Introduction to Communication Systems: Final Exam

Grading systems

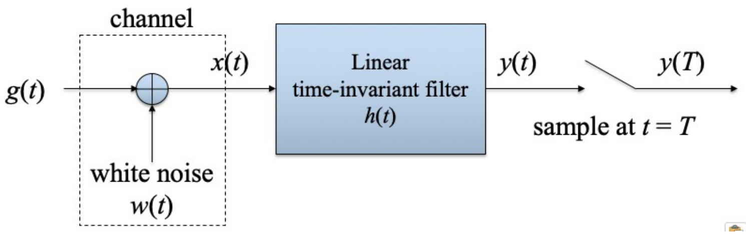
- 5 problems and each problem = 20 points.
  - Each problem awards you 14 (basic) points when a serious answer is provided, regardless of its correctness.
  - The remaining 6 (extra) points awards (partially or fully) to those who provide a (partially or fully) correct answer.
  - Please do not duplicate the answer from others.

1. (Matched filter over signal space)

   (a) (basic = 3 pts., extra = 1 pts.) For a single transmission, we have

   $$x(t) = a \cdot g(t) + w(t),$$

   where a is the digital message to be transmitted, and $w(t)$, $g(t)$ and $h(t)$ are all real-valued.

   Can we re-interpret the filtering and sampling operation as an inner product? Justify your answer. You shall clearly indicate what your definition of inner product is and what the two elements involved in the inner product are.

   (b) (basic = 7 pts., extra = 3 pts.) Based on the signal space viewpoint, the transmitter is modeled as the transmission of M waveforms, i.e., $s_1(t)$, $s_2(t)$, ..., $s_M(t)$, and the receiver employs "inner product" to transform the received waveform into a N-dimensional vector, based on which an estimate of the index of the transmitted waveform is made. In other words, the decision is made based on

   $$\begin{cases} \langle x(t), \phi_1(t)\rangle = \langle s_i(t), \phi_1(t)\rangle + \langle w(t), \phi_1(t)\rangle \\ \langle x(t), \phi_2(t)\rangle = \langle s_i(t), \phi_2(t)\rangle + \langle w(t), \phi_2(t)\rangle \\ \vdots \\ \underbrace{\langle x(t), \phi_j(t)\rangle}_{=x_j} = \underbrace{\langle s_i(t), \phi_j(t)\rangle}_{=s_{i,j}} + \underbrace{\langle w(t), \phi_j(t)\rangle}_{=n_j} \\ \vdots \\ \langle x(t), \phi_N(t)\rangle = \langle s_i(t), \phi_N(t)\rangle + \langle w(t), \phi_N(t)\rangle \end{cases}$$

   Suppose the noise power $E[|n_j|^2] = E[|\langle w(t), \phi_j(t)\rangle|^2]$ is proportional to the norm square $\|\phi_j(t)\|^2$, i.e.,

   $$E[|n_j|^2] = \beta\|\phi_j(t)\|^2 \text{ for a constant } \beta \text{ functionally independent of } j. \quad (1)$$

Assume $\|s_i(t)\| > 0$ and $\|\phi_j(t)\| = 1$. Let the signal-to-noise ratio for the transmission of waveform $s_i(t)$ be equal to

$$\eta_i \triangleq \frac{\sum_{j=1}^{N} |s_{i,j}|^2}{\sum_{j=1}^{N} \mathbf{1}\{s_{i,j} \neq 0\} E[n_j^2]} = \frac{\sum_{1\le j\le N: s_{i,j}\neq 0} |s_{i,j}|^2}{\sum_{1\le j\le N: s_{i,j}\neq 0} E[n_j^2]},$$

where the receiver reasonably exclude x_j from its decision making operation if it knows x_j contains no information but simply noise. What is the combination of N and $\{\phi_j\}_{j=1}^{N}$ such that the average SNR

$$\eta_{\text{ave}} \triangleq \frac{1}{M} \sum_{j=i}^{M} \eta_i$$

is maximized if $\{s_i(t)\}_{i=1}^{M}$ are orthogonal to each other in the sense that $\langle s_{i_1}(t), s_{i_2}(t)\rangle = 0$ for every $i_1 \neq i_2$?

