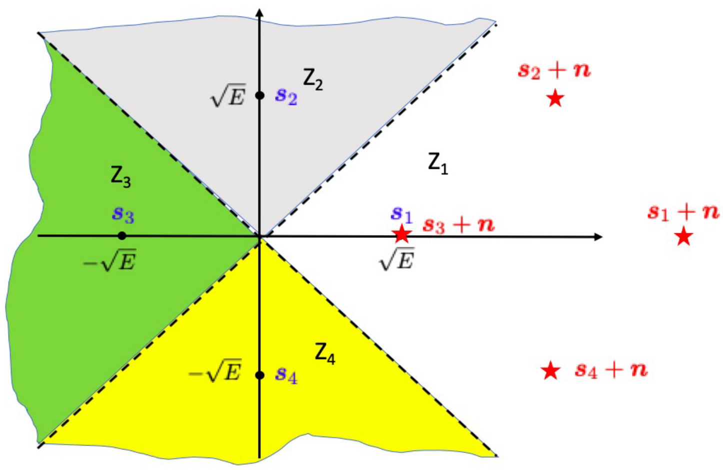
(b) The two decision boundary lines are $y = x + \sqrt{E}$ and $y = -x + \sqrt{E}$.



(c) From (a), we know that the decision rule for zero-mean additive noise is

$$d_{\mathrm{ML}}(\boldsymbol{x}) = \arg\min_{1\le m\le 4} \|\boldsymbol{x}-\boldsymbol{s}_m\|^2$$

There are only four possible receptions as marked in red in the below figure, and $d_{\mathrm{ML}}(\boldsymbol{x}) = 1$ for all four cases.



Since only the transmission of $\boldsymbol{s}_1$ will result in a correct decision, the probability of making the correct decision is given by

$$\begin{aligned} P_c &= \Pr[\boldsymbol{s}_1]\int_{Z_1} f(\boldsymbol{x}|\boldsymbol{s}_1)d\boldsymbol{x} + \Pr[\boldsymbol{s}_2]\int_{Z_2} f(\boldsymbol{x}|\boldsymbol{s}_2)d\boldsymbol{x} \\ &\quad + \Pr[\boldsymbol{s}_3]\int_{Z_3} f(\boldsymbol{x}|\boldsymbol{s}_3)d\boldsymbol{x} + \Pr[\boldsymbol{s}_4]\int_{Z_4} f(\boldsymbol{x}|\boldsymbol{s}_4)d\boldsymbol{x} \\ &= \Pr[\boldsymbol{s}_1] + \Pr[\boldsymbol{s}_2]\cdot 0 + \Pr[\boldsymbol{s}_3]\cdot 0 + \Pr[\boldsymbol{s}_4]\cdot 0 \\ &= \frac{1}{4}. \end{aligned}$$

Hence, the probability of symbol error is $P_e = 1 - P_c = \frac{3}{4}$.

Note: Even if the noise has zero variance, a high error probability can still be resulted from this mistreatment of noise mean.