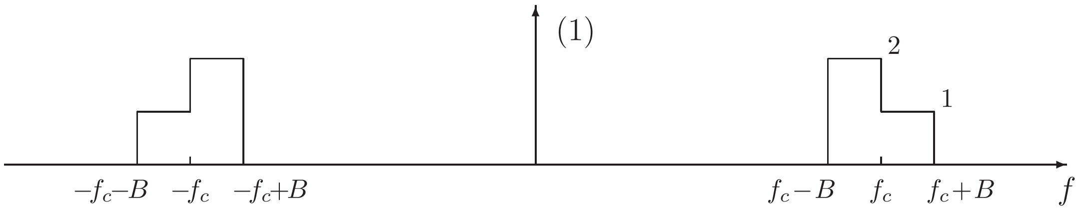
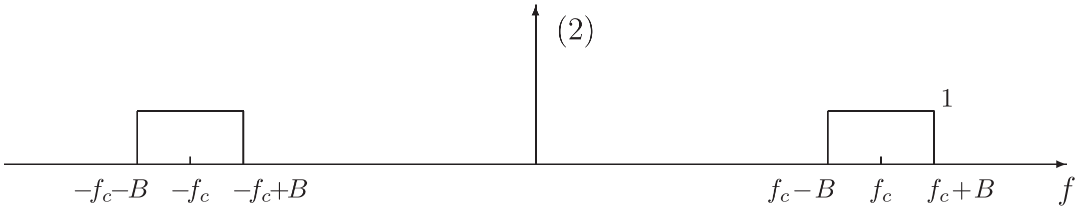
1. (Spectrum drawing)





(a) (4%) Suppose $G(f) = \mathcal{F}\{g(t)\}$ is real and symmetric as plotted in (1), and

$$\tilde{G}(f) = G_+(f + f_c),$$

where $G_+(f) = 2u(f)G(f)$ and $u(f)$ is the unit step function. Plot $\tilde{G}(f)$. Is $\tilde{g}(t) = \mathcal{F}^{-1}\{\tilde{G}(f)\}$ a real-valued function? Justify your answer by Property 10 in Table A6.2.

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$

(b) (3%) Continue from (a). Plot the Fourier transform of $g(t) - j\hat{g}(t)$, where $\hat{g}(t)$ is the Hilbert transform of $g(t)$.
Hint: $\hat{G}(f) = H_{\text{Hilbert}}(f)G(f) = (-j\text{sgn}(f))G(f)$.

(c) (4%) Suppose the PSD $S_N(f)$ of a WSS random process $N(t)$ is plotted as (2). Re-express $N(t)$ as

$$N(t) = N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t)$$

and assume $N_I(t)$ and $N_Q(t)$ are jointly WSS. We thus obtain (from our lectures)

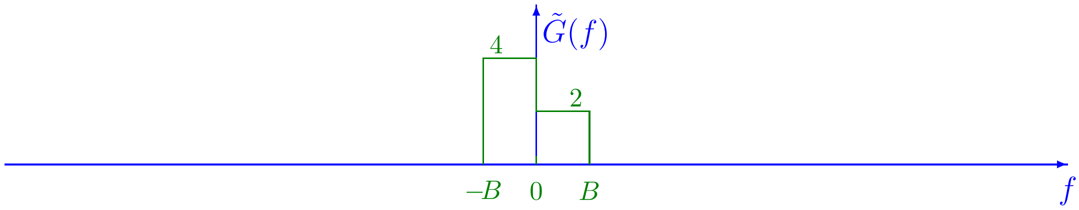
$$S_{N_I,N_Q}(f) = \begin{cases} j(S_N(f+f_c) - S_N(f-f_c)), & |f| < B; \\ 0, & \text{otherwise} \end{cases}$$

Plot $\frac{1}{j}S_{N_I,N_Q}(f)$. Is $N_I(t+\tau)$ and $N_Q(t)$ orthogonal for arbitrary t and τ? Justify your answer.
Note: $N_I(t+\tau)$ and $N_Q(t)$ is said to be orthogonal if $E[N_I(t+\tau)N_Q(t)] = 0$.

