Part 6 Pulse Modulation, Quantization and Line Coding

analog modulation

(continuous in both time and value)

→ *pulse modulation*

(discrete in time but could be continuous in value)

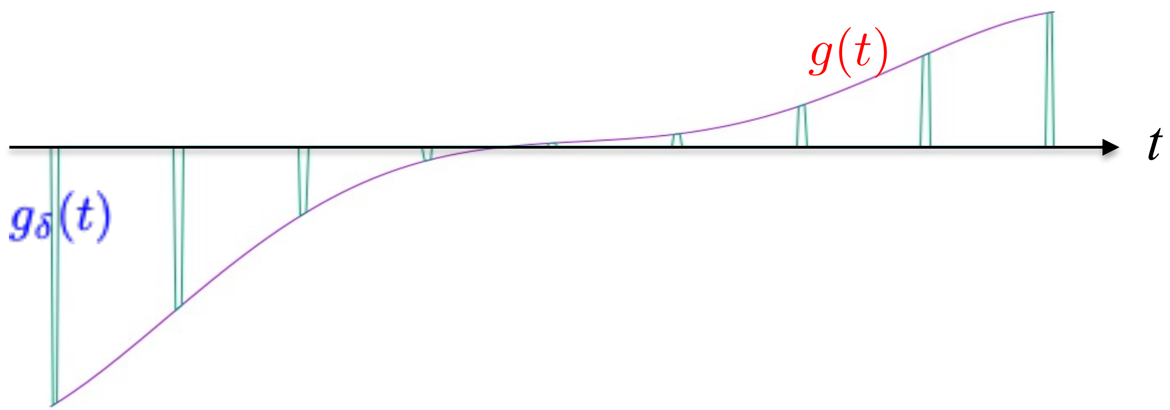
→ *digital modulation*

(discrete in both time and value)

Pulse Modulation

- Families of pulse modulation
  - Analog pulse modulation
    - A periodic pulse train is used as carriers (similar to sinusoidal carriers)
    - Some characteristic feature of each pulse, such as amplitude, duration, or position, is varied in a continuous matter in accordance with the sampled message signal.
  - Digital pulse modulation
    - Some characteristic feature of carriers is varied in a digital manner in accordance with the sampled, digitized message signal.

Sampling Theorem



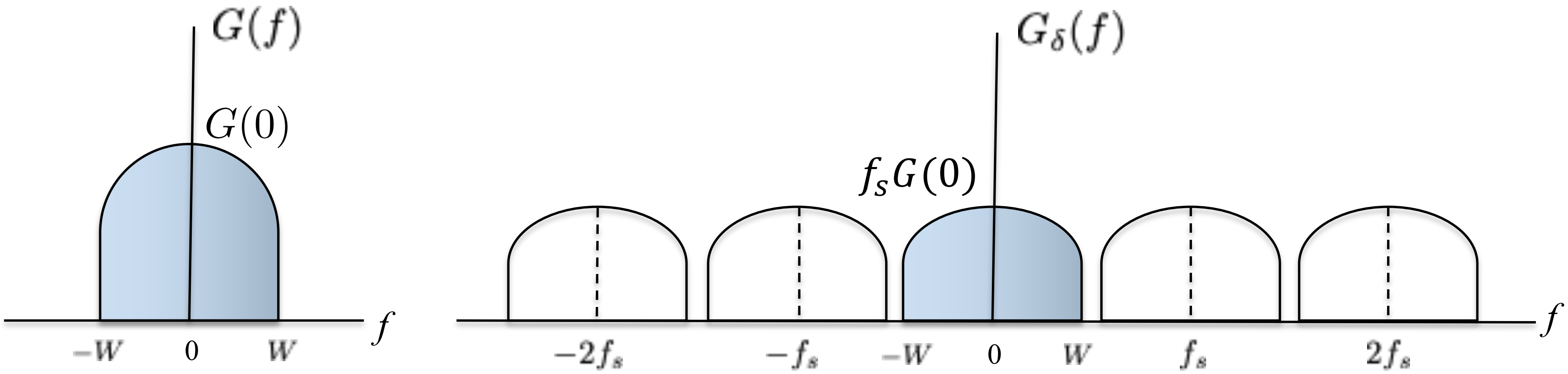
- T_s sampling period
- $f_s = 1/T_s$ sampling rate

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\int_{-\infty}^{\infty} \delta(t - nT_s)\exp(-j2\pi ft)dt = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nT_s f)$$

Sampling Theorem

- Given: $G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nT_s f)$
- Claim: $G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$



In this figure, $f_s > 2W$.

Spectrum of Sampled Signal

Proof:

Let $L(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$, and notice that it is periodic with period f_s.

$\Rightarrow$ By Fourier Series Expansion,

$$L(f) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j2\pi \frac{n}{f_s} f \right), \text{where } c_n = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} L(f) \exp\left(-j2\pi \frac{n}{f_s} f \right) df$$

$$\Rightarrow \; c_n = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} L(f) \exp\left(-j \frac{2\pi n}{f_s} f \right) df$$

$$= \int_{-f_s/2}^{f_s/2} \left(\sum_{m=-\infty}^{\infty} G(f - mf_s) \right) \exp\left(-j \frac{2\pi n}{f_s} f \right) df$$

$$c_n = \sum_{m=-\infty}^{\infty} \int_{-f_s/2}^{f_s/2} G(f - mf_s) \exp\left(-j\frac{2\pi n}{f_s} f\right) df, \quad s = f - mf_s$$

$$= \sum_{m=-\infty}^{\infty} \int_{-f_s/2-mf_s}^{f_s/2-mf_s} G(s) \exp\left(-j\frac{2\pi n}{f_s}(s + mf_s)\right) ds$$

$$= \sum_{m=-\infty}^{\infty} \int_{-f_s/2-mf_s}^{f_s/2-mf_s} G(s) \exp\left(-j\frac{2\pi n}{f_s} s\right) ds$$

$$= \int_{-\infty}^{\infty} G(s) \exp\left(-j\frac{2\pi n}{f_s} s\right) ds$$

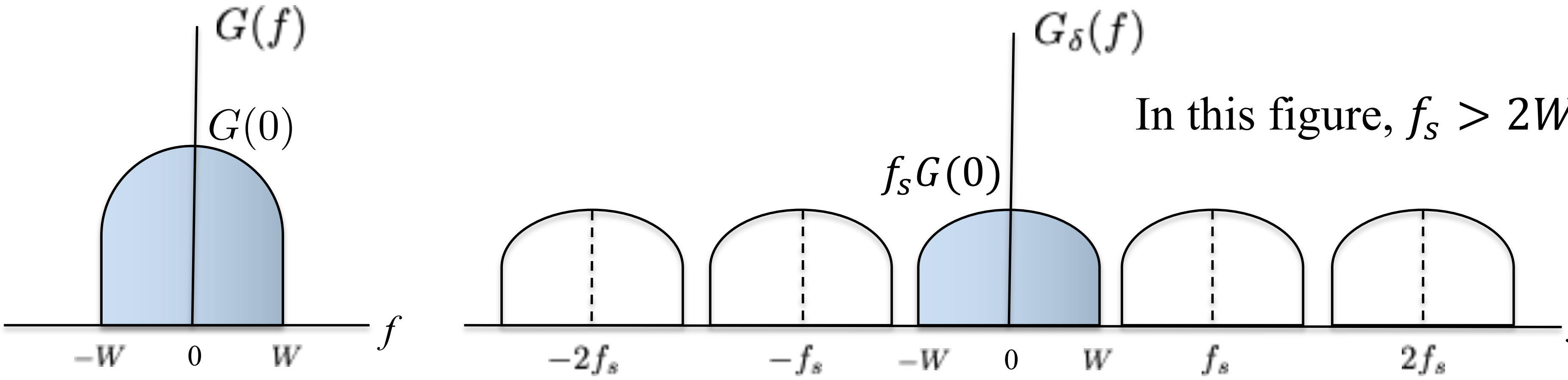
$$= g(-nT_s)$$

$$\Rightarrow L(f) = \sum_{n=-\infty}^{\infty} g(-nT_s) \exp\left(j2\pi\frac{n}{f_s} f\right)$$

$$= \sum_{m=-\infty}^{\infty} g(mT_s) \exp\left(-j2\pi mT_s f\right), \text{ where } m = -n.$$

Q.E.D.

First Important Conclusion from Sampling



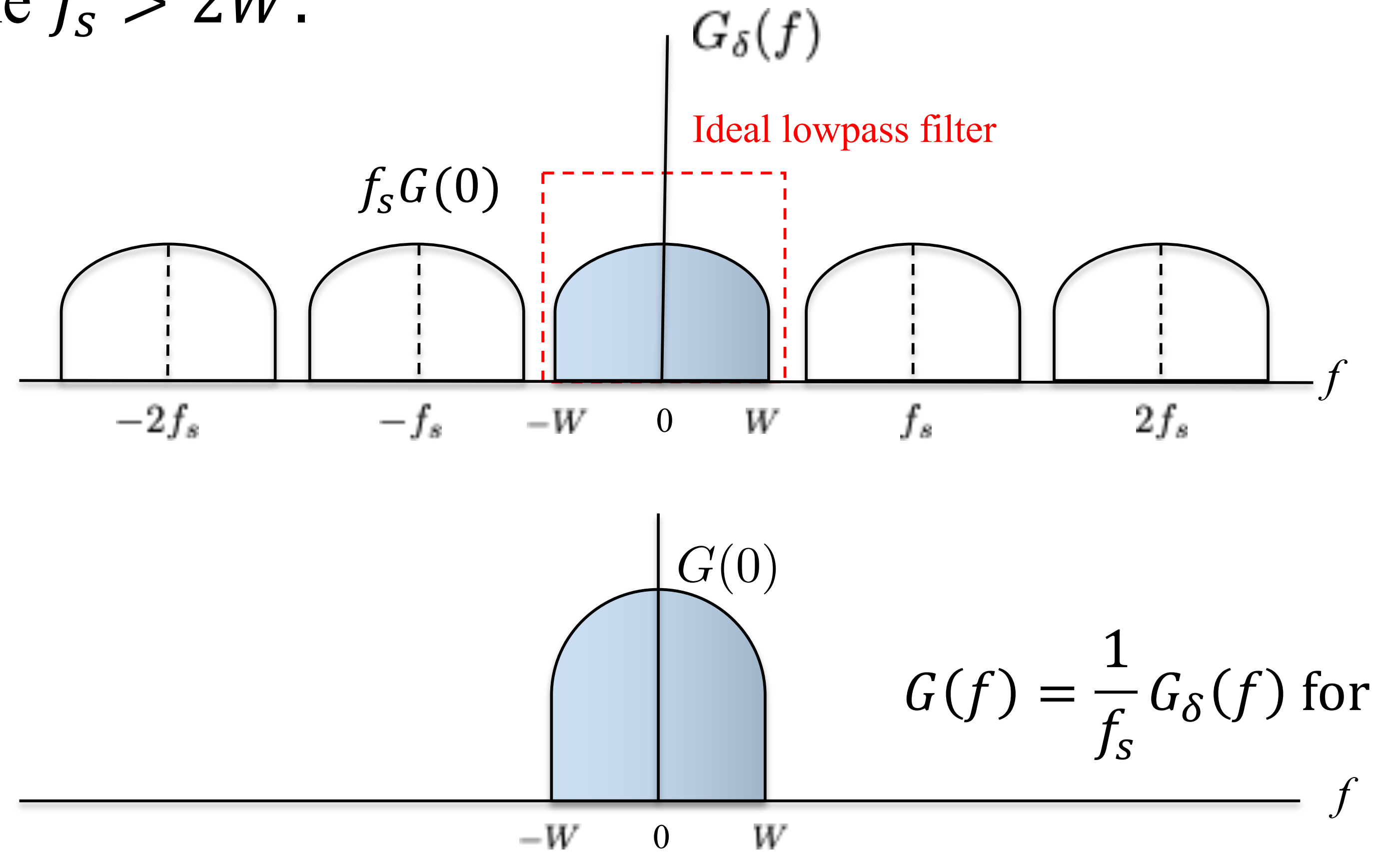
In this figure, $f_s > 2W$.

- *Uniform sampling at the time domain* results in *a periodic spectrum with a period equal to the sampling rate.*

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \Rightarrow G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$$

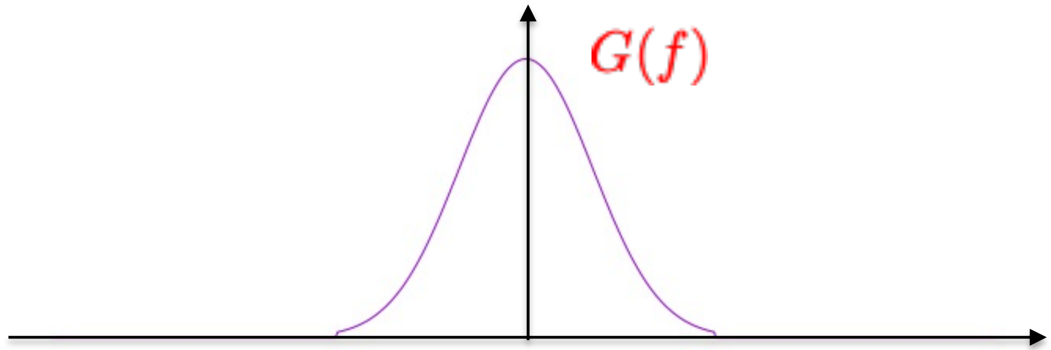
Reconstruction from Sampling

Take $f_s > 2W$.