Hint: The Cauchy Schwartz inequality states that

$$|\langle f_1(t), f_2(t)\rangle| \le \|f_1(t)\| \cdot \|f_2(t)\|$$

with equality holding iff $f_1(t) = c \cdot f_2(t)$. Also,[1] for non-negative $\{a_j, b_j\}_{j=1}^{N}$,

$$\frac{\sum_{j=1}^{N} a_j}{\sum_{j=1}^{N} b_j} \le \max_{1\le j\le N: a_j>0} \frac{a_j}{b_j}. \tag{2}$$

(c) (basic = 4 pts., extra = 2 pts.) Prove that $E[n_j^2] = \frac{N_0}{2}\|\phi_j(t)\|^2$, provided $w(t)$ is zero-mean white with PSD $\frac{N_0}{2}$, and the inner product of two (generally complex) functions f_1 and f_2 is defined as

$$\langle f_1(t), f_2(t)\rangle = \int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt.$$

Note: This is to confirm (1) can possibly hold.

Solution.

(a) Define the inner product of two (generally complex) functions f_1 and f_2 as

$$\langle f_1(t), f_2(t)\rangle = \int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt.$$

Then, we can equivalently transform the system to

$$y(T) = a\,\langle g(t), \phi(t)\rangle + \langle w(t), \phi(t)\rangle,$$

where $\phi(t) = h(T - t)$.

[1] Here, we allow $\frac{a_j}{b_j} = \infty$ to be an upper bound under $a_j > 0$ and $b_j = 0$.

(b) First, we note from the Cauchy-Schwartz inequality that

$$s_{i,j}^2 = |\langle s_i(t), \phi_j(t)\rangle|^2 \le \|s_i(t)\|^2 \, \|\phi_j(t)\|^2$$

Thus, using $E[|n_j|^2] = \beta\|\phi_j(t)\|^2$, we derive

$$\begin{aligned}
\eta_i &= \frac{\sum_{1\le j\le N: s_{i,j}\neq 0} |s_{i,j}|^2}{\sum_{1\le j\le N: s_{i,j}\neq 0}^{N} E[n_j]^2} \\
&\le \frac{\sum_{1\le j\le N: s_{i,j}\neq 0} \|s_i(t)\|^2 \, \|\phi_j(t)\|^2}{\sum_{1\le j\le N: s_{i,j}\neq 0} \beta \, \|\phi_j(t)\|^2} \quad \text{(Cauchy-Schwartz inequality)} \\
&\le \max_{1\le j\le N: s_{i,j}\neq 0} \frac{\|s_i(t)\|^2 \, \|\phi_j(t)\|^2}{\beta \, \|\phi_j(t)\|^2} \quad \text{(Use (2).)} \\
&= \frac{\|s_i(t)\|^2}{\beta},
\end{aligned}$$

where the upper bound has nothing to do with the choice of N and $\{\phi_j(t)\}_{j=1}^N$ (as long as there is at least one non-zero element in $\{s_{i,j}\}_{j=1}^N$). Consequently, if we can choose N and $\{\phi_j(t)\}_{j=1}^N$ such that $\eta_i = \frac{\|s_i(t)\|^2}{\beta}$ for each i (i.e., η_i achieves the upper bound), then η_{ave} is maximized. It is obvious that by letting $N = M$ and $\phi_j(t) = \frac{s_j(t)}{\|s_j(t)\|}$, we can equate both the Cauchy-Schwartz inequality and (2).

Note: Since I gave the wrong equality condition as $f_1(t) = c \cdot f_2(t)$, an answer of $\phi_j(t) = \frac{s_j^*(t)}{\|s_j(t)\|}$ is also regarded "correct."

(c)

$$\begin{aligned}
E[n_j^2] = E[(\langle w(t), \phi_j(t)\rangle)^2] &= E\left[\left(\int_{-\infty}^{\infty} w(t)\phi_j^*(t)dt\right)\left(\int_{-\infty}^{\infty} w(s)\phi_j^*(s)ds\right)^*\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[w(t)w^*(s)]\phi_j^*(t)\phi_j(s)\,dt\,ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(t-s)\phi_j^*(t)\phi_j(s)\,dt\,ds \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} \phi_j(t)\phi_j^*(t)\,dt \\
&= \frac{N_0}{2}\langle \phi_j(t), \phi_j(t)\rangle \\
&= \frac{N_0}{2}\|\phi_j(t)\|^2
\end{aligned}$$

2. A general correlative level coding scheme that follows

$$c_k = w_0 a_k + w_1 a_{k-1} + \cdots + w_{N-1} a_{k-N+1} \quad \text{(as shown in Slide 8-73)}$$

with $w_0 \neq 0$ may suffer serious error propagation if the estimate of a_k is determined via

$$\hat{a}_k = \frac{1}{w_0}(c_k - (w_1\hat{a}_{k-1} + \cdots + w_{N-1}\hat{a}_{k-N+1})).$$

This can be solved by the precoding technique.