(d) (4%) Re-do (c) by taking $S_N(f)$ to be the plot in (1) instead.
Note: $\int_0^B je^{j2\pi f\tau}\mathrm{d}f - \int_{-B}^0(-j)e^{j2\pi f\tau}\mathrm{d}f = j\frac{\sin(2\pi B\tau)}{\pi\tau} = j2B\,\mathrm{sinc}(2B\tau)$

Solution.

(a)

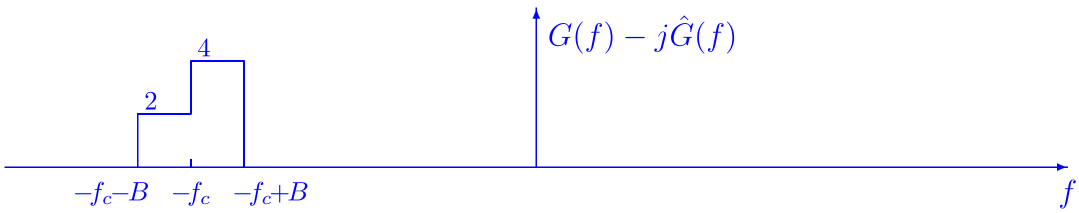


From Property 10 in Table A6.2, we know $\tilde{g}(t) = \tilde{g}^*(t)$ iff $\tilde{G}(f) = \tilde{G}^*(-f)$. Since $\tilde{G}(f) \neq \tilde{G}^*(-f)$ for $|f| < B$, $\tilde{g}(t)$ cannot be a real-valued function.

(b) Since

$$\begin{aligned} \mathcal{F}\{g(t) - j\hat{g}(t)\} &= G(f) - j\hat{G}(f) \\ &= G(f) - j(-j\text{sgn}(f))G(f) \\ &= (1 - \text{sgn}(f))G(f) \\ &= \begin{cases} 0, & f > 0; \\ 2G(f), & f < 0, \end{cases} \end{aligned}$$

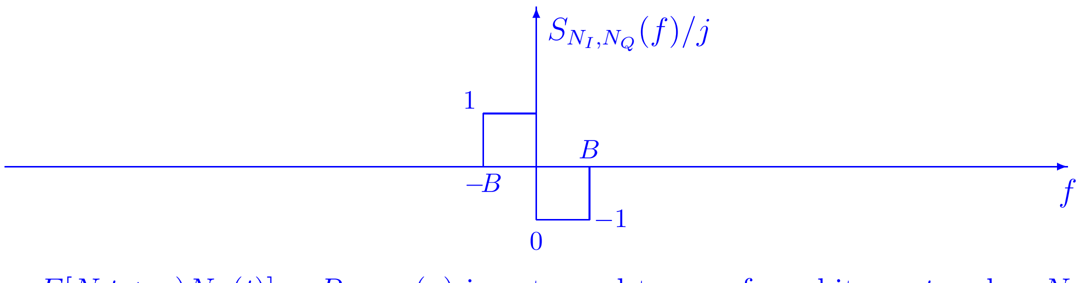
we obtain:



(c) Apparently, $\frac{1}{j}S_{N_I,N_Q}(f) = 0$ for $f \in \Re$, implying $R_{N_I,N_Q}(\tau) = 0$ for $\tau \in \Re$. Therefore, $N_I(t+\tau)$ and $N_Q(t)$ are orthogonal for arbitrary t and τ.

Note:For filtered white noise process, $N_I(t_1)$ and $N_Q(t_2)$ are always orthogonal for arbitrary t_1 and t_2.

(d)



Since $E[N_(t+\tau)N_Q(t)] = R_{N_I,N_Q}(\tau)$ is not equal to zero for arbitrary t and τ, $N_I(t)$ and $N_Q(t)$ are not orthogonal for arbitrary t and τ.

Note: More specifically,

$$R_{N_I,N_Q}(\tau) = \int_{-\infty}^{\infty} S_{N_I,N_Q}(f)e^{j2\pi f\tau}\text{d}f = j2B\,\text{sinc}(2B\tau) = 0$$

only when $\tau = \frac{k}{2B}$ for non-zero integer k.

2. (Parts 1 & 2) Check whether the statement are true or false.

   (a) (3%) For a stationary random process X_1, X_2, X_3, …, X_n, it is guaranteed that the time average converges to the ensemble average, i.e.,

$$\Pr\left[\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = E[X_1]\right] = 1.$$

   (b) (3%) If $X(t)$ is stationary, then its autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X^*(t_2)]$ is only a function of the time difference $(t_1 - t_2)$.