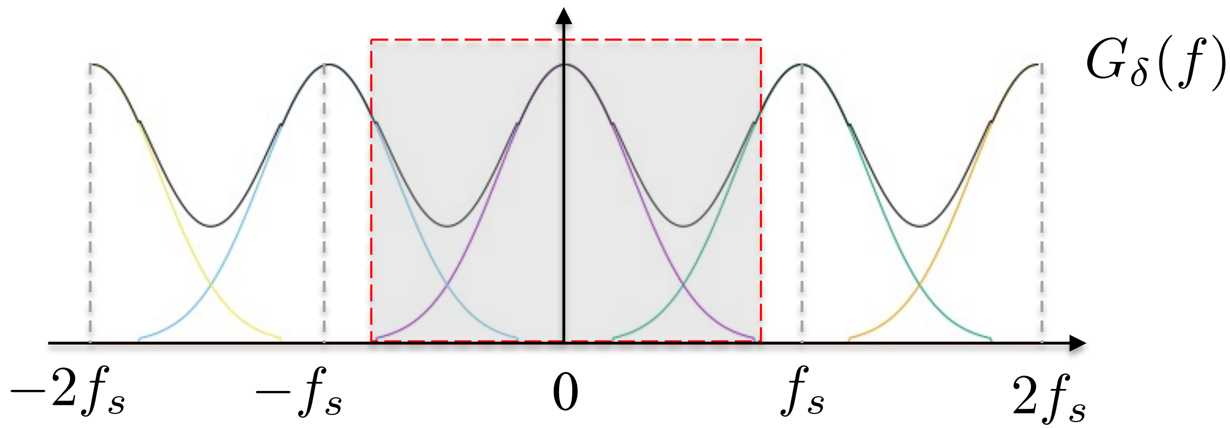


$$G(f) = \frac{1}{f_s} G_\delta(f) \text{ for } |f| \le W.$$

Aliasing due to Sampling



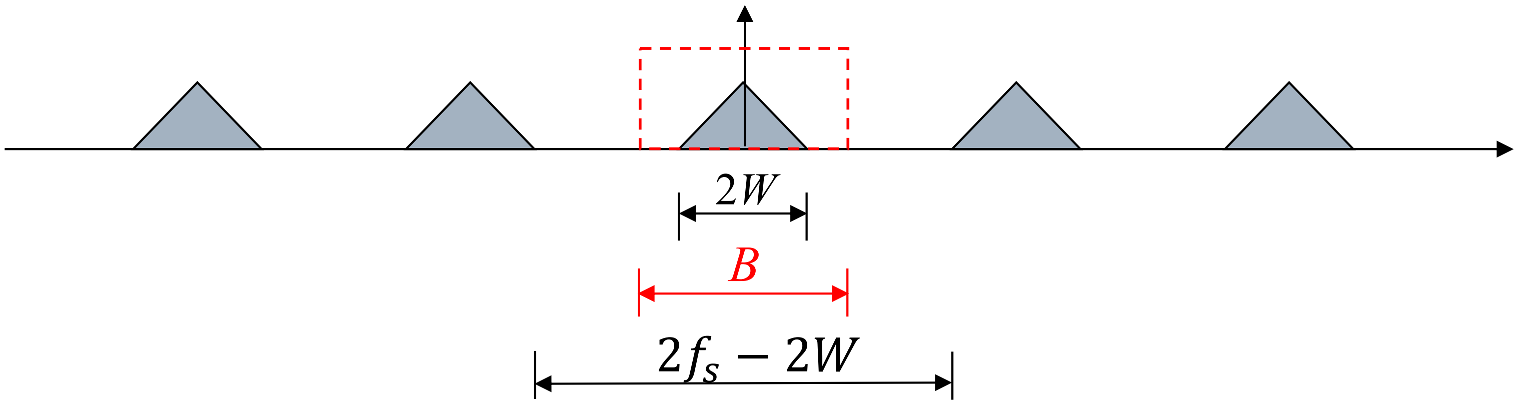
When $f_s < 2W$, $G(f)$ cannot be reconstructed by undersampled samples.



Second Important Conclusion from Sampling

- A band-limited signal of finite energy with bandwidth W can be completely described by its samples of sampling rate $f_s \geq 2W$.
  - $2W$ is commonly referred to as the *Nyquist rate*.

- How to reconstruct a band-limited signal from its samples?



$$
\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\mathrm{d}f \\
&= \int_{-B/2}^{B/2} G(f)e^{j2\pi ft}\mathrm{d}f \quad \text{(Because } B \geq 2W\text{)} \\
&= \int_{-B/2}^{B/2} \left(\frac{1}{f_s} \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi nT_s f} \right) e^{j2\pi ft}\mathrm{d}f \\
&\qquad \text{(Because } G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi nT_s f} \text{ and } G(f) = \frac{1}{f_s}G_\delta(f) \text{ for } |f| \leq W \leq \frac{B}{2} \\
&= \frac{1}{f_s} \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-B/2}^{B/2} e^{j2\pi f(t-nT_s)}\mathrm{d}f \\
&= \sum_{n=-\infty}^{\infty} g(nT_s) \cdot BT_s \operatorname{sinc}(B(t-nT_s))
\end{aligned}
$$

As a result, $BT_s\text{sinc}(B(t-nT_s))$ plays the role of an interpolation function.

Band-Unlimited Signals

- The signal encountered in practice is often not strictly band-limited.
- Hence, there is always "aliasing" after sampling.
- To combat the effects of aliasing, a low-pass *anti-aliasing* filter is used to attenuate the frequency components outside $[-f_s/2, f_s/2]$.
- In this case, the signal after passing the anti-aliasing filter is often treated as bandlimited with bandwidth $W = f_s/2$.

Interpolation in terms of Filtering

- Observe that

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\, BT_s \operatorname{sinc}(B(t - nT_s))$$

is indeed a **convolution** between $g_\delta(t)$ and $BT_s \operatorname{sinc}(Bt)$.

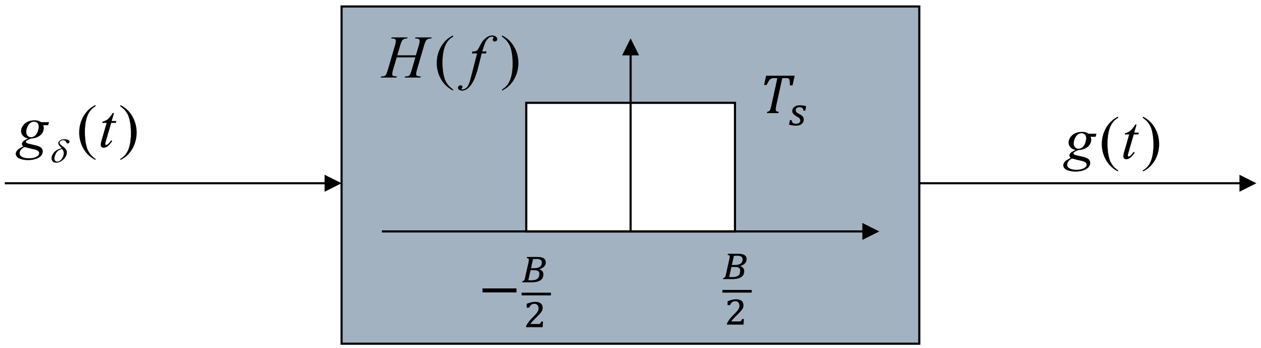
$$\begin{aligned}
g_\delta(t) \star BT_s \operatorname{sinc}(Bt) &= \int_{-\infty}^{\infty} g_\delta(\tau) \cdot BT_s \operatorname{sinc}(B(t-\tau)) d\tau \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} g(nT_s) \cdot \delta(\tau - nT_s) \right) BT_s \operatorname{sinc}(B(t-\tau)) d\tau \\
&= \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s) BT_s \operatorname{sinc}(B(t-\tau)) d\tau
\end{aligned}$$

(Continue from the previous slide.)

$$g_\delta(t) * BT_s \text{sinc}(Bt) = \sum_{n=-\infty}^{\infty} g(nT_s)\, BT_s\, \text{sinc}(B(t - nT_s))$$

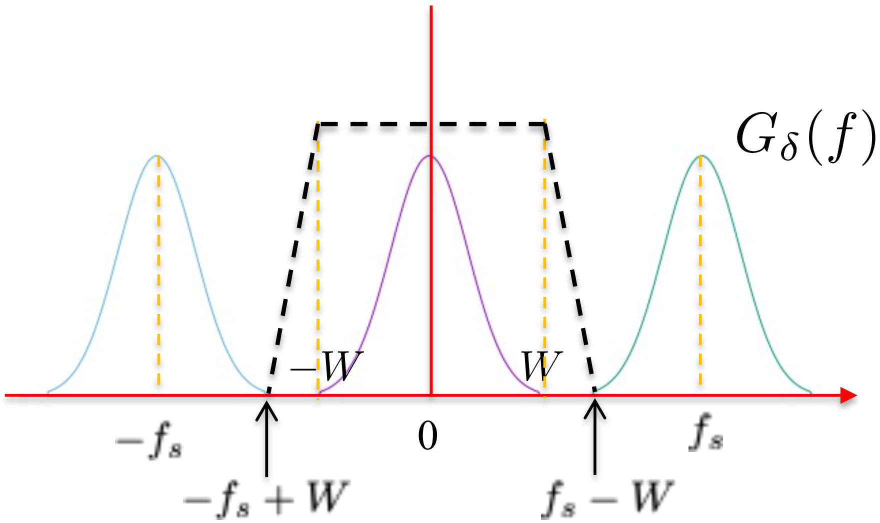
$\Longrightarrow$Reconstruction (interpolation) filter $h(t) = BT_s \text{sinc}(Bt)$

$\Longrightarrow H(f) = T_s \text{rect}\left(\frac{f}{B}\right)$

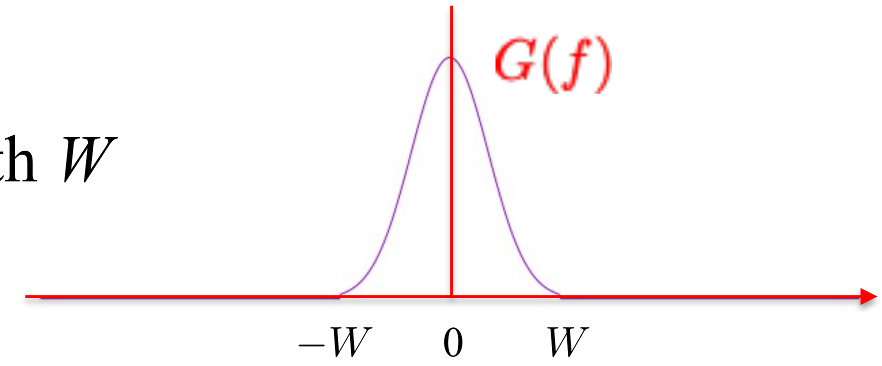


Physical Realization of Reconstruction Filter

- An ideal lowpass filter is not physically realizable.
- Instead, we can use an anti-aliasing filter of bandwidth W, and use a sampling rate $f_s > 2W$. Then, the spectrum of a reconstruction filter can be shaped like:

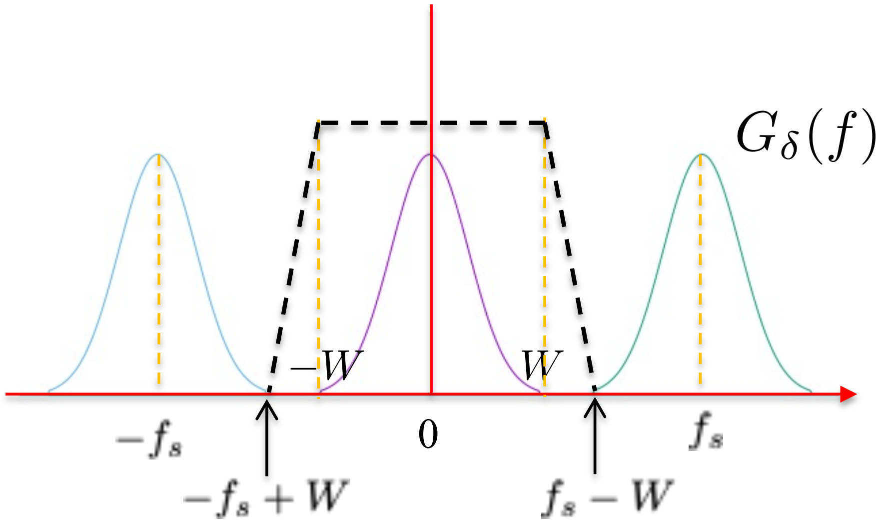


Signal spectrum with bandwidth W



Signal spectrum after sampling with $f_s > 2W$

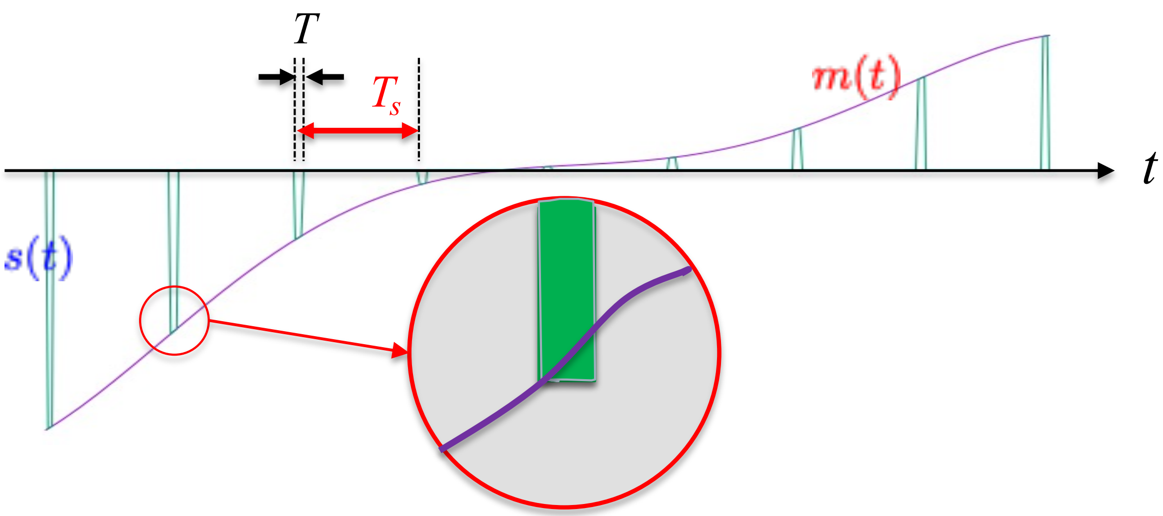
The physically realizable reconstruction filter



$$g_\delta(t) * h_{\text{relizable}}(t) \rightleftharpoons G_\delta(f)H_{\text{realizable}}(f) = G_\delta(f)H_{\text{ideal}}(f) \rightleftharpoons g_\delta(t) * h_{\text{ideal}}(t)$$

Pulse-Amplitude Modulation (PAM)

- PAM
  - The amplitude of regularly spaced pulses is varied in proportion to the corresponding sample values of a continuous message signal.