(a) (basic = 4 pts., extra = 2 pts.) Subject to $N = 3$, can we design a precoder that makes b_k only a function of c_k for arbitrary w_0, w_1 and w_2 in $\{0, 1\}$, where

$$\begin{cases} b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} & \text{for } \alpha_j, \tilde{b}_k \in \{0, 1\}; \\ a_k = 2\tilde{b}_k - 1; \\ c_k = w_0 a_k + w_1 a_{k-1} + w_2 a_{k-2} \end{cases}$$

If your answer is positive, give the design. If your answer is negative, give a counterexample.

Hint: $\alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} = (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2$

(b) (basic = 4 pts., extra = 2 pts.) Subject to $N = 3$, can we design a precoder that makes b_k only a function of c_k for arbitrary integers w_0, w_1 and w_2, where

$$\begin{cases} b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \text{ for } \alpha_j, \tilde{b}_k \in \{0, 1\}; \\ a_k = 2\tilde{b}_k - 1; \\ c_k = w_0 a_k + w_1 a_{k-1} + w_2 a_{k-2} \end{cases}$$

If your answer is positive, give the design. If your answer is negative, give a counterexample.

Hint: $\alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} = (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2$

(c) (basic = 6 pts., extra = 2 pts.) Subject to $N = 3$, can we design a precoder that makes b_k only a function of c_k for arbitrary rationals w_0, w_1 and w_2, where

$$\begin{cases} b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \text{ for } \alpha_j, \tilde{b}_k \in \{0, 1\}; \\ a_k = 2\tilde{b}_k - 1; \\ c_k = w_0 a_k + w_1 a_{k-1} + w_2 a_{k-2} \end{cases}$$

If your answer is positive, give the design. If your answer is negative, give a counterexample.

Hint: $\alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} = (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2$

Solution.

(a) If $w_0 = w_1 = w_2 = 0$, then c_k is always zero. Hence, a two-value b_k cannot be made a function of (and uniquely recovered from) a single-value c_k.

Then, we proceed to work on the cases that at least one of w_0, w_1 and w_2 is non-zero. First, we derive

$$\begin{aligned} c_k &= w_0 a_k + w_1 a_{k-1} + w_2 a_{k-2} \\ &= w_0(2\tilde{b}_k - 1) + w_1(2\tilde{b}_{k-1} - 1) + w_2(2\tilde{b}_{k-2} - 1) \\ &= 2(w_0 \tilde{b}_k + w_1 \tilde{b}_{k-1} + w_2 \tilde{b}_{k-2}) - (w_0 + w_1 + w_2) \end{aligned}$$

Next we note

$$b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} = (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2$$

Thus, it is obvious that if $w_i \in \{0,1\}$, we can let $\alpha_i = w_i$ and obtain

$$\begin{aligned} b_k &= (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2 \\ &= (w_0 \tilde{b}_k + w_1 \tilde{b}_{k-1} + w_2 \tilde{b}_{k-2}) \bmod 2 \\ &= \left(\frac{c_k + (w_0 + w_1 + w_2)}{2} \right) \bmod 2 \end{aligned}$$

Consequently, if we know c_k, we can uniquely determine b_k.

Note: If $w_0 + w_1 + w_2$ is odd, then c_k is also odd. If, however, $w_0 + w_1 + w_2$ is even, then c_k is also even.

(b) If $w_0 = w_1 = w_2 = 0$, then c_k is always zero. Hence, a two-value b_k cannot be made a function of (and uniquely recovered from) a single-value c_k.

Then, we proceed to work on the cases that at least one of w_0, w_1 and w_2 is non-zero. First, we derive

$$\begin{aligned} c_k &= w_0 a_k + w_1 a_{k-1} + w_2 a_{k-2} \\ &= w_0(2\tilde{b}_k - 1) + w_1(2\tilde{b}_{k-1} - 1) + w_2(2\tilde{b}_{k-2} - 1) \\ &= 2(w_0 \tilde{b}_k + w_1 \tilde{b}_{k-1} + w_2 \tilde{b}_{k-2}) - (w_0 + w_1 + w_2) \end{aligned}$$

Next we note

$$b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} = (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2$$

Thus, if w_i is an integer, we can let $\alpha_i = w_i \bmod 2$ and obtain

$$\begin{aligned} b_k &= (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2 \\ &= (w_0 \tilde{b}_k + w_1 \tilde{b}_{k-1} + w_2 \tilde{b}_{k-2}) \bmod 2 \\ &= \left(\frac{c_k + (w_0 + w_1 + w_2)}{2} \right) \bmod 2 \end{aligned}$$

Consequently, if we know c_k, we can uniquely determine b_k.