   (c) (3%) If $X(t)$ is a real-valued WSS process, then the absolute value of its autocorrelation function always peaks at the origin, i.e., $|R_X(\tau)| \le R_X(0)$.

   (d) (3%) For a stable LTI system, a WSS input always induces a WSS output.

   (e) (3%) For a stable LTI system, the time-average PSD of the system output $Y(t)$ only depends on the magnitude of the transfer function but not the phase. In other words, $\bar{S}_Y(f)$ is only a function of $|H(f)|$.

Solution.

   (a) False. (We also need ergodicity for the validity of the statement.)

   (b) True. (See Slide 1-17.)

   (c) True. (See Slide 1-25.)

   (d) True. (See Slide 1-66.)

   (e) True. (See Slide 2-31.)

3. (Part 2)

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a\rvert} G(\frac{f}{a})$ where a is a constant
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$

   (a) (5%) Explain based on Property 2 in Table A6.2 what the time-bandwidth product is.

   (b) (6%) Argue based on the two properties in Table A6.2 above that $R_X^*(\tau) = R_X(-\tau)$ implies that PSD $S_X(f)$ must be real-valued.

   Hint: Use Property 2 with $a = -1$.

   (c) (6%) Argue based on Proper 10 in Table A6.2 and the result from (b) that if $R_X(\tau)$ is real, then $S_X(f)$ is symmetric.

Solution.

(a) Property 2 in Table A6.2 indicates that reducing the time-scale by a factor of a extends the bandwidth by the same factor. This suggests that the product of time- and frequency- parameters should remain constant, which is named the time-bandwidth product or bandwidth-duration product.

(b) By Property 2 with $a = -1$, we know that $\mathcal{F}\{R_X(-\tau)\} = S_X(-f)$. Property 10 indicates $\mathcal{F}\{R_X^*(\tau)\} = S_X^*(-f)$. Thus, $R_X^*(\tau) = R_X(-\tau)$ iff $S_X^*(-f) = S_X(-f)$. Thus, $R_X^*(\tau) = R_X(-\tau)$ implies $S_X(-f)$ is real.

(c) By Property 10, $R_X(\tau) = R_X^*(\tau)$ implies $S_X(f) = S_X^*(-f)$. Since from (a), $S_X(f)$ is real, we obtain $S_X(f) = S_X(-f)$.

4. (Part 3) Suppose

$$X(t) = \text{Re}\Big\{ \underbrace{(X_I(t) + jX_Q(t))}_{=\tilde{X}(t)} e^{j2\pi f_c t} \Big\} = X_I(t)\cos(2\pi f_c t) - X_Q(t)\sin(2\pi f_c t),$$

and both $X(t)$ and $\tilde{X}(t)$ are WSS. Then, we have proven in our lectures that

$$\text{P1:} \quad \begin{cases} R_{X_I}(\tau) = R_{X_Q}(\tau) \\ R_{X_I,X_Q}(\tau) = -R_{X_Q,X_I}(\tau) \end{cases}$$

$$\text{P2:} \quad R_X(\tau) = R_{X_I}(\tau)\cos(2\pi f_c \tau) - R_{X_Q,X_I}(\tau)\sin(2\pi f_c \tau)$$

(a) (6%) Show that $R_{\tilde{X}}(\tau) = 2R_{X_I}(\tau) + j2R_{X_Q,X_I}(\tau)$.
<u>Note:</u>You shall indicate what properties have been applied in your derivation.

(b) (6%) Argue based on (a) and Property 2 that $R_X(\tau) = \frac{1}{2}\text{Re}\{R_{\tilde{X}}(\tau)e^{j2\pi f_c \tau}\}$.
<u>Note:</u>You shall indicate what properties have been applied in your derivation.