*Notably, the top of each pulse is maintained **flat**. So, this is PAM, not natural-sampling for which the message signal is directly multiplied by a periodic train of rectangular pulses.*

Pulse-Amplitude Modulation (PAM)

- The operation of generating a PAM modulated signal is often referred to as "sample and hold."
- This "sample and hold" process can also be analyzed through "filtering technique."

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) = m_\delta(t) * h(t)$$

$$\text{where } h(t) = \begin{cases} 1, & 0 < t < T \\ 1/2, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases} \text{ and } m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s).$$

Pulse-Amplitude Modulation (PAM)

- By taking "filtering" standpoint, the spectrum of $S(f)$ can be derived as:

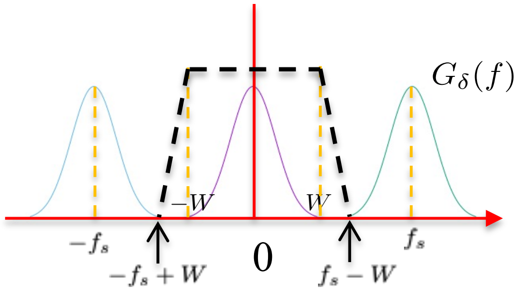
$$S(f) = M_\delta(f)H(f)$$

$$= \left(f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) \right) H(f)$$

$$= f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) H(f)$$

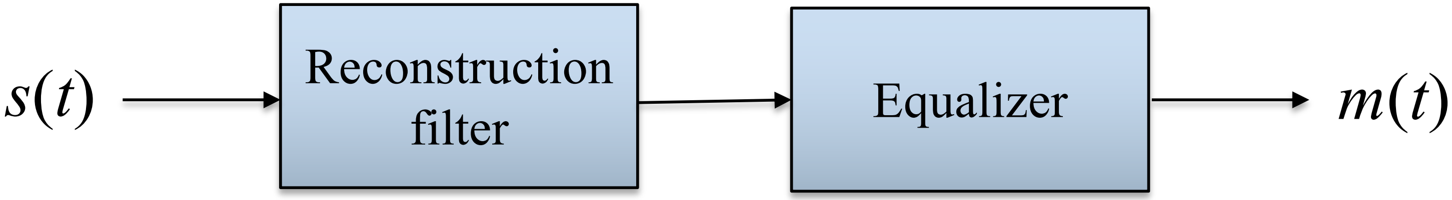
  - $M(f)$ is the message signal with bandwidth W (or having experienced an anti-aliasing filter of bandwidth W).
  - $f_s \geq 2W$.

Pulse-Amplitude Modulation (PAM)



(over the range $[-W, W]$ of $M(f)$)

$$\frac{1}{H(f)}$$



$$S(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)H(f)$$

$$= f_s M(f)H(f) + f_s \sum_{|k|\geq 1} M(f - kf_s)H(f)$$

$$\xrightarrow{\text{Reconstruction Filter}} M(f)H(f) \xrightarrow{\text{Equalizer}} M(f)$$

Feasibility of Equalizer Filter

- The distortion of $M(f)$ is due to $M(f)H(f)$,

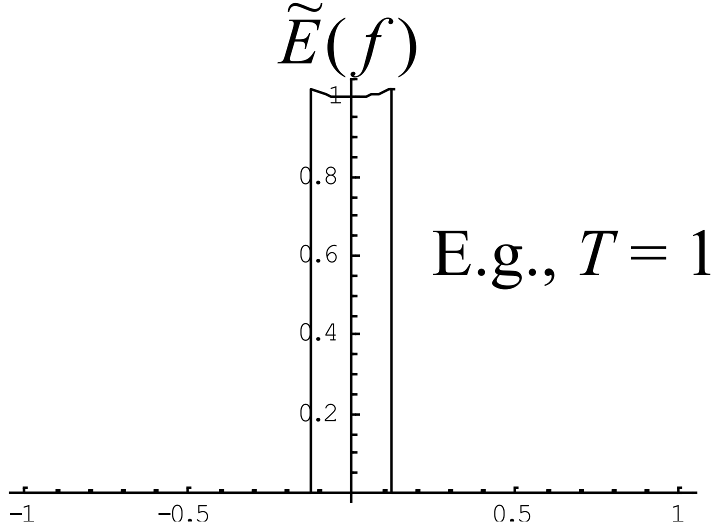
$$\text{where } h(t) = \begin{cases} 1, & 0 < t < T \\ 1/2, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases} \quad \text{or} \quad H(f) = T\operatorname{sinc}(fT)\exp(-j\pi fT)$$

$$\Rightarrow E(f) = \begin{cases} \dfrac{1}{H(f)} = \dfrac{1}{T\operatorname{sinc}(fT)}\exp(j\pi fT), & |f| \le W \\ 0, & \text{otherwise} \end{cases}$$

Question: Is the above $E(f)$ feasible or realizable?

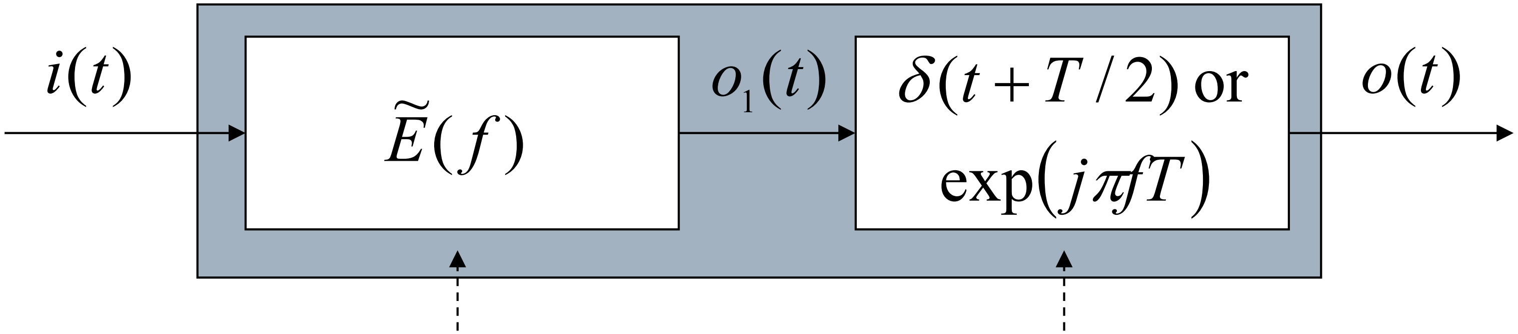
$$\boxed{\frac{1}{T} > \frac{1}{T_s} = f_s > 2W.}$$

$$\widetilde{E}(f) = \begin{cases} \dfrac{1}{T\operatorname{sinc}(fT)}, & |f| \le W \\ 0, & \text{otherwise} \end{cases}$$



E.g., $T = 1$, $W = 1/8$.

This gives an equalizer:

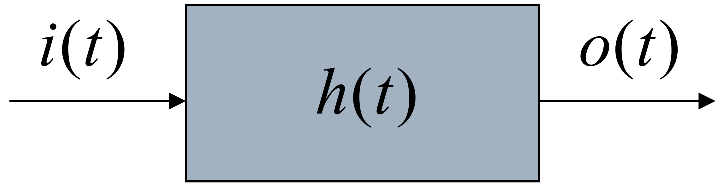


A lowpass filter **non-realizable! Why?**

Because "$o_1(t) = 0$ for $t < 0$" does not imply "$o(t) = 0$ for $t < 0$."

Feasibility of Equalizer Filter

- Causal



  - A reasonable assumption for a feasible linear filter system is that:

For any $i(t)$ satisfying $i(t) = 0$ for $t < 0$, we have $o(t) = 0$ for $t < 0$.

  - A necessary and sufficient condition for the above assumption to hold is that $h(t) = 0$ for $t < 0$.

Aperture Effect

- The distortion of $M(f)$ due to $M(f)H(f)$

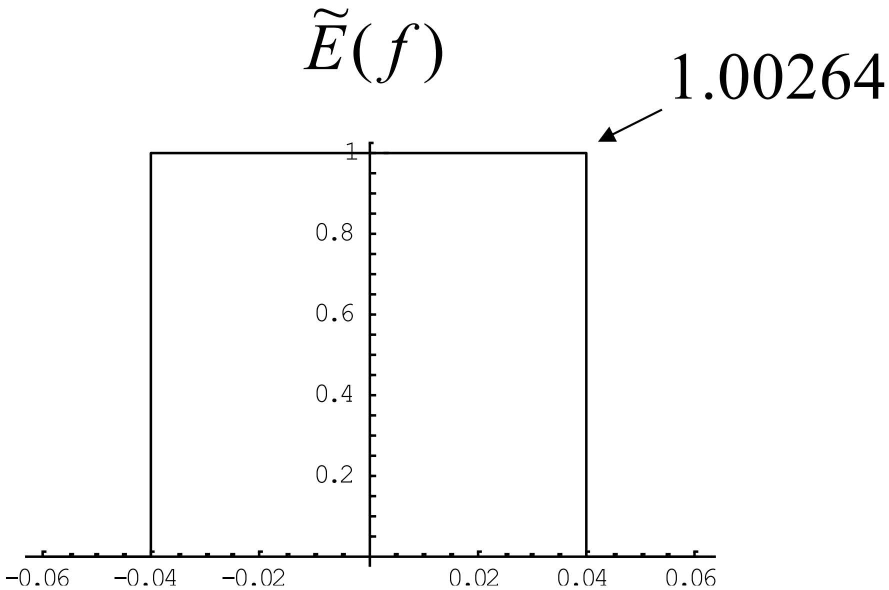
$$\text{where } h(t) = \begin{cases} 1, & 0 < t < T \\ 1/2, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases} \text{ or } H(f) = T\text{sinc}(fT)\exp(-j\pi fT)$$

is very similar to the distortion caused by the finite size of the scanning aperture in television. So, it is named the *aperture effect*.

- If $T/T_s \leqq 0.1$, the amplitude distortion is less than 0.5%; hence, the equalizer may not be necessary.

$$\widetilde{E}(f) = \begin{cases} \dfrac{1}{T\operatorname{sinc}(fT)}, & |f| \le W \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \frac{1}{T} > \frac{1}{T_s} = f_s > 2W.$$

$$\Rightarrow \widetilde{E}(f) = \begin{cases} \dfrac{1}{\operatorname{sinc}(f)}, & |f| \le 0.04 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } T = 1, T_s = 10, W = 0.04$$



Pulse-Amplitude Modulation

- Final notes on PAM
  - PAM is rather stringent in its system requirement, such as short duration of pulse.
  - Also, the noise performance of PAM may not be sufficient for long distance transmission.
  - Accordingly, PAM is often used as a mean of message processing for time-division multiplexing, from which conversion to some other form of pulse modulation is subsequently made. Details will be discussed in Part 7 (Section 3.9).

Other Forms of Pulse Modulation

- Pulse-Duration Modulation (or Pulse-Width Modulation)
  - Samples of the message signal are used to vary the duration of the pulses.
- Pulse-Position Modulation
  - The position of a pulse relative to its unmodulated time of occurrence is varied in accordance with the message signal.

Pulse trains

PDM

PPM

PDM and PPM

- Comparisons between PDM and PPM
  - PPM is more power efficient because excessive pulse duration consumes considerable power.
- Final note
  - It is expected that PPM is immune to additive noise, since additive noise only perturbs the amplitude of the pulses rather than the positions.
  - However, since the pulse cannot be made perfectly rectangular in practice (namely, there exists a non-zero transition time in pulse edge), the detection of pulse positions is somehow still affected by additive noise.

