Problems for Quiz 3 (March 22, 2021)

Name:________________ Student ID:________________ Score:____________

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a\rvert} G(\frac{f}{a})$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_0 t)g(t) \rightleftharpoons G(f - f_0)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)df$

1. Let $G(f) = \mathcal{F}\{g(t)\}$ be the Fourier transform of $g(t)$. Use the properties in Table A6.2 to derive the Fourier transform of the following functions.

   (a) (30%) $G(-2t)$.

   (b) (30%) $g(t - t_0)e^{j2\pi f_0 t}$.

   Solution.

   (a) By duality, $\mathcal{F}\{G(t)\} = g(-f)$. By time-scaling, $\mathcal{F}\{G(-2t)\} = \frac{1}{\lvert -2\rvert} g\left(-\left(\frac{f}{-2}\right)\right) = \frac{1}{2} G\left(\frac{f}{2}\right)$.

   (b) By time-shifting, $\mathcal{F}\{g(t - t_0)\} = G(f)e^{-j2\pi f t_0}$. By frequency shifting, $\mathcal{F}\{g(t - t_0)e^{j2\pi f_0 t}\} = G(f - f_0)e^{-j2\pi(f-f_0)t_0} = G(f - f_0)e^{j2\pi f_0 t_0}e^{-j2\pi f t_0}$.

2. (40%) Find the time-average PSD of a deterministic function $X(t) = \begin{cases} 1, & \lvert t\rvert < 20; \\ 0, & \text{otherwise.} \end{cases}$

   Hint: $\bar{S}_X(f) = \lim_{T\to\infty} \frac{1}{2T} E[X(f)X_{2T}^*(f)]$, where

$$X_{2T}(t) := \begin{cases} X(t), & \lvert t\rvert \le T; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad X_{2T}(f) = \mathcal{F}\{X_{2T}(t)\}.$$

   Solution. $X_{2T}(t) = X(t)$ for $T > 20$. Hence, $X_{2T}(f) = X(f)$, which implies

$$\bar{S}_X(f) = \lim_{T\to\infty} \frac{1}{2T} X(f)X_{2T}^*(f) = \lim_{T\to\infty} \frac{1}{2T} X(f)X^*(f) = \lim_{T\to\infty} \frac{1}{2T} \lvert X(f)\rvert^2 = 0,$$

   where the last step follows from the fact that $\lvert X(f)\rvert^2 = \lvert 40\text{sinc}(40f)\rvert^2 \le 1600$ is bounded (and is functionally independent of T).

   Note: For any process of finite duration, its time-average PSD is zero as long as its Fourier transform is bounded.