(c) (6%) Based on (b), use the properties in Table A6.2 to show that

$$S_X(f) = \frac{1}{4}\left(S_{\tilde{X}}(f - f_c) + S_{\tilde{X}}(-f - f_c)\right).$$

<u>Note:</u>You shall indicate what properties have been applied in your derivation.
<u>Hint:</u>

$$R_X(\tau) = \frac{1}{2}\text{Re}\{R_{\tilde{X}}(\tau)e^{j2\pi f_c \tau}\} = \frac{1}{4}\left(R_{\tilde{X}}(\tau)e^{j2\pi f_c \tau} + R_{\tilde{X}}^*(\tau)e^{-j2\pi f_c \tau}\right)$$

$$\Rightarrow S_X(f) = \mathcal{F}\{R_X(\tau)\} = \frac{1}{4}\left(\mathcal{F}\left\{R_{\tilde{X}}(\tau)e^{j2\pi f_c \tau}\right\} + \mathcal{F}\left\{R_{\tilde{X}}^*(\tau)e^{-j2\pi f_c \tau}\right\}\right)$$

TABLE A6.2 Summary of properties of the Fourier transform

Property	Mathematical Description
5. Frequency shifting	$\exp(j2\pi f_0 t)g(t) \rightleftharpoons G(f - f_0)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$

Solution.

(a)

$$\begin{aligned} R_{\tilde{X}}(\tau) &= E[\tilde{X}(t+\tau)\tilde{X}^*(t)] \\ &= E[(X_I(t+\tau)+jX_Q(t+\tau))(X_I(t)-jX_Q(t))] \\ &= R_{X_I}(\tau)+R_{X_Q}(\tau)+j[R_{X_Q,X_I}(\tau)-R_{X_I,X_Q}(\tau)] \\ &= 2R_{X_I}(\tau)+j2R_{X_Q,X_I}(\tau) \quad \text{(From P1)} \end{aligned}$$

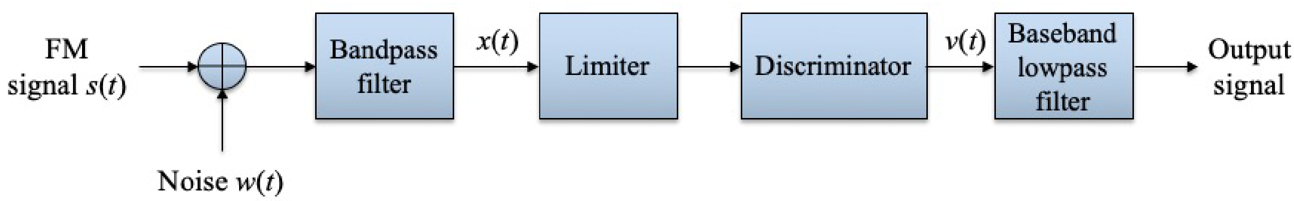
(b) It can be easily verified that

$$\begin{aligned} \frac{1}{2}\mathrm{Re}\{R_{\tilde{X}}(\tau)e^{j2\pi f_c\tau}\} &= \frac{1}{2}\mathrm{Re}\{[2R_{X_I}(\tau)+j2R_{X_Q,X_I}(\tau)]e^{j2\pi f_c\tau}\} \quad \text{(From (a))} \\ &= \mathrm{Re}\{[R_{X_I}(\tau)+jR_{X_Q,X_I}(\tau)]e^{j2\pi f_c\tau}\} \\ &= R_{X_I}(\tau)\cos(2\pi f_c t)-R_{X_Q,X_I}(\tau)\sin(2\pi f_c\tau) \\ &= R_X(\tau) \quad \text{(From P2)} \end{aligned}$$

(c)

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \frac{1}{4}\left(\mathcal{F}\left\{R_{\tilde{X}}(\tau)e^{j2\pi f_c\tau}\right\}+\mathcal{F}\left\{R^*_{\tilde{X}}(\tau)e^{-j2\pi f_c\tau}\right\}\right) \\ &= \frac{1}{4}\Bigg(\underbrace{S_{\tilde{X}}(f-f_c)}_{\text{Property 5 of Tab. 6.2}} + \underbrace{S^*_{\tilde{X}}(-(f+f_c))}_{\text{Properties 5 \& 10 of Tab. 6.2}}\Bigg) \\ &= \frac{1}{4}\Big(S_{\tilde{X}}(f-f_c)+S_{\tilde{X}}(-f-f_c)\Big) \end{aligned}$$

5. (Parts 4 & 5)



For an FM receiver in the above figure, we have

$$s(t) = A_c\cos(2\pi f_c t + 2\pi\phi(t)),$$

Denote the passband noise process (i.e., the noise process observed at the output of the bandpass filter) as

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t).$$

Due to the limiter, the amplitude of $x(t)$ is of no influence on its output, and only the phase remains. Hence, we can assume the amplitude at the output of the limiter is 1 for simplicity.