See Slide 5-46 figure-of-metric $\propto D^2 = \left(\frac{B_{T,\text{Carson}}}{2W} - 1\right)^2$

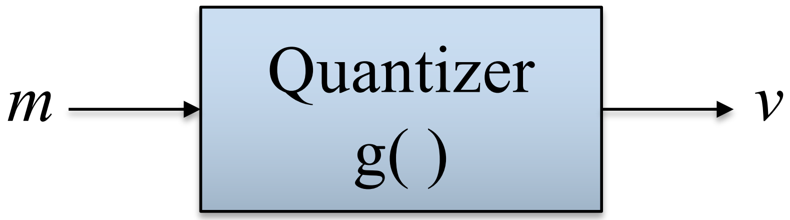
Trade-Off between Bandwidth and Performance

- PPM seems to be a better form for analog pulse modulation from noise performance standpoint. However, its noise performance is very similar to (analog) FM modulation as:
  - Its figure of merit is proportional to the square of transmission bandwidth (i.e., $1/T$) normalized with respect to the message bandwidth (W). $(I.e., B_n = B_T / W)$
  - There exists a threshold effect as SNR is reduced.

- Question: Can we do better than the "*square*" law in figure-of-merit improvement? Answer: Yes, by means of Digital Communication, we can realize an "*exponential*" law (with respect to error rates)!

Quantization

- Transform the continuous-amplitude $m = m(nT_s)$ to discrete approximate amplitude $v = v(nT_s)$

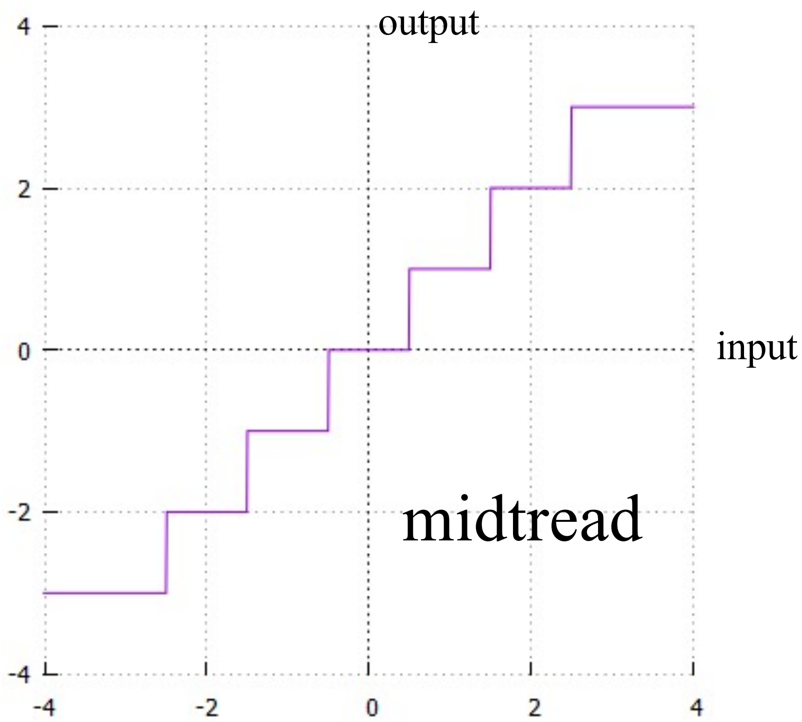


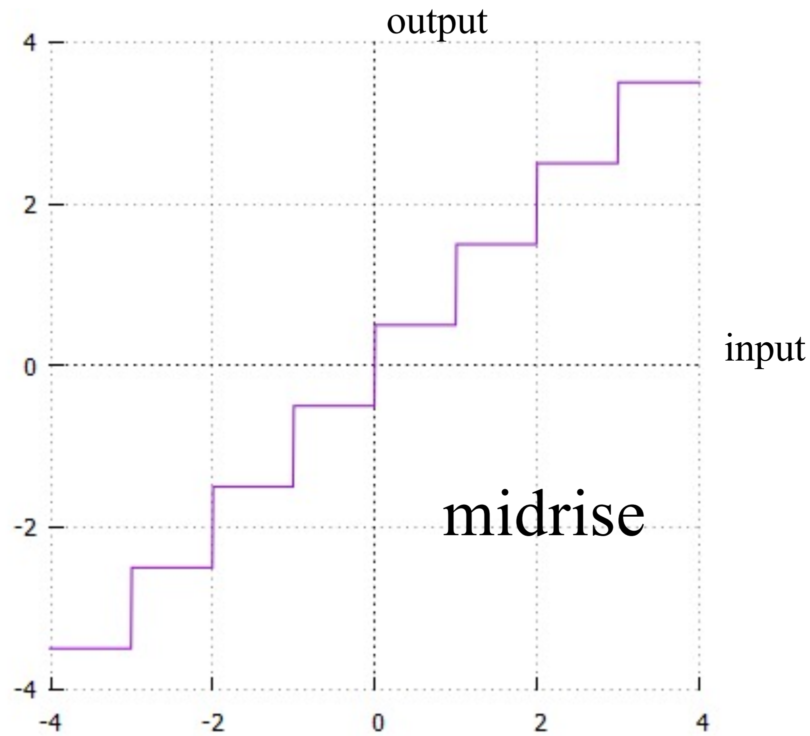
- Such a discrete approximate is adequately good in the sense that any human ear or eye can detect only finite intensity differences.

Quantization

- We may drop the time instance nT_s for convenience, when the quantization process is *memoryless* and *instantaneous* (hence, the quantization at time nT_s is not affected by earlier or later samples of the message signal.)
- Types of quantization
  - Uniform
    - Quantization step sizes are of equal length.
  - Non-uniform
    - Quantization step sizes are not of equal length.

- An alternative classification of quantization
  - Midtread
  - Midrise

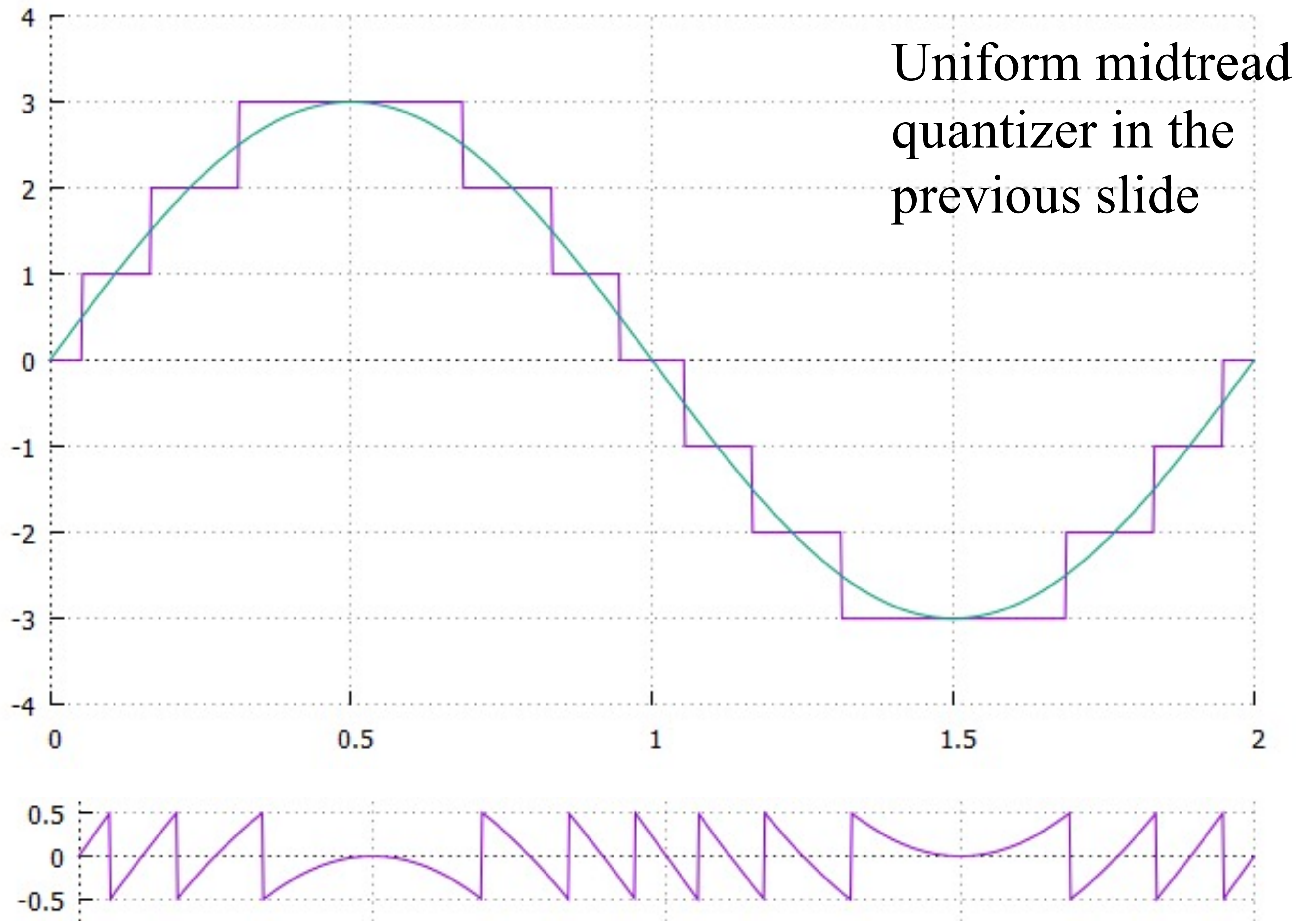




Quantization Noise

$m(t) = 3\sin(\pi * t)$

$m(t) - g(m(t))$



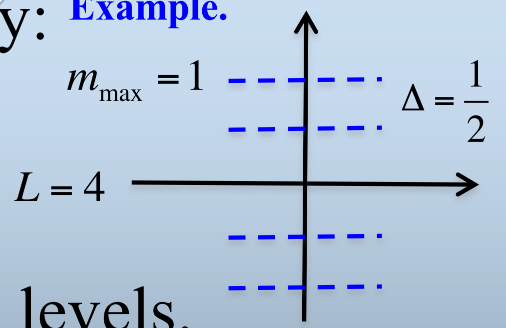
Quantization Noise

- Define the quantization noise to be $Q = M - V = M - g(M)$, where $g(\ )$ is the quantizer.
- Let the message M be uniformly distributed in $(-m_{\max}, m_{\max})$. So, M has zero mean.
- Assume $g(\ )$ is symmetric and of midrise type; then, $V = g(M)$ also has zero-mean, and so does $Q = M - V$.
- Then, the step size of the quantizer is given by:

$$\Delta = \frac{2m_{\max}}{L}$$

where L is the total number of representation levels.

Example.



Quantization Noise

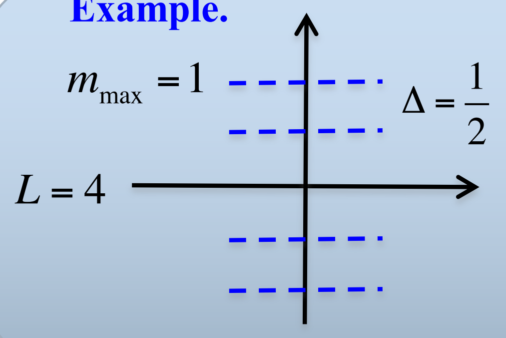
- Assume that $g(\,)$ assigns the midpoint of each step interval to be the representation level. Then,

$$V = M - M \bmod \Delta + \frac{\Delta}{2}$$
$$\Rightarrow Q = M - V = M \bmod \Delta - \frac{\Delta}{2}.$$

$$\Pr\{Q \le q\} = \Pr\left\{(M \bmod \Delta) - \frac{\Delta}{2} \le q\right\} = \begin{cases} 0, & q < -\frac{\Delta}{2} \\ \frac{q}{\Delta} + \frac{1}{2}, & -\frac{\Delta}{2} \le q < \frac{\Delta}{2} \\ 1, & q \ge \frac{\Delta}{2} \end{cases}$$

$$\text{pdf } f_Q(q) = \frac{1}{\Delta} \cdot \mathbf{1}\left\{-\frac{\Delta}{2} \le q < \frac{\Delta}{2}\right\}$$

Example.



Quantization Noise

$V = M - Q$

- So, the output signal-to-noise ratio is equal to:

$$SNR_O = \frac{P}{\int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq} = \frac{P}{\frac{1}{12}\Delta^2} = \frac{P}{\frac{1}{12}\left(\frac{2m_{\max}}{L}\right)^2} = \frac{3P}{m_{\max}^2} L^2$$

- The transmission bandwidth of a quantization system is conceptually proportional to the number of bits required per sample, i.e., $R = \log_2(L)$.
- We then conclude that $SNR_O \propto 4^R$, which increases *exponentially* with transmission bandwidth.

Sinusoidal Modulating Signal

- Let $m(t) = A_m \cos(2\pi f_m t)$. Then

$$P = \frac{A_m^2}{2} \text{ and } m_{\max} = A_m$$
$$\Rightarrow SNR_O = \frac{3(Am^2/2)}{A_m^2} L^2 = \frac{3}{2} 4^R = 10\log_{10}(3/2) + R \cdot 10\log_{10}(4) \text{ dB} \approx (1.8 + 6R) \text{ dB}$$

L	R	SNR_O (dB)
32	5	31.8
64	6	37.8
128	7	43.8
256	8	49.8

* Note that in this example, we assume a full-load quantizer, in which no quantization loss is encountered due to saturation.

Quantization Noise

- In the previous analysis of quantization error, we assume the quantizer assigns the mid-point of each step interval to be the representative level.
- Questions:
  - Can the power of quantization noise be further reduced by adjusting the representative levels?
  - Can the power of quantization noise be further reduced by adopting a non-uniform quantizer?

