Problems for Quiz 2 (March 15, 2021)

Name:________________ Student ID:________________ Score:____________

1. (a) (40%) Complete the Fourier transform pair below.

$$\begin{cases} G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\mathrm{d}t \\ g(t) &= \int_{-\infty}^{\infty} \Big(\qquad\qquad \Big)\mathrm{d}f \end{cases}$$

   (b) (60%) Given that $Y(f)$, $X(f)$ and $H(f)$ are Fourier transforms of $y(t)$, $x(t)$ and $h(t)$, respectively, prove that $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$ implies $Y(f) = H(f)X(f)$.
   Hint: Complete the following derivation.

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau\right)e^{-j2\pi ft}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} \quad \cdots \quad \mathrm{d}t\right)\mathrm{d}\tau \\ &= \cdots \end{aligned}$$

Solution.

(a) See Slide 2-2.

(b) See Slide 2-18.
Note: In fact, we shall have $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$ if and only if $Y(f) = H(f)X(f)$. The converse can be proved as follows.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau\right)e^{j2\pi ft}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{-j2\pi f\tau}e^{j2\pi ft}\mathrm{d}f\right)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\tau)}\mathrm{d}f\right)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau \end{aligned}$$