(a) (6%) Under $\phi(t) = 0$, i.e., zero transmission, show that the output of the limiter $x_{\text{limiter}}(t)$ is given by

$$x_{\text{limiter}}(t) = \cos(2\pi f_c t + \theta(t))$$

where

$$\theta(t) = \tan^{-1}\left(\frac{n_Q(t)}{A_c + n_I(t)}\right).$$

Hint: Complete the following derivation:

$$\begin{aligned} x(t) &= A_c \cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= (A_c + n_I(t))\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= \cdots \\ &\overset{\text{limiter}}{\rightarrow} \cos(2\pi f_c t + \theta(t)) \end{aligned}$$

(b) (5%) On the contrary, we assume $w(t) = 0$ (equivalently, $n(t) = 0$) but $\phi(t) = k_f \int_0^t m(\tau)d\tau$. What is the instantaneous frequency of the FM signal $s(t)$?

(c) (6%) Continue from (b). Further assume that $m(t) = A_m \cos(2\pi f_m t)$ and $f_c = 1000 f_m$, which results in

$$s(t) = \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$$

with $\beta \triangleq \frac{k_f A_m}{f_m}$. Let $S_{2T}(f)$ be the Fourier transform of $s_{2T}(t)$, where

$$s_{2T}(t) \triangleq s(t) \cdot \mathbf{1}\{|t| \le T\}$$

and $\mathbf{1}\{\cdot\}$ is the set indicator function. What is the value of $S_{2T}(f_c + k f_m)$ for each integer k, provided $2T f_m$ is an integer?

Hint: We have derived in our lectures that

$$\mathcal{F}\{s(t)\} = \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)],$$

$\mathcal{F}\{\mathbf{1}\{|t| \le T\}\} = 2T\,\text{sinc}(2Tf)$ and $S_{2T}(f) = \mathcal{F}\{s(t)\} \star \mathcal{F}\{\mathbf{1}\{|t| \le T\}\}$, where "$\star$" denotes the convolution operation.

Solution.

(a)

$$\begin{aligned} x(t) &= A_c \cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= (A_c + n_I(t))\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= \sqrt{(A_c + n_I(t))^2 + n_Q^2(t)}\,(\cos(\theta(t))\cos(2\pi f_c t) - \sin(\theta(t))\sin(2\pi f_c t)) \\ &= \sqrt{(A_c + n_I(t))^2 + n_Q^2(t)}\,\cos(2\pi f_c t + \theta(t)) \\ &\overset{\text{limiter}}{\rightarrow} \cos(2\pi f_c t + \theta(t)) \quad \Big(= x_{\text{limiter}}(t)\Big) \end{aligned}$$

(b)

$$f_i(t) = f_c + \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = f_c + k_f m(t)$$

(c)

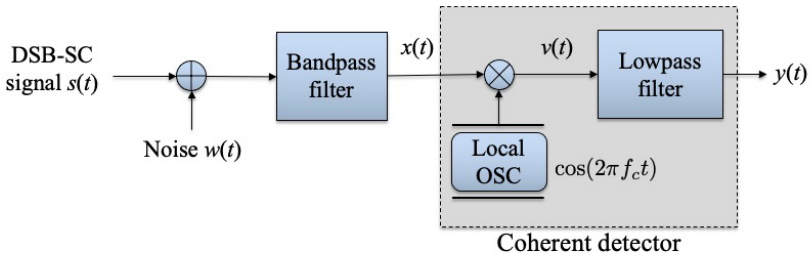
$$\begin{aligned}
S_{2T}(f) &= \mathcal{F}\{s_{2T}(t)\} \\
&= \mathcal{F}\{s(t)\cdot \mathbf{1}\{|t|\le T\}\} \\
&= \mathcal{F}\{s(t)\} \star \mathcal{F}\{\mathbf{1}\{|t|\le T\}\} \\
&= \left(\frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)]\right) \star (2T\,\text{sinc}(2Tf)) \\
&= T\sum_{n=-\infty}^{\infty} J_n(\beta)[\text{sinc}(2T(f-f_c-nf_m))+\text{sinc}(2T(f+f_c+nf_m))]
\end{aligned}$$