Optimality of Scalar Quantizers

Representation level	v_1	v_2	…	v_{L-1}	v_L
Partitions	I_1	I_2	…	I_{L-1}	I_L

$$\bigcup_{k=1}^{L} I_k = [-A, A)$$

Notably, interval I_k may not be a "consecutive" single interval.

- Let $d(m, v_k)$ be the distortion by representing m by v_k.
- **Goal**: To find $\{I_k\}$ and $\{v_k\}$ such that the average distortion $D = E[d(M, g(M))]$ is minimized.

Optimality of Scalar Quantizers

- Solution:

$$\min_{\{v_k\}} \min_{\{I_k\}} D = \min_{\{v_k\}} \min_{\{I_k\}} \sum_{k=1}^{L} \int_{I_k} d(m, v_k) f_M(m) dm$$

(I) For fixed $\{v_k\}$, determine the optimal $\{I_k\}$.
(II) For fixed $\{I_k\}$, determine the optimal $\{v_k\}$.

(I) If $d(m, v_k) \leqq d(m, v_j)$, then m should be assigned to I_k rather than I_j.

$$\Rightarrow I_k = \{m \in [-A, A) : d(m, v_k) \leq d(m, v_j) \text{ for all } 1 \leq j \leq L\}$$

(II) For fixed $\{I_k\}$, determine the optimal $\{v_k\}$.

$$\min_{\{v_k\}} \sum_{k=1}^{L} \int_{I_k} d(m, v_k) f_M(m) dm$$

Since $\frac{\partial}{\partial v_j}\left(\sum_{k=1}^{L} \int_{I_k} d(m, v_k) f_M(m) dm \right) = \frac{\partial}{\partial v_j}\left(\int_{I_j} d(m, v_j) f_M(m) dm \right)$

$$= \int_{I_j} \frac{\partial d(m, v_j)}{\partial v_j} f_M(m) dm$$

a necessary condition for the optimal v_j is:

$$\int_{I_j} \frac{\partial d(m, v_j)}{\partial v_j} f_M(m) dm = 0.$$

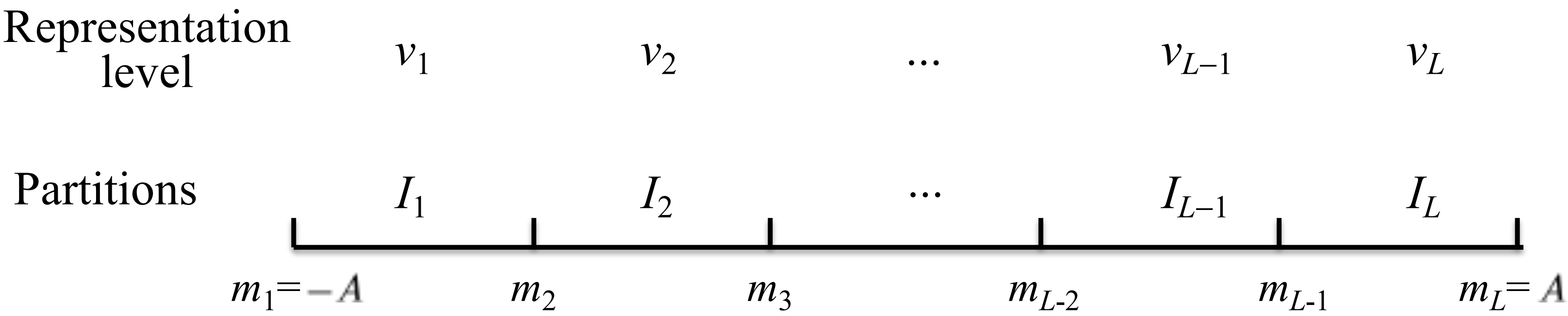
Lloyd-Max algorithm is to repetitively apply (I) and (II) for the search of the optimal quantizer.

Mean-Square Distortion

- $d(m, v_k) = (m - v_k)^2$

  (I) $I_k = \{m \in [-A, A) : (m - v_k)^2 \le (m - v_j)^2 \text{ for all } 1 \le j \le L\}$

  should be a consecutive interval.



Mean-Square Distortion

(II) A necessary condition for the optimal v_j is :

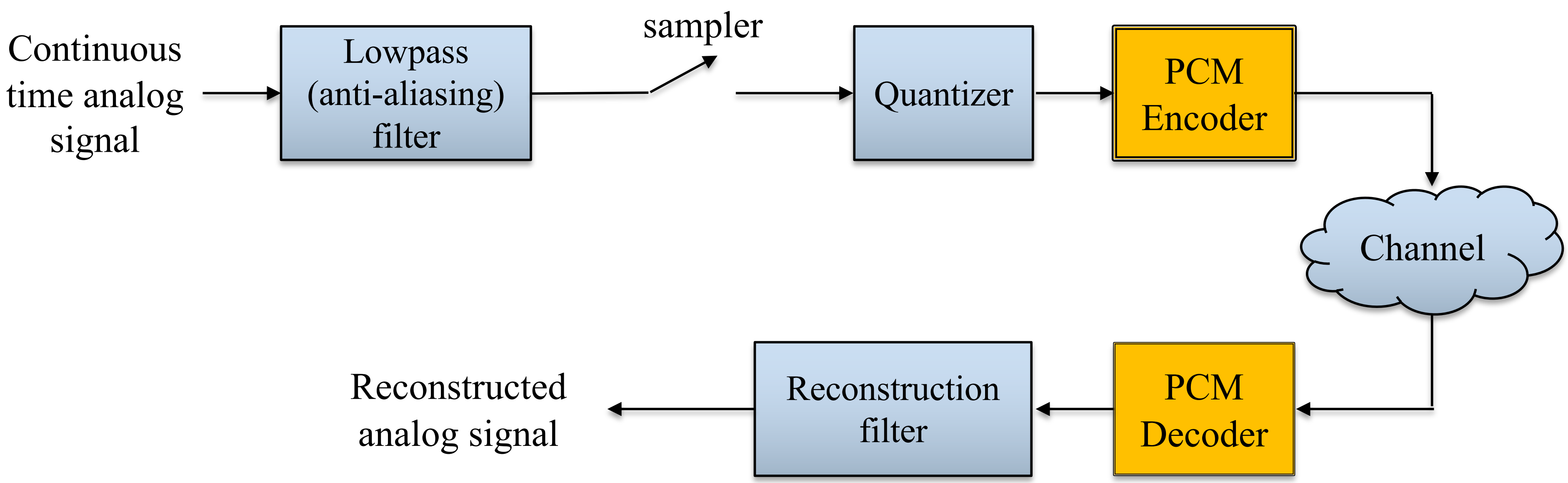
$$\int_{m_j}^{m_{j+1}} \frac{\partial (m-v_j)^2}{\partial v_j} f_M(m) dM = -2 \int_{m_j}^{m_{j+1}} (m-v_j) f_M(m) dm = 0.$$

$$\Rightarrow v_{j,\text{optimal}} = \frac{\int_{m_j}^{m_{j+1}} m f_M(m) dm}{\int_{m_j}^{m_{j+1}} f_M(m) dm} = E[M \mid m_j \le M < m_{j+1}]$$

Exercise: What is the best $\{m_k\}$ and $\{v_k\}$ if M is uniformly distributed over $[-A, A)$.

Hint: $\min_{\{I_k\}} \min_{\{v_k\}} D = \frac{1}{2A} \min_{\{m_k\}} \sum_{k=1}^{L} \int_{m_k}^{m_{k+1}} \left(m - \frac{m_k + m_{k+1}}{2} \right)^2 dm.$

Pulse-Code Modulation (PCM)



Pulse-Code Modulation (PCM)

- Non-uniform quantizers used for telecommunication (ITU-T G.711)
  - ITU-T G.711: Pulse Code Modulation (PCM) of Voice Frequencies (1972)
    - It consists of two laws: A-law (mainly used in Europe) and μ-law (mainly used in US and Japan)
  - This design helps to protect weak signal, which occurs more frequently in, say, human voice.

Quantization Laws

- Quantization Laws
  - A-law
    - 13-bit uniformly quantized
    - Conversion to 8-bit code
  - μ-law
    - 14-bit uniformly quantized
    - Conversion to 8-bit code.
  - These two are referred to as *compression laws* since they use 8-bit to (lossily) represent 13-(or 14-)bit information.

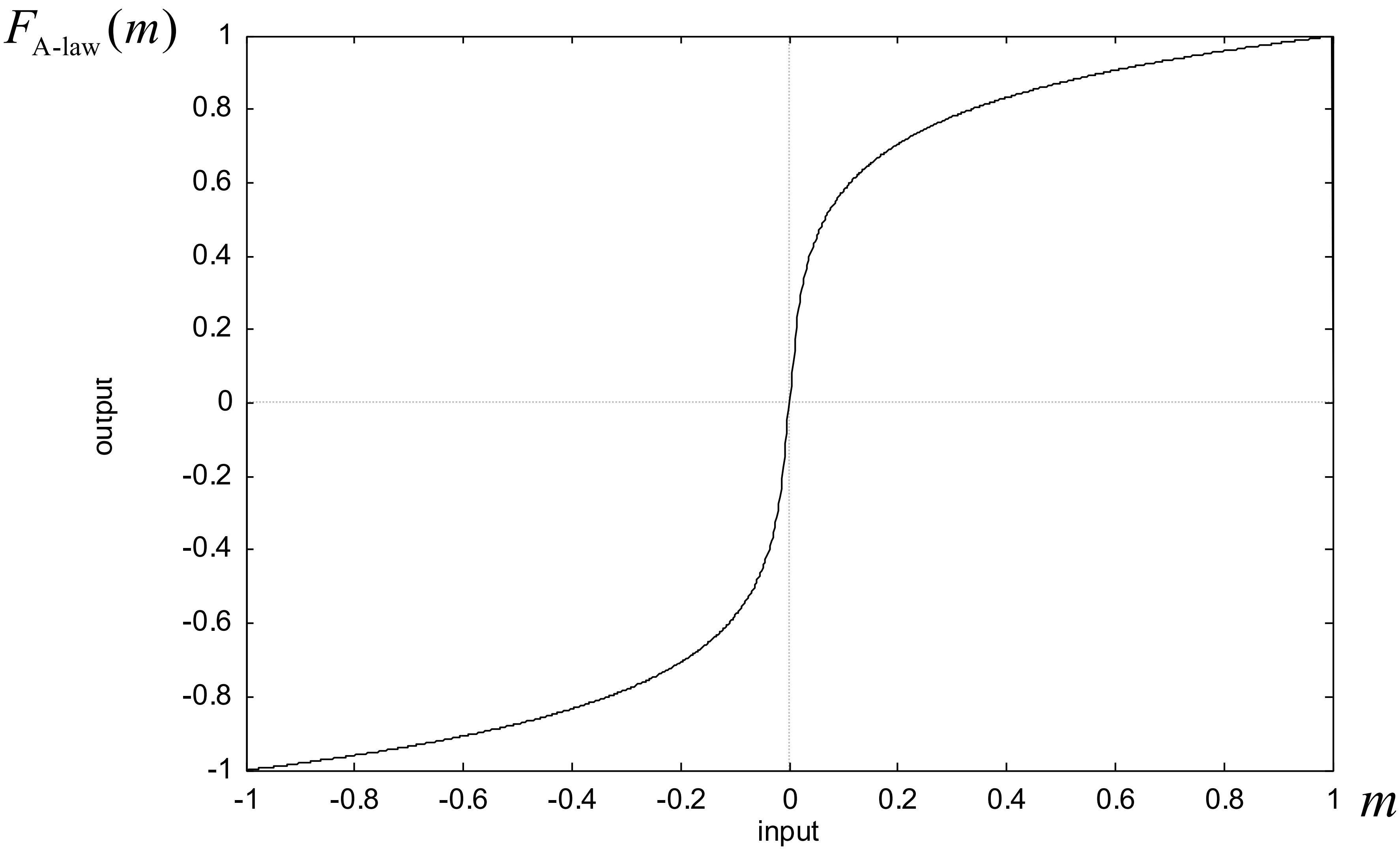
A-law in G.711

- A-law (A=87.6)

$$F_{\text{A-law}}(m) = \begin{cases} \dfrac{A}{1+\log(A)} m, & |m| \le \dfrac{1}{A} \\ \text{sgn}(m)\left[\dfrac{1+\log(A|m|)}{1+\log(A)}\right], & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