Note: If $w_0 + w_1 + w_2$ is odd, then c_k is also odd. If, however, $w_0 + w_1 + w_2$ is even, then c_k is also even.

(c) If $w_0 = w_1 = w_2 = 0$, then c_k is always zero. Hence, a two-value b_k cannot be made a function of (and uniquely recovered from) a single-value c_k.

Then, we proceed to work on the cases that at least one of w_0, w_1 and w_2 is non-zero. First, we derive

$$\begin{aligned} c_k &= w_0 a_k + w_1 a_{k-1} + w_2 a_{k-2} \\ &= w_0(2\tilde{b}_k - 1) + w_1(2\tilde{b}_{k-1} - 1) + w_2(2\tilde{b}_{k-2} - 1) \\ &= \frac{2}{W}(W w_0 \tilde{b}_k + W w_1 \tilde{b}_{k-1} + W w_2 \tilde{b}_{k-2}) - (w_0 + w_1 + w_2), \end{aligned}$$

where W is an integer that makes $W w_0$, $W w_1$ and $W w_2$ integers. The existence of such integer W can be guaranteed because w_0, w_1 and w_2 are rational. Next we note

$$b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} = (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2$$

Thus, we can let $\alpha_i = (Ww_i) \bmod 2$ and obtain

$$\begin{aligned} b_k &= (\alpha_0 \tilde{b}_k + \alpha_1 \tilde{b}_{k-1} + \alpha_2 \tilde{b}_{k-2}) \bmod 2 \\ &= (Ww_0 \tilde{b}_k + Ww_1 \tilde{b}_{k-1} + Ww_2 \tilde{b}_{k-2}) \bmod 2 \\ &= \left(\frac{W(c_k + (w_0 + w_1 + w_2))}{2} \right) \bmod 2 \end{aligned}$$

Consequently, if we know c_k, we can uniquely determine b_k.

3. Continue from the previous problem. Given

$$b_k = \alpha_0 \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \quad \text{for } \alpha_j, \tilde{b}_k \in \{0, 1\}.$$

(a) (basic = 7 pts., extra = 3 pts.) Subject to $\alpha_0 = 1$ and $\Pr(b_k = 0) = \Pr(b_k = 1) = \frac{1}{2}$, determine $\Pr(\tilde{b}_k = 0)$ for arbitrary $\alpha_1, \alpha_2 \in \{0, 1\}$.

(b) (basic = 7 pts., extra = 3 pts.) Continue from (a). Show that $\tilde{b}_k$ is independent of $\{\tilde{b}_{k-i}\}_{i=1}^{k-1}$.
Hint: Show that $\Pr(\tilde{b}_k = \beta_k | \tilde{b}_{k-1} = \beta_{k-1}, \tilde{b}_{k-2} = \beta_{k-2}, \ldots, \tilde{b}_1 = \beta_1)$ has nothing to do with $\beta_{k-1}, \beta_{k-2}, \ldots, \beta_1$.

Solution.

(a) Since

$$b_k = \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2}$$

implies

$$b_k \oplus \underbrace{b_k \oplus \tilde{b}_k} = \tilde{b}_k \oplus \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus \underbrace{b_k \oplus \tilde{b}_k},$$

(equivalently, $\tilde{b}_k = \alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus b_k$) we obtain

$$\begin{aligned} \Pr(\tilde{b}_k = 0) &= \Pr(\alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus b_k = 0) \\ &= \Pr(\tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 0) \Pr(\alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus b_k = 0 | \tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 0) \\ &\quad + \Pr(\tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 1) \Pr(\alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus b_k = 0 | \tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 1) \\ &\quad + \Pr(\tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 0) \Pr(\alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus b_k = 0 | \tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 0) \\ &\quad + \Pr(\tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 1) \Pr(\alpha_1 \tilde{b}_{k-1} \oplus \alpha_2 \tilde{b}_{k-2} \oplus b_k = 0 | \tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 1) \\ &= \Pr(\tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 0) \Pr(b_k = 0) \\ &\quad + \Pr(\tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 1) \Pr(b_k = \alpha_2) \\ &\quad + \Pr(\tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 0) \Pr(b_k = \alpha_1) \\ &\quad + \Pr(\tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 1) \Pr(b_k = \alpha_1 \oplus \alpha_2) \\ &= \Pr(\tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 0) \frac{1}{2} + \Pr(\tilde{b}_{k-1} = 0 \text{ and } \tilde{b}_{k-2} = 1) \frac{1}{2} \\ &\quad + \Pr(\tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 0) \frac{1}{2} + \Pr(\tilde{b}_{k-1} = 1 \text{ and } \tilde{b}_{k-2} = 1) \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

