Problems for the 6th Quiz (April 15, 2021)

Name:________________ Student ID:________________ Score:____________

1. (a) (50%) Prove that $2\cos\left(2\pi f_c t + \beta_1 \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t)\right)$ can be re-expressed as:

$$2\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\beta_1)J_m(\beta_2)\cos(2\pi f_c t + 2\pi n f_1 t + 2\pi m f_2 t).$$

Hint: $e^{j\beta\sin(\phi)} = \sum_{n=-\infty}^{\infty} J_n(\beta)e^{jn\phi}$ and

$$\begin{aligned} &2\cos\left(2\pi f_c t + \beta_1 \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t)\right) \\ &= e^{j2\pi f_c t}e^{j\beta_1\sin(2\pi f_1 t)}e^{j\beta_2\sin(2\pi f_2 t)} + e^{-j2\pi f_c t}e^{j\beta_1\sin(-2\pi f_1 t)}e^{j\beta_2\sin(-2\pi f_2 t)} \end{aligned}$$

(b) (50%) What is the instantaneous frequency of $2\cos\left(2\pi f_c t + \beta_1 \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t)\right)$?
Hint: The instantaneous frequency of $A\cos(2\pi\phi(t))$ is $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t)$.

Solutions.

(a)

$$\begin{aligned} &2\cos\left(2\pi f_c t + \beta_1 \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t)\right) \\ &= e^{j2\pi f_c t}e^{j\beta_1\sin(2\pi f_1 t)}e^{j\beta_2\sin(2\pi f_2 t)} + e^{-j2\pi f_c t}e^{j\beta_1\sin(-2\pi f_1 t)}e^{j\beta_2\sin(-2\pi f_2 t)} \\ &= e^{j2\pi f_c t}\sum_{n=-\infty}^{\infty} J_n(\beta_1)e^{jn(2\pi f_1 t)}\sum_{m=-\infty}^{\infty} J_m(\beta_2)e^{jm(2\pi f_1 t)} \\ &\quad + e^{-j2\pi f_c t}\sum_{n=-\infty}^{\infty} J_n(\beta_1)e^{jn(-2\pi f_1 t)}\sum_{m=-\infty}^{\infty} J_m(\beta_2)e^{jm(-2\pi f_2 t)} \qquad (1) \\ &= 2\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\beta_1)J_m(\beta_2)\cos(2\pi f_c t + 2\pi n f_1 t + 2\pi m f_2 t) \end{aligned}$$

Note: I have missed the multiplicative 2 in the problem statement. Thus, as long as you have put down (1) in your derivation, you get the full credit of this problem (I hope by this problem, you realize how to apply the Bessel function to eliminate "sin(·)" from the exponent.)

(b)

$$\begin{aligned} &\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left(2\pi f_c t + \beta_1 \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t)\right) \\ &= \frac{1}{2\pi}\left(2\pi f_c + 2\pi f_1\beta_1\cos(2\pi f_1 t) + 2\pi f_2\beta_2\cos(2\pi f_2 t)\right) \\ &= f_c + \underbrace{f_1\beta_1}_{(\Delta f)_1}\cos(2\pi f_1 t) + \underbrace{f_2\beta_2}_{(\Delta f)_2}\cos(2\pi f_2 t) \end{aligned}$$

Note: We use this problem to explain what would happen when a multiple-tone signal (or a general $m(t)$) is FM-modulated.

Suppose $f_1 > f_2$ and there exists t^* such that $\cos(2\pi f_1 t^*) = \cos(2\pi f_2 t^*) = 1$. Then, as in Slide 4-60, $m(t) = A_1\cos(2\pi f_1 t) + A_2\cos(2\pi f_2 t)$ implies

$$f_i(t) = f_c + k_f m(t) = f_c + k_f A_1\cos(2\pi f_1 t) + k_f A_2\cos(2\pi f_2 t).$$

Thus, the frequency deviation Δf is given by

$$\Delta f = \max_{t \in \Re} \{k_f A_1 \cos(2\pi f_1 t) + k_f A_2 \cos(2\pi f_2 t)\} = k_f(A_1 + A_2).$$

As a result, Slide 4-82 indicates $W = \max\{f_1, f_2\} = f_1$ and the deviation ratio equals

$$D = \frac{\Delta f}{W} = \frac{k_f A_1}{f_1} + \frac{k_f A_2}{f_1}.$$

As we learn from Slide 4-84 that *i*) FM radio in North America requires the maximum frequency derivation $\Delta f = 75$ kHz; *ii*) The message bandwidth is $W = 15$ kHz; and *iii*) The deviation ratio is $D = \Delta f / W = 75/5 = 5$, we conclude

$$A_1 + A_2 = \frac{5W}{k_f}.$$

Consequently, for single-tone transmission, $A_2 = \frac{5W}{k_f}$; however, when a two-tone message is transmitted, we shall have $A_1 + A_2 = \frac{5W}{k_f}$ (i.e., the sum of amplitudes of all tones should be a constant).

In summary, we state in Slide 4-82 that "for non-sinusoidal modulation, the deviation ratio $D = \Delta f / W$ is used instead of the modulation index β." Thus, if

$$m(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t) \quad \text{and} \quad W = \max_{1 \le i \le N} f_i = f_1,$$

then

$$D = \frac{\Delta f}{W} = \frac{\Delta f}{f_1},$$

as long as $\sum_{i=1}^{N} A_i$ remains constant and equals $\frac{DW}{k_f}$. Therefore, the amplitude A_1 of the "worst-case" tone should "share" its value with other tones if a multiple-tone message is transmitted..