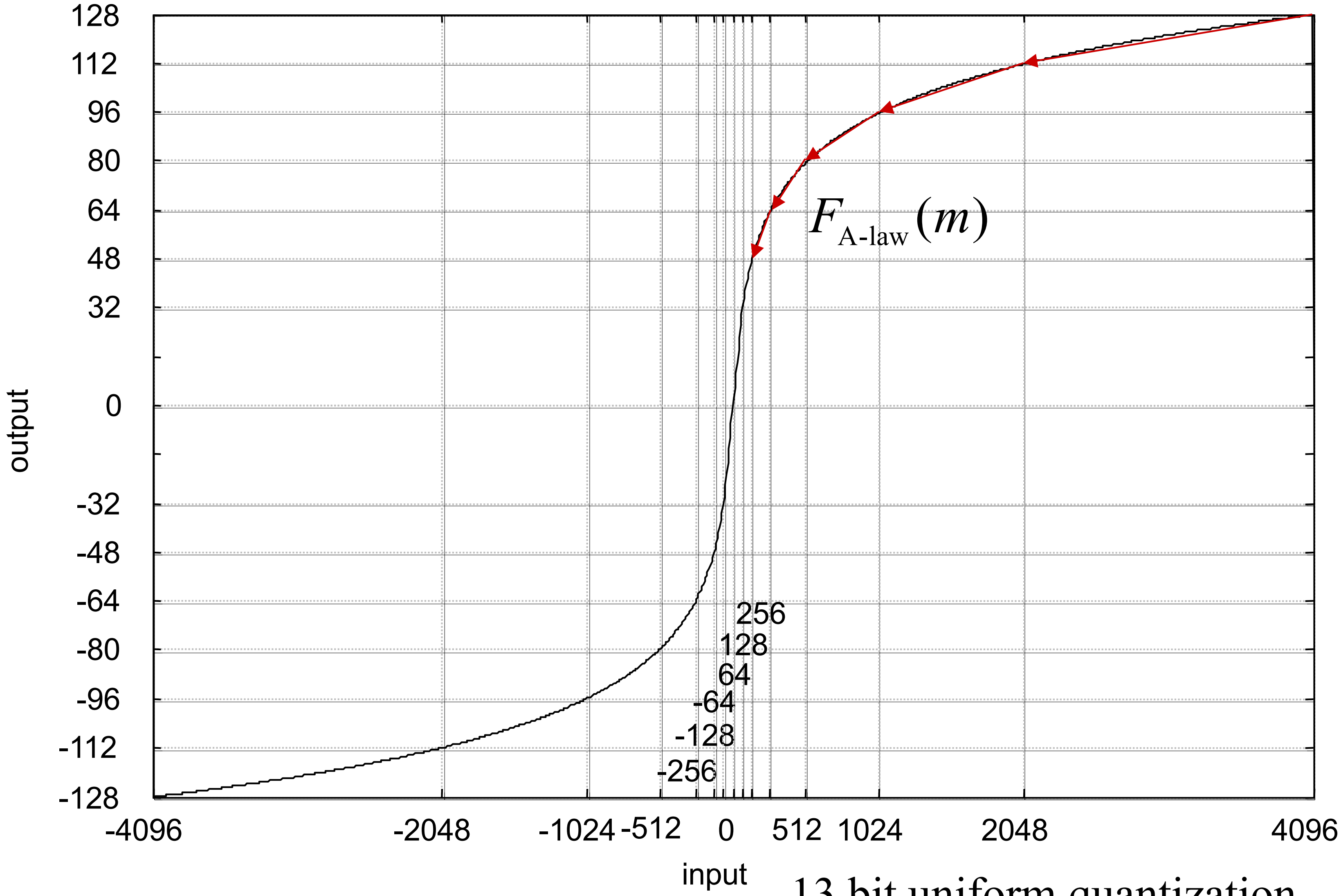
Linear mapping

Logarithmic mapping


$F_{\text{A-law}}(m)$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
output
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
input
m

A piecewise linear approximation to the law.

8 bit PCM code



13 bit uniform quantization

Compressor of A-law (assume nonnegative *m*)

Input Values												Compressed Code Word						
												Chord			Step			
Bits:11	10	9	8	7	6	5	4	3	2	1	0	Bits: 6	5	4	3	2	1	0
0	0	0	0	0	0	0	a	b	c	d	x	0	0	0	a	b	c	d
0	0	0	0	0	0	1	a	b	c	d	x	0	0	1	a	b	c	d
0	0	0	0	0	1	a	b	c	d	x	x	0	1	0	a	b	c	d
0	0	0	0	1	a	b	c	d	x	x	x	0	1	1	a	b	c	d
0	0	0	1	a	b	c	d	x	x	x	x	1	0	0	a	b	c	d
0	0	1	a	b	c	d	x	x	x	x	x	1	0	1	a	b	c	d
0	1	a	b	c	d	x	x	x	x	x	x	1	1	0	a	b	c	d
1	a	b	c	d	x	x	x	x	x	x	x	1	1	1	a	b	c	d

E.g. $(3968)_{10}$ ––> $(1111,1000,0000)_2$ ––> $(111,1111)_2$ ––> $(127)_{10}$

E.g. $(2176)_{10}$ ––> $(1000,1000,0000)_2$ ––> $(111,0001)_2$ ––> $(113)_{10}$

Expander of A-law (assume nonnegative *m*)

Compressed Code Word: Chord	Compressed Code Word: Step	Output Values
Bits: 6 5 4	Bits: 3 2 1 0	Bits: 11 10 9 8 7 6 5 4 3 2 1 0
0 0 0	a b c d	0 0 0 0 0 0 0 a b c d 1
0 0 1	a b c d	0 0 0 0 0 0 1 a b c d 1
0 1 0	a b c d	0 0 0 0 0 1 a b c d 1 0
0 1 1	a b c d	0 0 0 0 1 a b c d 1 0 0
1 0 0	a b c d	0 0 0 1 a b c d 1 0 0 0
1 0 1	a b c d	0 0 1 a b c d 1 0 0 0 0
1 1 0	a b c d	0 1 a b c d 1 0 0 0 0 0
1 1 1	a b c d	1 a b c d 1 0 0 0 0 0 0

E.g. $(113)_{10} \to (111,0001)_2 \to (1000,1100,0000)_2 \to (2240)_{10}$

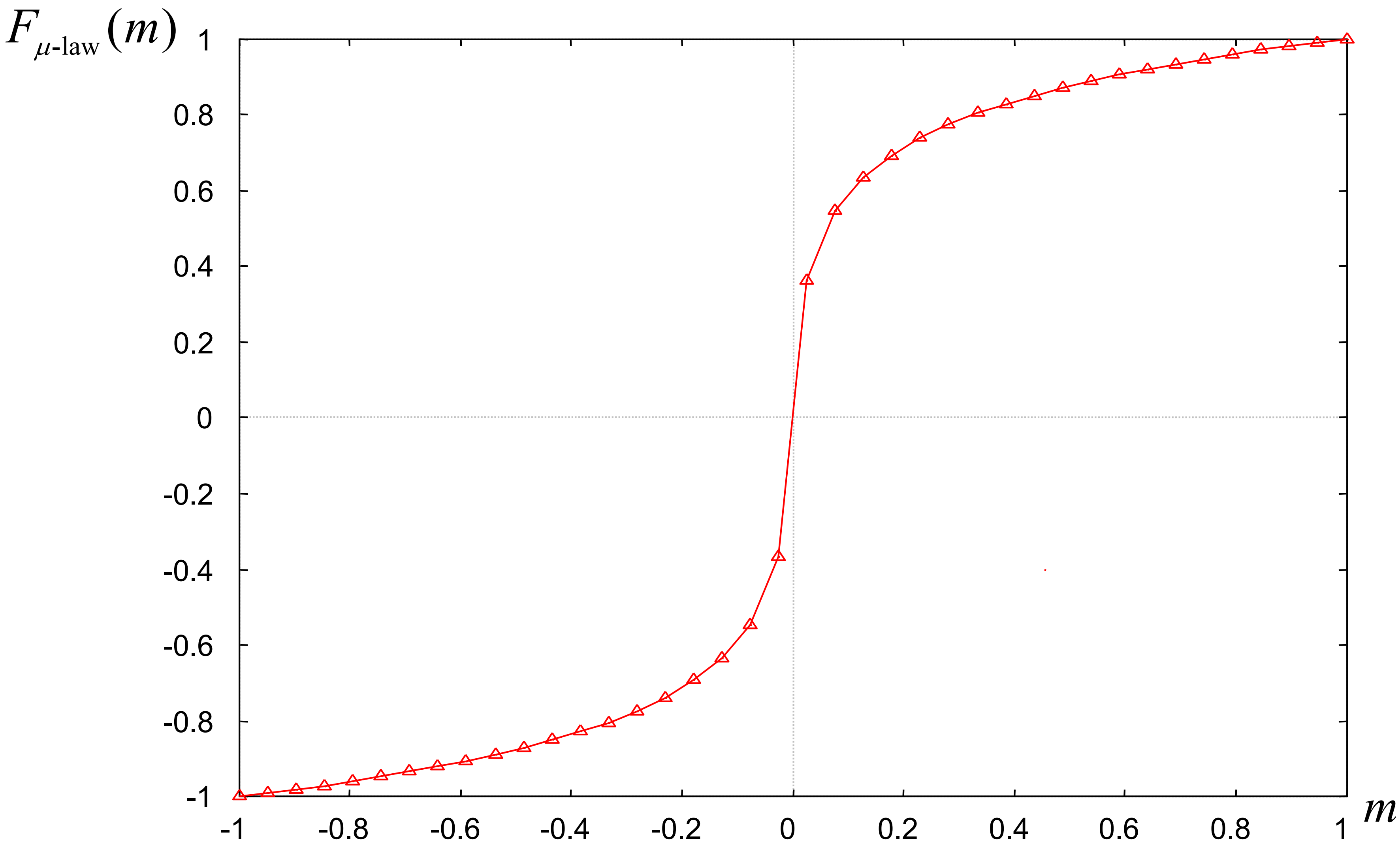
In other words, $\dfrac{(1001,0000,0000)_2 + (1000,1000,0000)_2}{2} = \dfrac{(2304)_{10} + (2176)_{10}}{2} = (2240)_{10}$

μ-law in G.711

- μ-law ($\mu = 255$)

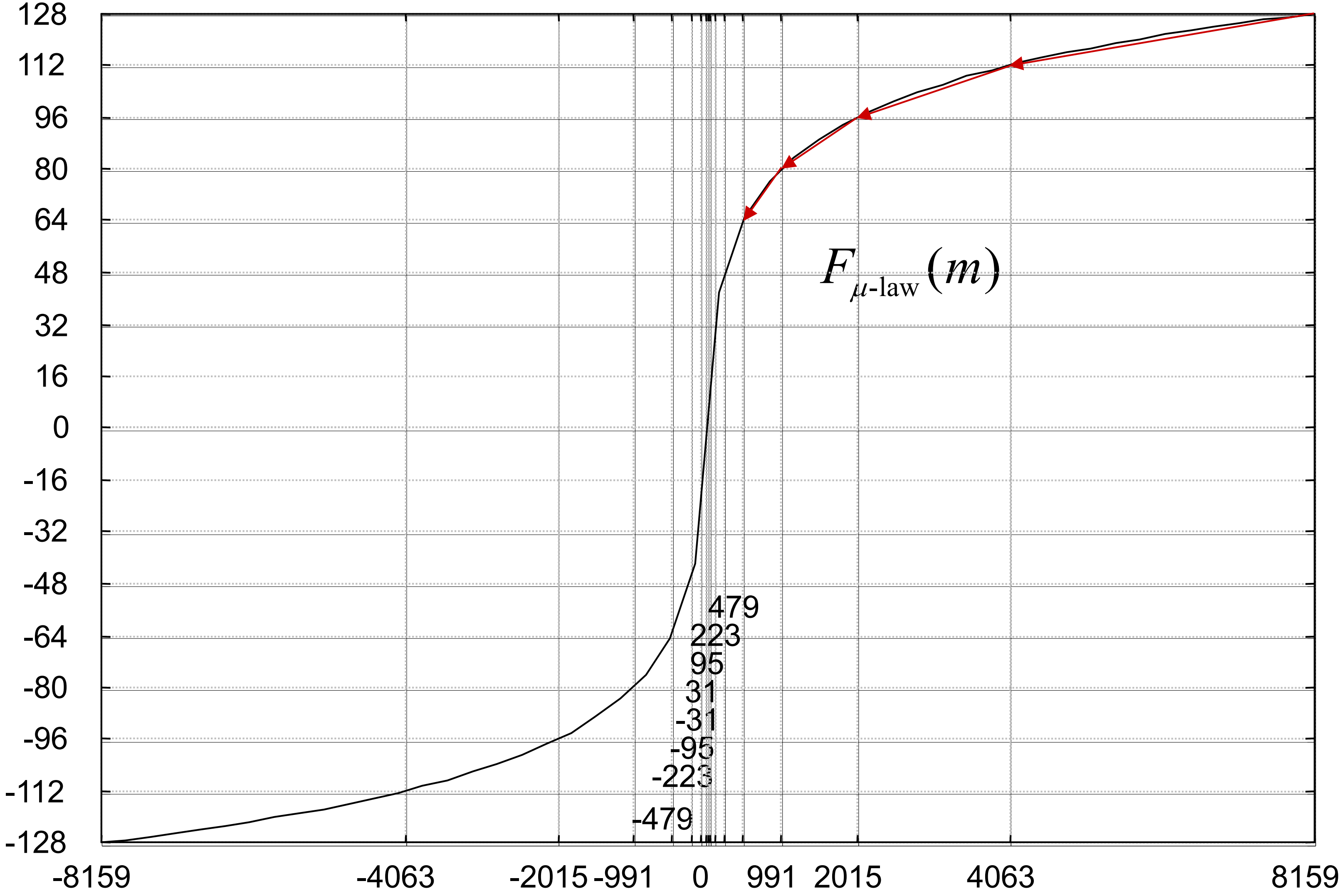
$$F_{\mu\text{-law}}(m) = \text{sgn}(m)\frac{\log(1+\mu|m|)}{1+\log(\mu)} \text{ for } |m| \le 1.$$

  - It is approximately linear at low m.
  - It is approximately logarithmic at large m.


$F_{\mu\text{-law}}(m)$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
m

8 bit PCM code

A piecewise linear approximation to the law.