(b)

$$\begin{aligned}
&\Pr(\tilde{b}_k=\beta_k|\tilde{b}_{k-1}=\beta_{k-1},\tilde{b}_{k-2}=\beta_{k-2},\ldots,\tilde{b}_1=\beta_1)\\
&= \Pr(\tilde{b}_k=\beta_k|\tilde{b}_{k-1}=\beta_{k-1},\tilde{b}_{k-2}=\beta_{k-2}) \quad (\text{Because } \tilde{b}_k=\alpha_1\tilde{b}_{k-1}\oplus\alpha_2\tilde{b}_{k-2}\oplus b_k)\\
&= \Pr(b_k=\beta_k\oplus\alpha_1\beta_{k-1}\oplus\alpha_2\beta_{k-2})\\
&= \frac{1}{2}
\end{aligned}$$

The above result holds regardless of $\{\beta_{k-i}\}_{i=1}^{k-1}$. Hence, $\tilde{b}_k$ is independent of $\{\tilde{b}_{k-i}\}_{i=1}^{k-1}$.

4. (Signal-space-based constellation design that minimizes the union bound) Suppose $\boldsymbol{x}=\boldsymbol{s}_m+\boldsymbol{n}$, where $1\le m\le M=3$, and $\boldsymbol{x}$, $\boldsymbol{s}_m$ and $\boldsymbol{n}$ are 2-dimensional vectors. Assume the prior probability for each $\boldsymbol{s}_m$ is 1/3 and $\boldsymbol{n}$ is zero-mean Gaussian distributed with covariance matrix $\sigma^2\mathbb{I}$, where $\mathbb{I}$ is the 2×2 identity matrix.

   (a) (basic = 4 pts., extra = 2 pts.) Subject to $\boldsymbol{s}_1+\boldsymbol{s}_2+\boldsymbol{s}_3=\boldsymbol{0}$, show that

$$\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2+\|\boldsymbol{s}_1-\boldsymbol{s}_3\|^2+\|\boldsymbol{s}_2-\boldsymbol{s}_3\|^2=3\|\boldsymbol{s}_1\|^2+3\|\boldsymbol{s}_2\|^2+3\|\boldsymbol{s}_3\|^2.$$

   (b) (basic = 4 pts., extra = 2 pts.) Let

$$P_2(\boldsymbol{s}_i,\boldsymbol{s}_j)\triangleq\int_{d_{i,j}/2}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}dx=\int_{-\infty}^{-d_{i,j}/2}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}dx=\Phi\left(-\frac{d_{i,j}}{2\sigma}\right),$$

   where $\Phi(\cdot)$ denotes the standard normal cdf, and $d_{i,j}=\|\boldsymbol{s}_i-\boldsymbol{s}_j\|$. Show that subject to $a^2+b^2=$ constant with non-negative a and b,

$$\Phi\left(-\frac{a}{2\sigma}\right)+\Phi\left(-\frac{b}{2\sigma}\right)$$

   is minimized when $a=b$.

   (c) (basic = 6 pts., extra = 2 pts.) Subject to a fixed unit average transmission power, i.e.,

$$\frac{1}{3}\sum_{m=1}^{3}\|\boldsymbol{s}_m\|^2=1,$$

   and $\sum_{m=1}^{3}\boldsymbol{s}_m=\boldsymbol{0}$, find the best $\{\boldsymbol{s}_m\}_{m=1}^{3}$ that minimizes the union bound

$$\frac{1}{3}\sum_{i=1}^{3}\sum_{j=1,j\ne i}^{3}P_2(\boldsymbol{s}_i,\boldsymbol{s}_j)=\frac{2}{3}\sum_{i=1}^{3}\sum_{j=i+1}^{3}P_2(\boldsymbol{s}_i,\boldsymbol{s}_j).$$

Solution.