Hence,

$$\begin{aligned}
S_{2T}(f_c+kf_m) &= T\sum_{n=-\infty}^{\infty} J_n(\beta)[\text{sinc}(2T(k-n)f_m)+\text{sinc}(2T(2000+k+n)f_m)] \\
&= T\big(J_k(\beta)+J_{-k-2000}(\beta)\big)
\end{aligned}$$

6. (Part 5)

(a) (6%) In the figure below, we suppose $s(t) = m(t)\cos(2\pi f_c t)$, $m(t)$ is a zero-mean WSS process with $E[m^2(t)] = P$ and bandwidth W, and $w(t)$ is an additive white Gaussian noise with two-sided PSD $N_0/2$. Find the SNR_O of the coherent receiver for DSB-SC modulation.

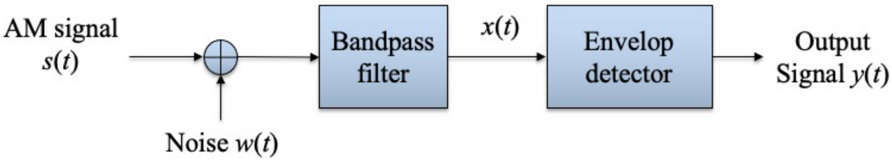


Hint: $E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)] = 2WN_0$ and

$$\begin{cases}
x(t) = s(t) + n(t) \\
\quad \text{where } n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \text{ is the filtered white noise} \\
v(t) = x(t)\cos(2\pi f_c t) \\
y(t) = v(t) \text{ passing through an ideal lowpass filter}
\end{cases}$$

You need to find the expression of the final output $y(t)$. Then, SNR_O is the average power of the *signal content* in $y(t)$ divided by the average power of the *noise content* in $y(t)$.

(b) (6%) In the figure below, we suppose $s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$, $m(t)$ is a zero-mean WSS process with $E[m^2(t)] = P$ and bandwidth W, and $w(t)$ is an additive white Gaussian noise with two-sided PSD $N_0/2$. Find the SNR_O of the envelop detector with DC remover for DSB-C modulation, provided $A_c[1+k_a m(t)] \gg |\tilde{n}(t)|$.



Hint: The expression of the final output $y(t)$ is given below. Then, SNR_O is the average power of the *signal content* in $y(t)$ divided by the average power of the *noise content* in $y(t)$. Note that $E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)] = 2WN_0$ and

$$y(t)\big|_{\text{block DC}} = \left(|\tilde{x}(t)|\right)\big|_{\text{block DC}} = \sqrt{(A_c[1 + k_a m(t)] + n_I(t))^2 + \cancel{n_Q^2(t)}}\Big|_{\text{block DC}} \approx A_c k_a m(t) + n_I(t)$$

(c) (6%) Show that under fixed N_0, W and A_c, the SNR_O of DSB-C with envelop detector is bounded above by $\frac{A_c^2}{2WN_0}$ and hence cannot be improved by increasing P, due to the constraint $|k_a m(t)| \leq 1$.

Solution.

(a)

$$y(t) = \left[m(t)\cos^2(2\pi f_c t)\right]_{\text{Lowpass}} + \left[n_I(t)\cos^2(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t)\right]_{\text{Lowpass}}$$

$$\Rightarrow \boxed{y(t) = \tfrac{1}{2}m(t) + \tfrac{1}{2}n_I(t)}$$

$$\Rightarrow \text{SNR}_{\text{O,DSB-SC/coherent receiver}} = \frac{E[m^2(t)/4]}{E[n_I^2(t)/4]} = \frac{P}{E[n^2(t)]} = \frac{P}{2WN_0}$$

(b)

$$\text{SNR}_{\text{O,DSB-C/envelop detector}} = \frac{E[A_c^2 k_a^2 m^2(t)]}{E[n_I^2(t)]} = \frac{A_c^2 k_a^2 P}{E[n^2(t)]} = \frac{A_c^2 k_a^2 P}{2WN_0}$$

(c)

$$\text{SNR}_{\text{O,DSB-C/envelop detector}} = \frac{A_c^2 k_a^2 P}{2WN_0} \leq \frac{A_c^2}{2WN_0}$$