14 bit uniform quantization ($2^{13} = 8192$)

Compressor of μ-law (assume nonnegative m)

Raised Input Values	*Compressed Code Word*	
	Chord	*Step*
Bits:12 11 10 9 8 7 6 5 4 3 2 1 0	*Bits: 6 5 4 3 2 1 0*	
0 0 0 0 0 0 0 1 a b c d x	0 0 0 a b c d	
0 0 0 0 0 0 1 a b c d x x	0 0 1 a b c d	
0 0 0 0 0 1 a b c d x x x	0 1 0 a b c d	
0 0 0 0 1 a b c d x x x x	0 1 1 a b c d	
0 0 0 1 a b c d x x x x x	1 0 0 a b c d	
0 0 1 a b c d x x x x x x	1 0 1 a b c d	
0 1 a b c d x x x x x x x	1 1 0 a b c d	
1 a b c d x x x x x x x x	1 1 1 a b c d	

Raised Input = *Input* + $(33)_{10}$ = *Input* + *21H*

(For negative m, the raised input becomes (input – 33).)

An additional 7th bit is used to indicate whether the input signal is positive (1) or negative (0).

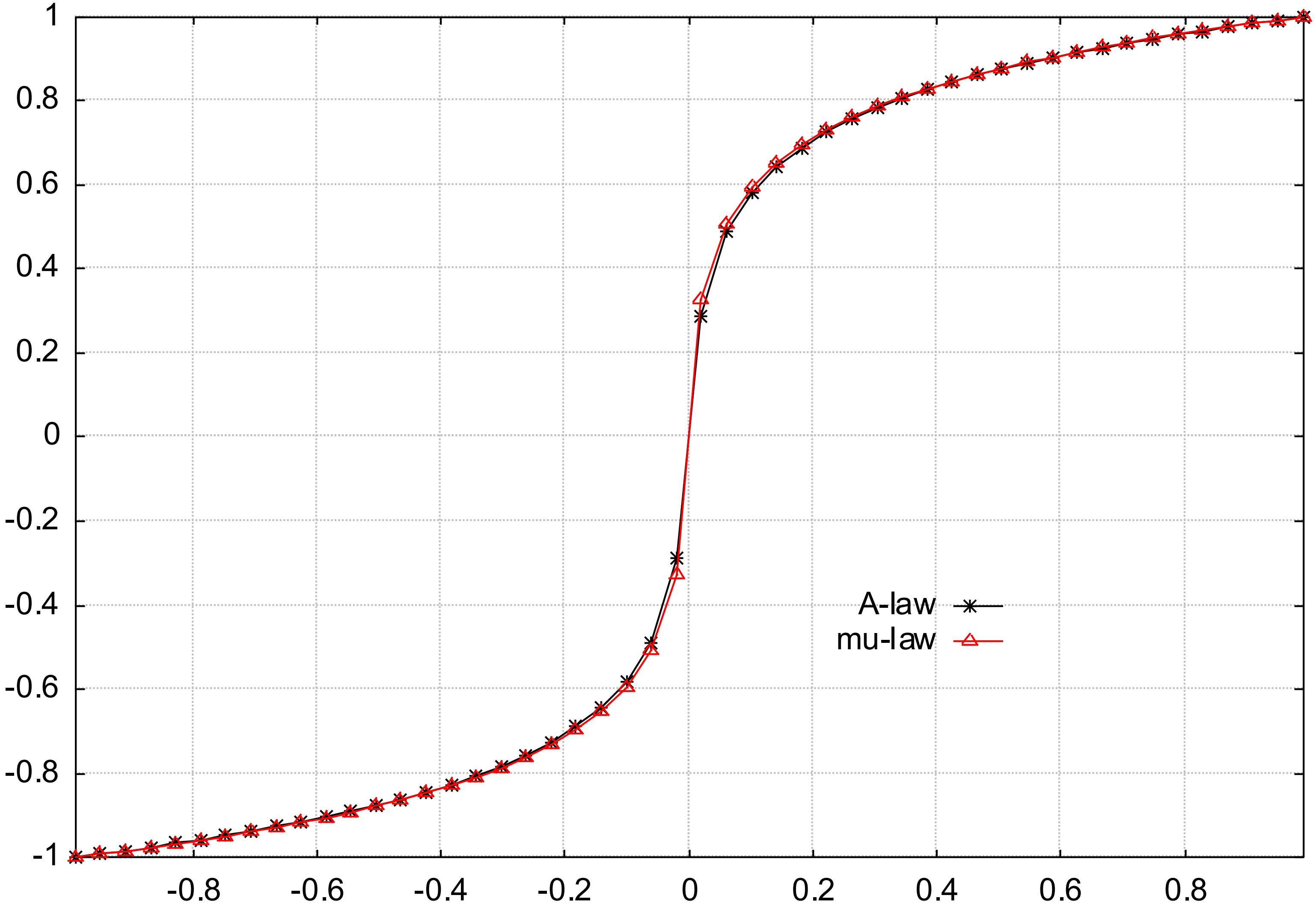
Expander of μ-law (assume nonnegative m)

Compressed Code Word		Raised Output Values
Chord	Step	
Bits: 6 5 4	3 2 1 0	Bits: 12 11 10 9 8 7 6 5 4 3 2 1 0
0 0 0	a b c d	0 0 0 0 0 0 0 1 a b c d 1
0 0 1	a b c d	0 0 0 0 0 0 1 a b c d 1 0
0 1 0	a b c d	0 0 0 0 0 1 a b c d 1 0 0
0 1 1	a b c d	0 0 0 0 1 a b c d 1 0 0 0
1 0 0	a b c d	0 0 0 1 a b c d 1 0 0 0 0
1 0 1	a b c d	0 0 1 a b c d 1 0 0 0 0 0
1 1 0	a b c d	0 1 a b c d 1 0 0 0 0 0 0
1 1 1	a b c d	1 a b c d 1 0 0 0 0 0 0 0

Output = Raised Output – 33

Note that the combination of a *compressor* and an *expander* is called a *compander*.

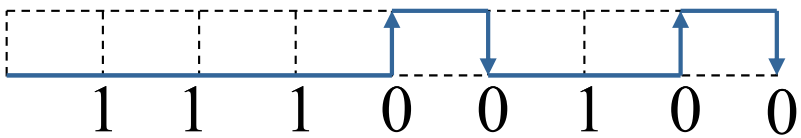
Comparison of A-law and μ-law specified in G.711.



Coding

- After the quantizer provides a symbol, representing one of 256 possible levels (8 bits of information) at each sampled time, the encoder will transform the symbol (or several symbols) into a code character (or code word) that is *suitable* for transmission over a *noisy* channel.
- Example. Binary code.

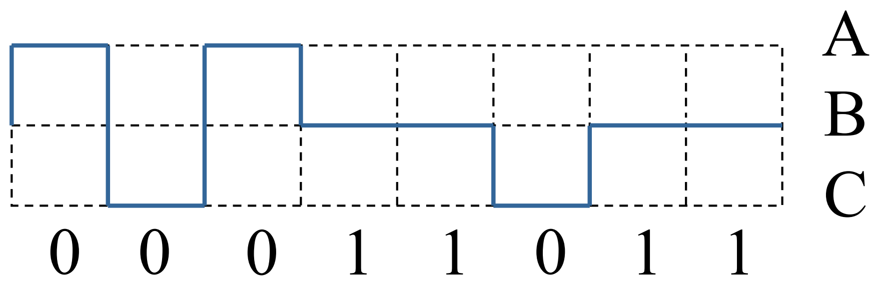
11100100



0 = change
1 = unchange

Coding

- Example. Ternary code (Pseudo-binary code).

00011011→ 

00011011→ACABBCBB

Through the help of coding, the receiver may be able to detect (or even correct) the transmission errors due to noise. For example, it is impossible to receive ABABBABB, since this is not a legitimate code word (character).

Coding

- Example of error correcting code : *Three-times repetition code* (to protect Bluetooth packet header).

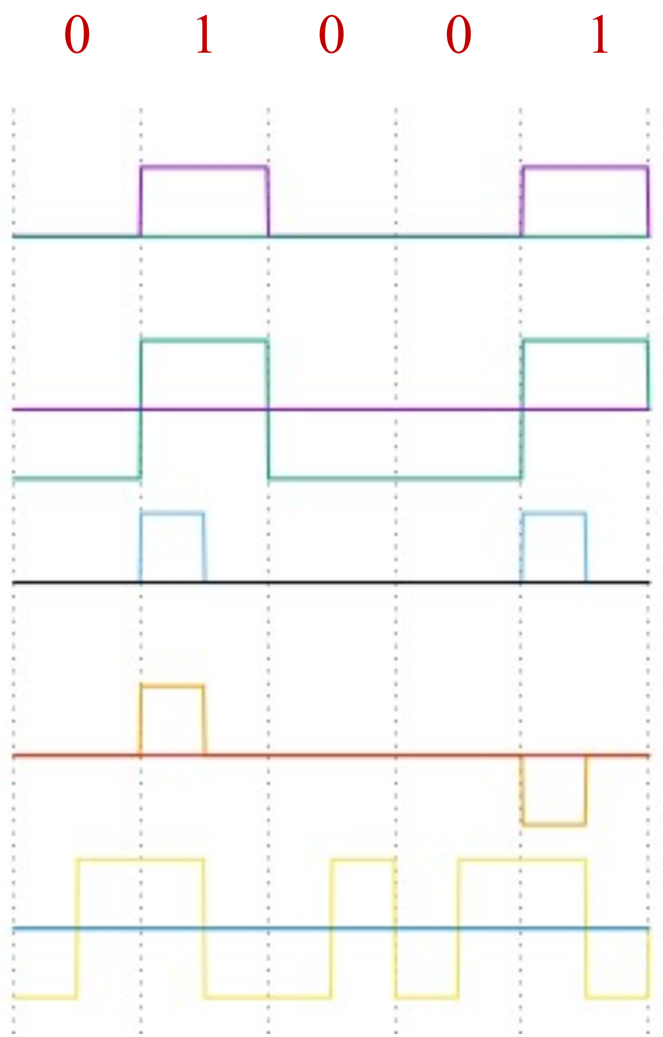
  $$00011011 \rightarrow 000,000,000,111,111,000,111,111$$

  Then, the so-called *majority law* can be applied at the receiver to correct one-bit error.

  - **Channel** (error correcting) **codes** are designed to compensate the channel noise, while **line codes** are simply used as the electrical representation of a binary data stream over the electrical line.

Line Codes

(a) Unipolar nonreturn-to-zero (NRZ) signaling
(b) Polar nonreturn-to-zero (NRZ) signaling
(c) Unipolar return-to-zero (RZ) signaling
(d) Bipolar return-to-zero (BRZ) signaling
(e) Split-phase (Manchester code)



Derivation of PSD

- From Slide 2-30, we obtain that the general PSD formula is:

$$\overline{\text{PSD}} = \lim_{T\to\infty} \frac{1}{2T} E[S(f)S_{2T}^*(f)], \text{ where } s_{2T}(t) = s(t)\cdot \mathbf{1}\{|t| \le T\}.$$

For a line coded signal, $s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b)$, where $g(t) = 0$ outside $[0, T_\text{b})$.

Hence, $S(f) = G(f) \sum_{n=-\infty}^{\infty} a_n e^{-j2\pi f n T_\text{b}}$ and $S_{2NT_\text{b}}(f) = G(f) \sum_{n=-N}^{N-1} a_n e^{-j2\pi f n T_\text{b}}$.

$$\Rightarrow \overline{\text{PSD}} = \lim_{N\to\infty} \frac{1}{2NT_\text{b}} |G(f)|^2 \left(\sum_{n=-\infty}^{\infty} \sum_{m=-N}^{N-1} E[a_n a_m^*] e^{-j2\pi f(n-m)T_\text{b}} \right).$$

$$\text{PSD} = \lim_{N\to\infty} \frac{1}{2NT_b} |G(f)|^2 \left(\sum_{n=-\infty}^{\infty} \sum_{m=-N}^{N-1} E[a_n a_m^*] e^{-j2\pi f(n-m)T_b} \right)$$

$$= |G(f)|^2 \lim_{N\to\infty} \frac{1}{2NT_b} \left(\sum_{m=-N}^{N-1} \sum_{n=-\infty}^{\infty} \phi_a(n-m) e^{-j2\pi f(n-m)T_b} \right)$$

$$= |G(f)|^2 \lim_{N\to\infty} \frac{1}{2NT_b} \left(\sum_{m=-N}^{N-1} \sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fkT_b} \right)$$

$$= |G(f)|^2 \frac{1}{T_b} \left(\sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fkT_b} \right)$$

For i.i.d. $\{a_n\}$, $\frac{1}{T_b}\left(\sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fkT_b} \right) = \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b} \sum_{k=-\infty}^{\infty} e^{-j2\pi fkT_b}$

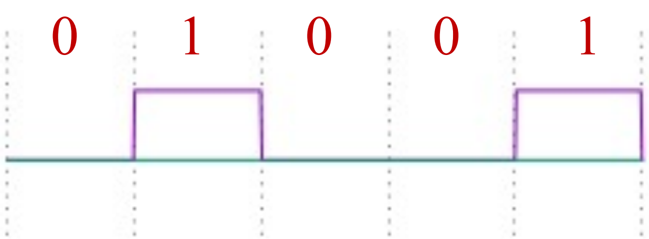
(See Slide 6-4.) $= \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b)$

(i.i.d. = independent and identically distributed)