(a)

$$\begin{aligned}
&\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2+\|\boldsymbol{s}_1-\boldsymbol{s}_3\|^2+\|\boldsymbol{s}_2-\boldsymbol{s}_3\|^2\\
&= 2\|\boldsymbol{s}_1\|^2+2\|\boldsymbol{s}_2\|^2+2\|\boldsymbol{s}_3\|^2-2\boldsymbol{s}_1\cdot\boldsymbol{s}_2-2\boldsymbol{s}_1\cdot\boldsymbol{s}_3-2\boldsymbol{s}_2\cdot\boldsymbol{s}_3\\
&= 2\|\boldsymbol{s}_1\|^2+2\|\boldsymbol{s}_2\|^2+2\|\boldsymbol{s}_3\|^2-(\boldsymbol{s}_1\cdot\boldsymbol{s}_2+\boldsymbol{s}_1\cdot\boldsymbol{s}_3)-(\boldsymbol{s}_2\cdot\boldsymbol{s}_1+\boldsymbol{s}_2\cdot\boldsymbol{s}_3)-(\boldsymbol{s}_3\cdot\boldsymbol{s}_1+\boldsymbol{s}_3\cdot\boldsymbol{s}_2)\\
&= 2\|\boldsymbol{s}_1\|^2+2\|\boldsymbol{s}_2\|^2+2\|\boldsymbol{s}_3\|^2-\boldsymbol{s}_1\cdot(\boldsymbol{s}_2+\boldsymbol{s}_3)-\boldsymbol{s}_2\cdot(\boldsymbol{s}_1+\boldsymbol{s}_3)-\boldsymbol{s}_3\cdot(\boldsymbol{s}_1+\boldsymbol{s}_2)\\
&= 2\|\boldsymbol{s}_1\|^2+2\|\boldsymbol{s}_2\|^2+2\|\boldsymbol{s}_3\|^2+\|\boldsymbol{s}_1\|^2+\|\boldsymbol{s}_2\|^2+\|\boldsymbol{s}_3\|^3\\
&= 3\|\boldsymbol{s}_1\|^2+3\|\boldsymbol{s}_2\|^2+3\|\boldsymbol{s}_3\|^2
\end{aligned}$$

(b) Let $a^2+b^2=C$. Then, the problem becomes to find a such that

$$\Phi\left(-\frac{a}{2\sigma}\right)+\Phi\left(-\frac{\sqrt{C-a^2}}{2\sigma}\right)$$

is minimized. Taking the derivative of the above equation, we obtain

$$-\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{a^2}{2\sigma^2}}+\frac{a}{2\sqrt{C-a^2}}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{C-a^2}{2\sigma^2}}=0, \tag{3}$$

which can be equivalently simplified to

$$\frac{a^2}{C-a^2}=e^{\frac{C-2a^2}{\sigma^2}}.$$

Noting that $\frac{a^2}{C-a^2}$ is monotonically increasing in $a^2\in[0,C]$ and $e^{\frac{C-2a^2}{\sigma^2}}$ is monotonically decreasing in $a^2\in[0,C]$, a unique solution of (3) is $a=\sqrt{\frac{C}{2}}$; hence, the solution satisfies $a=b$.

(c) Based on (a), the problem becomes the minimization of

$$P_2(\boldsymbol{s}_1,\boldsymbol{s}_2)+P_2(\boldsymbol{s}_1,\boldsymbol{s}_3)+P_2(\boldsymbol{s}_2,\boldsymbol{s}_3)=\Phi\left(-\frac{d_{1,2}}{2\sigma}\right)+\Phi\left(-\frac{d_{1,3}}{2\sigma}\right)+\Phi\left(-\frac{d_{2,3}}{2\sigma}\right)$$

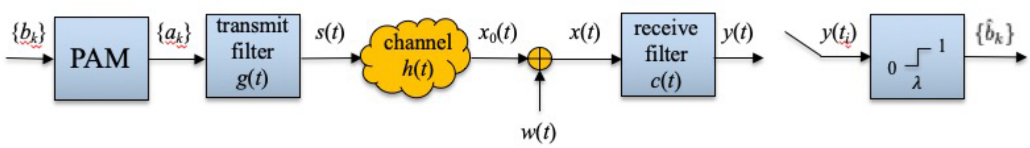
subject to $d_{1,2}^2+d_{1,3}^2+d_{2,3}^2=9$. Since for fixed $d_{1,2}$ (respectively, $d_{1,3}$ and $d_{2,3}$), the union bound is minimized when $d_{1,3}=d_{2,3}$ (respectively, $d_{1,2}=d_{2,3}$ and $d_{1,2}=d_{1,3}$), the optimal solution should satisfy

$$d_{1,2}=d_{2,3}=d_{1,3}=\sqrt{3}.$$

Thus, a best constellation design is

$$\begin{cases}\boldsymbol{s}_1=(1,0)\\ \boldsymbol{s}_2=(-\frac{1}{2},\frac{\sqrt{3}}{2})\\ \boldsymbol{s}_3=(-\frac{1}{2},-\frac{\sqrt{3}}{2})\end{cases}$$

5. (Matched filter and MMSE equalizer)



The above diagram shows that for sequence transmission,

$$x_0(t)=\sum_{k=-\infty}^{\infty}a_k\,q(t-kT_b),\ \text{where}\ q(t)=g(t)\star h(t),$$

and

$$y(t)=x_0(t)\star c(t)+w(t)\star c(t).$$

From Sample Problem 3 for Final Exam, the MMSE equalizer is given by

$$C_{\text{MMSE}}(f) = \frac{Q^*(f)}{S_q(f) + S_w(f)},$$

where from Slide 8-115, we derive

$$S_q(f) = Q^*(f) \cdot \frac{1}{T_b} \sum_{k=-\infty}^{\infty} Q\left(f + \frac{k}{T_b}\right)$$

provided that

$$R_q(\tau) \triangleq \sum_{k=-\infty}^{\infty} q(kT_b - \tau) q(kT_b)$$

is only a function of time difference τ.

(a) (basic = 4 pts., extra = 2 pts.) Does $G(f)H(f)C_{\text{MMSE}}(f)$ satisfy Nyquist's Criterion when $S_w(f) = 0$? Justify your answer.
Hint: From Sample Problem 3(f) for Final Exam, we obtain that

$$A_{\text{MMSE}}(f) = -\frac{|Q(f)|^2}{S_q(f) + S_w(f)},$$

implying

$$J_{i,\text{MMSE}} = 1 - \int_{-\infty}^{\infty} \frac{|Q(f)|^2}{S_q(f) + S_w(f)} df.$$

(b) (basic = 4 pts., extra = 2 pts.) When $S_w(f) = 0$, determine a $G(f)$ that minimizes the minimum mean square error $J_{i,\text{MMSE}}$, subject to the constraint that $G(f) = 0$ for $f > \frac{1}{2T_b}$.
Hint: $G(f) = 0$ for $f > \frac{1}{2T_b}$ implies

$$Q^*(f) \cdot \frac{1}{T_b} \sum_{k=-\infty}^{\infty} Q\left(f + \frac{k}{T_b}\right) = \frac{1}{T_b}|Q(f)|^2$$

for $|f| < \frac{1}{2T_b}$ and check whether $J_{i,\text{MMSE}} = 0$ when $S_w(f) = 0$.

(c) (basic = 6 pts., extra = 2 pts.) When $S_w(f) > 0$, subject to bandlimited constraint of $G(f) = 0$ for $f > \frac{1}{2T_b}$, and fixed transmission power:

$$\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{1}{T_b}|G(f)|^2 df = P,$$

find the $G(f)$ that minimizes the minimum mean square error $J_{i,\text{MMSE}}$.
Hint: $G(f) = 0$ for $f > \frac{1}{2T_b}$ implies

$$Q^*(f) \cdot \frac{1}{T_b} \sum_{k=-\infty}^{\infty} Q\left(f + \frac{k}{T_b}\right) = \frac{1}{T_b}|Q(f)|^2$$

for $|f| < \frac{1}{2T_b}$ and apply the Cauchy-Schwarz inequality

Solution.

(a) Since

$$\begin{aligned} P(f) &= G(f)H(f)C_{\text{MMSE}}(f) \\ &= Q(f)C_{\text{MMSE}}(f) \\ &= Q(f)\cdot\frac{Q^*(f)}{\underbrace{S_q(f)}_{=Q^*(f)\cdot\frac{1}{T_b}\sum_{k=-\infty}^{\infty}Q\left(f+\frac{k}{T_b}\right)}+\underbrace{S_w(f)}_{=0}} \\ &= \frac{Q(f)}{\frac{1}{T_b}\sum_{k=-\infty}^{\infty}Q\left(f+\frac{k}{T_b}\right)}, \end{aligned}$$

we derive

$$\begin{aligned} \sum_{\ell=-\infty}^{\infty}P\left(f-\frac{\ell}{T_b}\right) &= \sum_{\ell=-\infty}^{\infty}\frac{Q\left(f-\frac{\ell}{T_b}\right)}{\frac{1}{T_b}\sum_{k=-\infty}^{\infty}Q\left(\left(f-\frac{\ell}{T_b}\right)+\frac{k}{T_b}\right)} \\ &= \frac{\sum_{\ell=-\infty}^{\infty}Q\left(f-\frac{\ell}{T_b}\right)}{\frac{1}{T_b}\sum_{j=-\infty}^{\infty}Q\left(f-\frac{j}{T_b}\right)} \quad (\text{Set } j=\ell-k.) \\ &= T_b. \end{aligned}$$