PSD of Line Codes

- Unipolar nonreturn-to-zero (NRZ) signaling
  - Also named *on-off signaling*.
  - Disadvantage: Waste of power due to the non-zero-mean nature (i.e., PSD does not approach zero at zero frequency).

$$s(t)=\sum_{n=-\infty}^{\infty} a_n g(t-nT_b), \text{ where } \begin{cases} \{a_n\}_{n=-\infty}^{\infty} \text{ is zero/one i.i.d.,} \\ g(t)=\begin{cases} A, & 0\le t<T_b \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

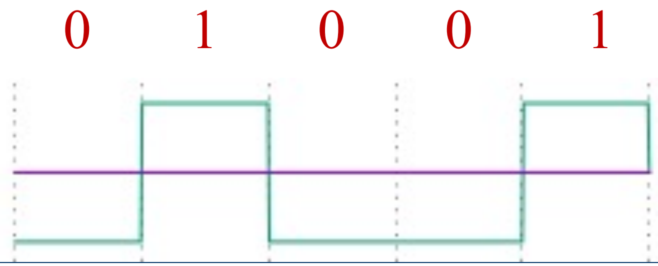


PSD of Line Codes

- PSD of Unipolar NRZ

$$
\begin{aligned}
\text{PSD}_{\text{U-NRZ}} &= |G(f)|^2 \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= A^2 T_b^2 \text{sinc}^2(f T_b) \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= \frac{A^2 T_b}{4} \text{sinc}^2(f T_b) \left(1 + \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= \frac{A^2 T_b}{4} \text{sinc}^2(f T_b) \left(1 + \frac{1}{T_b} \delta(f) \right)
\end{aligned}
$$

PSD of Line Codes

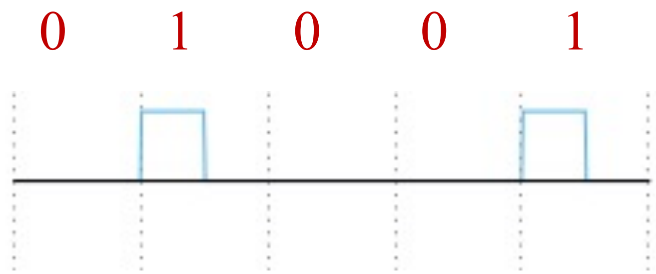


- Polar nonreturn-to-zero (NRZ) signaling
  - The previous PSD of Unipolar NRZ suggests that a zero-mean data sequence is preferred.

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \text{ where } \begin{cases} \{a_n\}_{n=-\infty}^{\infty} \text{ is } \pm 1 \text{ i.i.d.,} \\ g(t) = \begin{cases} A, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

$$\begin{aligned} \text{PSD}_{\text{P-NRZ}} &= |G(f)|^2 \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\ &= A^2 T_b \text{sinc}^2(fT_b) \end{aligned}$$

PSD of Line Codes



- Unipolar return-to-zero (RZ) signaling
  - An attractive feature of this line code is the presence of delta functions at $f = -1/T_b$, 0, $1/T_b$ in the PSD, which can be used for bit-timing recovery at the receiver.
  - Disadvantage: It requires 3dB more power than polar return-to-zero signaling.

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \text{ where } \begin{cases} \{a_n\}_{n=-\infty}^{\infty} \text{ is zero/one i.i.d.,} \\ g(t) = \begin{cases} A, & 0 \le t < T_b/2 \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

PSD of Line Codes

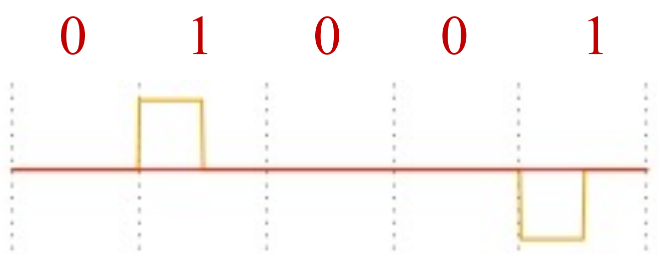
- PSD of Unipolar RZ

$$
\begin{aligned}
\text{PSD}_{\text{U-RZ}} &= |G(f)|^2 \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= \frac{A^2 T_b^2}{4} \text{sinc}^2\left(\frac{fT_b}{2}\right) \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= \frac{A^2 T_b}{16} \text{sinc}^2\left(\frac{fT_b}{2}\right) \left(1 + \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= \frac{A^2 T_b}{16} \text{sinc}^2\left(\frac{fT_b}{2}\right) \left(1 + \frac{1}{T_b} \sum_{k=\infty}^{\infty} \delta(f - k/T_b) \right)
\end{aligned}
$$

PSD of Line Codes

- Bipolar return-to-zero (BRZ) signaling
  - Also named *alternate mark inversion* (AMI) signaling
  - No DC component and relatively insignificant low-frequency components in PSD.

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \text{ where } g(t) = \begin{cases} A, & 0 \le t < T_b/2 \\ 0, & \text{otherwise} \end{cases}$$



PSD of Line Codes

- PSD of BRZ
  - $\{a_n\}$ is no longer i.i.d.

$$E[a_n^2] = (0)\frac{1}{2} + (-1)^2\frac{1}{4} + (+1)^2\frac{1}{4} = \frac{1}{2}$$

$$E[a_n a_{n+1}] = (-1)\frac{1}{4} = -\frac{1}{4}$$

$$E[a_n a_{n+2}] = (1)(1)\frac{1}{16} + (1)(-1)\frac{1}{16} + (-1)(1)\frac{1}{16} + (-1)(-1)\frac{1}{16} = 0$$

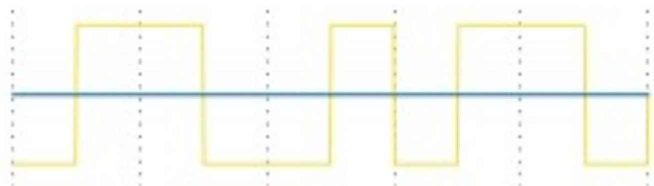
$$\vdots$$

$$E[a_n a_{n+m}] = 0 \text{ for } m > 1.$$

PSD of Line Codes

$$
\begin{aligned}
\mathrm{PSD}_{\mathrm{BRZ}} &= |G(f)|^2 \frac{1}{T_b}\left(\sum_{k=-\infty}^{\infty}\phi_a(k)e^{-j2\pi fkT_b}\right)\\
&= \frac{A^2T_b^2}{4}\mathrm{sinc}^2\left(\frac{fT_b}{2}\right)\cdot\frac{1}{T_b}\left(-\frac{1}{4}e^{j2\pi fT_b}+\frac{1}{2}-\frac{1}{4}e^{-j2\pi fT_b}\right)\\
&= \frac{A^2T_b^2}{4}\mathrm{sinc}^2\left(\frac{fT_b}{2}\right)\cdot\frac{1}{T_b}\left(\frac{1}{2}-\frac{1}{2}\cos(2\pi fT_b)\right)\\
&= \frac{A^2T_b}{4}\mathrm{sinc}^2\left(\frac{fT_b}{2}\right)\sin^2(\pi fT_b)
\end{aligned}
$$

PSD of Line Codes



- Split-phase (Manchester code)
  - This signaling suppressed the DC component, and has relatively insignificant low-frequency components, **regardless of the signal statistics**.
  - Notably, for P-NRZ and BRZ, the DC component is suppressed only when the signal has the right statistics.

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \text{ where } \begin{cases} \{a_n\}_{n=-\infty}^{\infty} \text{ is } \pm 1 \text{ i.i.d.,} \\ g(t) = \begin{cases} A, & 0 \le t < T_b/2 \\ -A, & T_b/2 \le t < T_b \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

PSD of Line Codes

- PSD of Manchester code

$$\begin{aligned}
\overline{\text{PSD}}_{\text{Manchester}} \\
&= |G(f)|^2 \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= A^2 T_b^2 \text{sinc}^2 \left(\frac{f T_b}{2} \right) \sin^2 \left(\frac{\pi f T_b}{2} \right) \left(\frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \\
&= A^2 T_b \text{sinc}^2 \left(\frac{f T_b}{2} \right) \sin^2 \left(\frac{\pi f T_b}{2} \right)
\end{aligned}$$

Let T_b=1, and adjust A such that the total power of each line code is 1. This gives a fair comparison among line codes.

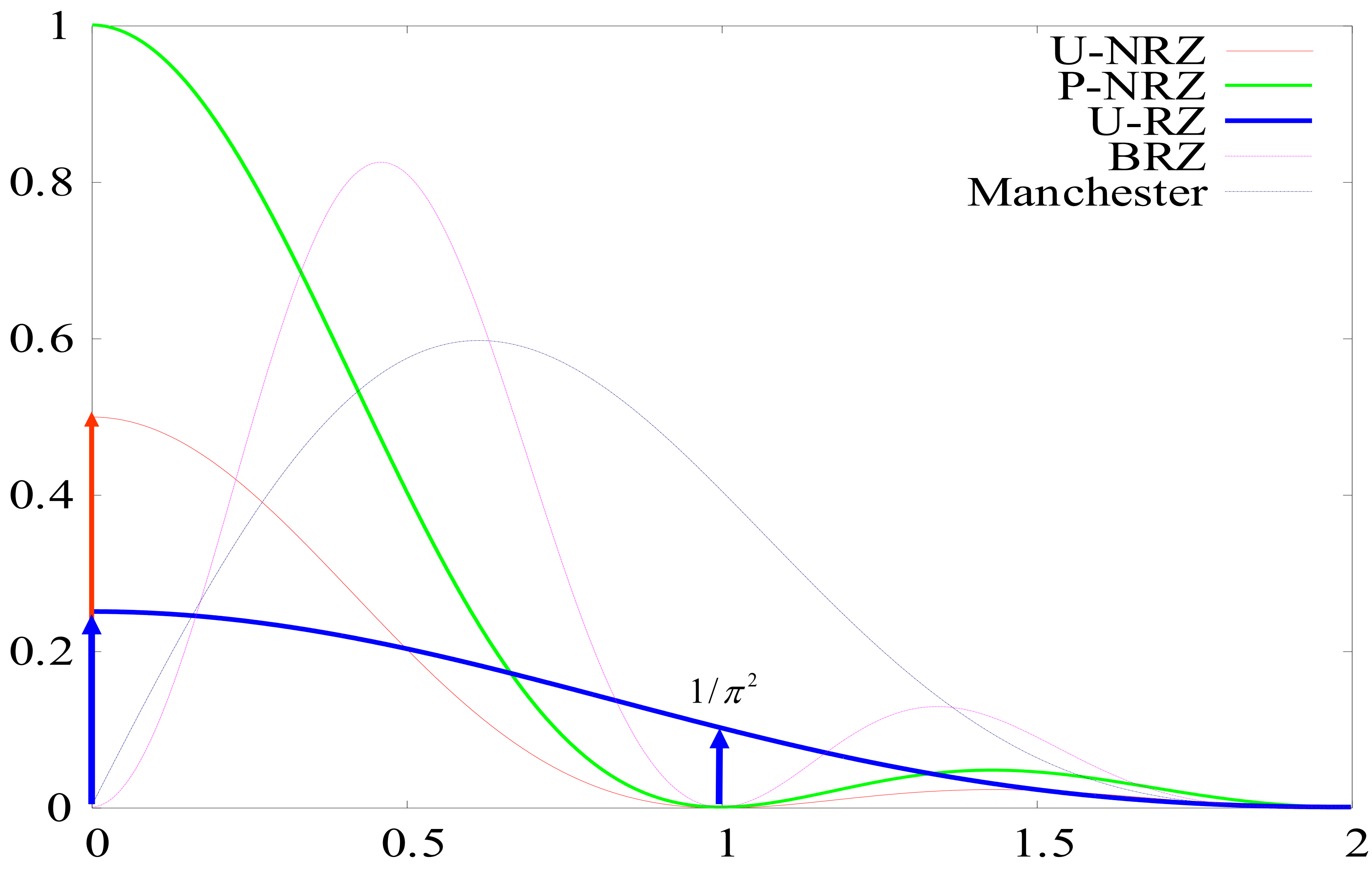
power= $\frac{1}{2} + \frac{1}{2}$ $\quad \text{PSD}_{\text{U-NRZ}} = \frac{1}{2}\text{sinc}^2(f) + \frac{1}{2}\delta(f)$ $\quad A = \sqrt{2}$

power= 1 $\quad \text{PSD}_{\text{P-NRZ}} = \text{sinc}^2(f)$ $\quad A = 1$

power= $\frac{1}{2} + \frac{1}{2}$ $\quad \text{PSD}_{\text{U-RZ}} = \frac{1}{4}\text{sinc}^2\left(\frac{f}{2}\right) + \frac{1}{4}\sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k}{2}\right)\delta(f-k)$ $\quad A = 2$

power= 1 $\quad \text{PSD}_{\text{BRZ}} = \text{sinc}^2\left(\frac{f}{2}\right)\sin^2(\pi f)$ $\quad A = 2$

power= 1 $\quad \text{PSD}_{\text{Manchester}} = \text{sinc}^2\left(\frac{f}{2}\right)\sin^2\left(\frac{\pi f}{2}\right)$ $\quad A = 1$



U-NRZ
P-NRZ
U-RZ
BRZ
Manchester
$1/\pi^2$
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2

Summary

- Sampling – transform analog waveform to discrete-time continuous wave
  - Nyquist rate
- Quantization – transform discrete-time continuous wave to discrete data.
  - Human can only detect finite intensity difference.
- PAM, PDM and PPM
- Line coding and its PSD