Thus, adopting $C_{\text{MMSE}}(f)$ will automatically satisfy Nyquist's Criterion when $S_w(f)$ is reduced to zero.

(b) Noting that $Q(f)=G(f)H(f)=0$ for $|f|>\frac{1}{2T_b}$ is also bandlimited, we derive

$$\begin{aligned} J_{i,\text{MMSE}} &= 1-\int_{-\infty}^{\infty}\frac{|Q(f)|^2}{S_q(f)+S_w(f)}df \\ &= 1-\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}}\frac{|Q(f)|^2}{Q^*(f)\cdot\frac{1}{T_b}\sum_{k=-\infty}^{\infty}Q\left(f+\frac{k}{T_b}\right)+\underbrace{S_w(f)}_{=0}}df \\ &= 1-\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}}\frac{|Q(f)|^2}{\frac{1}{T_b}|Q(f)|^2}df \\ &= 1-\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}}T_b df = 0. \end{aligned}$$

The zero minimum mean square error can be achieved by any (bandlimited) $G(f)$.
Note: Since the transmission is noiseless, the minimum mean square error should be zero as anticipated.

(c) Noting that $Q(f) = G(f)H(f) = 0$ for $|f| > \frac{1}{2T_b}$ is also bandlimited, we derive

$$\begin{aligned}
J_{i,\text{MMSE}} &= 1 - \int_{-\infty}^{\infty} \frac{|Q(f)|^2}{S_q(f) + S_w(f)} df \\
&= 1 - \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{|Q(f)|^2}{Q^*(f) \cdot \frac{1}{T_b} \sum_{k=-\infty}^{\infty} Q\left(f + \frac{k}{T_b}\right) + S_w(f)} df \\
&= 1 - \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{|Q(f)|^2}{\frac{1}{T_b}|Q(f)|^2 + S_w(f)} df \\
&= 1 - \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \left(T_b - \frac{T_b S_w(f)}{\frac{1}{T_b}|Q(f)|^2 + S_w(f)} \right) df \\
&= T_b \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{S_w(f)}{\frac{1}{T_b}|G(f)|^2|H(f)|^2 + S_w(f)} df \quad \text{(Because } Q(f) = G(f)H(f).\text{)} \\
&= T_b \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{\frac{S_w(f)}{|H(f)|^2}}{\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2}} df
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\left(\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \sqrt{\frac{S_w(f)}{|H(f)|^2}} df \right)^2 &= \left(\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \sqrt{\frac{\frac{S_w(f)}{|H(f)|^2}}{\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2}}} \cdot \sqrt{\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2}} df \right)^2 \\
&\leq \left(\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{\frac{S_w(f)}{|H(f)|^2}}{\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2}} df \right) \left(\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \left(\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2} \right) df \right) \\
&= \left(\frac{1}{T_b} J_{i,\text{MMSE}} \right) \left(P + \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{S_w(f)}{|H(f)|^2} df \right).
\end{aligned}$$

Thus,

$$J_{i,\text{MMSE}} \geq \frac{T_b \left(\int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \sqrt{\frac{S_w(f)}{|H(f)|^2}} df \right)^2}{P + \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \frac{S_w(f)}{|H(f)|^2} df}.$$

Equality holds when for constant c,

$$c \cdot \sqrt{\frac{\frac{S_w(f)}{|H(f)|^2}}{\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2}}} = \sqrt{\frac{1}{T_b}|G(f)|^2 + \frac{S_w(f)}{|H(f)|^2}}.$$

Accordingly,

$$|G_{\text{optimal}}(f)|^2 = T_b \left(c \frac{\sqrt{S_w(f)}}{|H(f)|} - \frac{S_w(f)}{|H(f)|^2} \right).$$

Note: The constant c must be the one to fulfill

$$\int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} \frac{1}{T_b} |G_{\text{optimal}}(f)|^2 df = P.$$