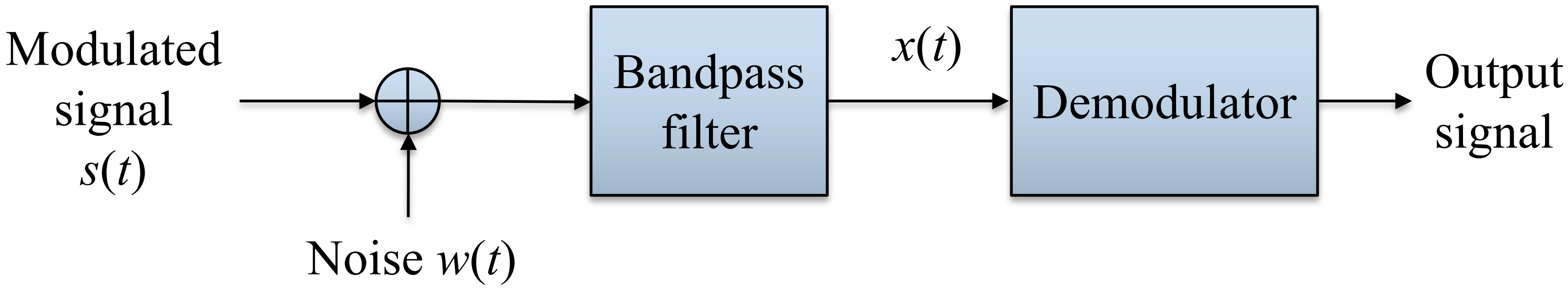
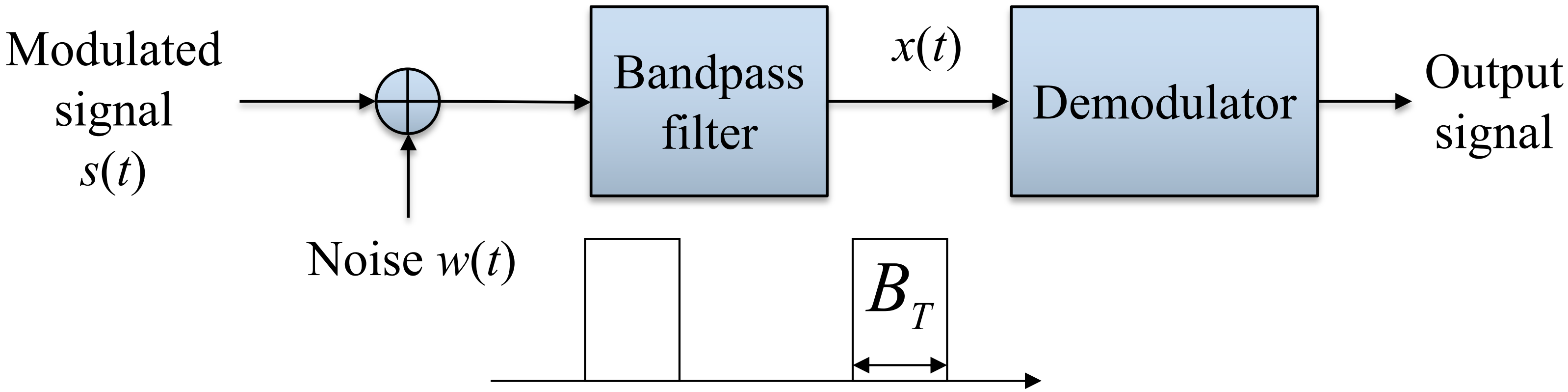
# Part 5 Analog Demodulation with Noise

# Impact of Additive Noise in Analog Modulation Systems




- To simplify the system analysis, we assume:
  - ideal bandpass filter that is just wide enough to pass the modulated signal $s(t)$ without distortion,
  - ideal demodulator,
  - Gaussian distributed white noise process.
- So, the only source of imperfection is from the noise.

# Impact of Additive Noise in Analog Modulation Systems




As a result, after passing through the ideal bandpass filter, $s(t)$ is unchanged but $w(t)$ becomes a narrowband noise $n(t)$. Hence,

$$x(t) = s(t) + n(t),$$

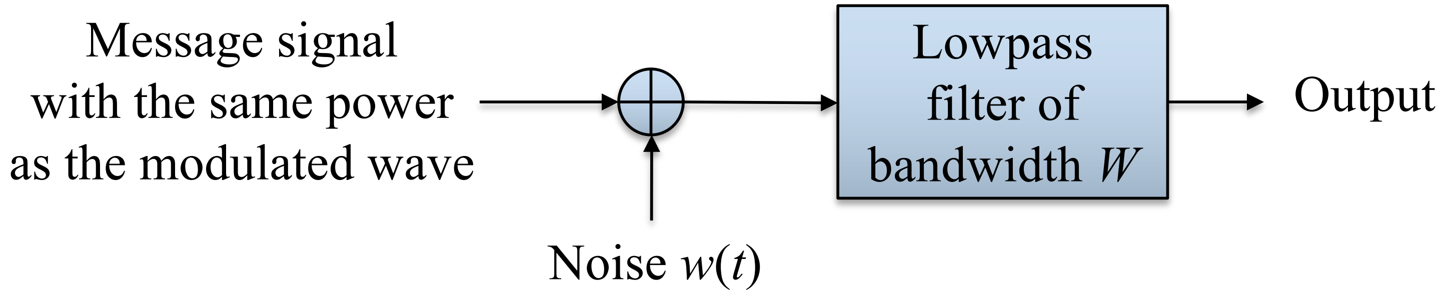
where $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$.

# Impact of Additive Noise in Analog Modulation Systems

- Input signal-to-noise (power) ratio ($SNR_I$)
  - The ratio of the average power of the *modulated signal* $s(t)$ to the average power of the *filtered noise* $n(t)$.
- Output signal-to-noise (power) ratio ($SNR_O$)
  - The ratio of the average power of the *demodulated message signal* to the average power of the *noise*, *measured at the receiver output*.

# Impact of Additive Noise in Analog Modulation Systems

- It is sometimes advantageous to look at the lowpass equivalent model.
- Channel signal-to-noise (power) ratio ($SNR_C$)
  - The ratio of the average power of the modulated signal $s(t)$ to the average power of the *channel noise in the message bandwidth*, measured at the receiver input (as illustrated below).



# Impact of Additive Noise in Analog Modulation Systems

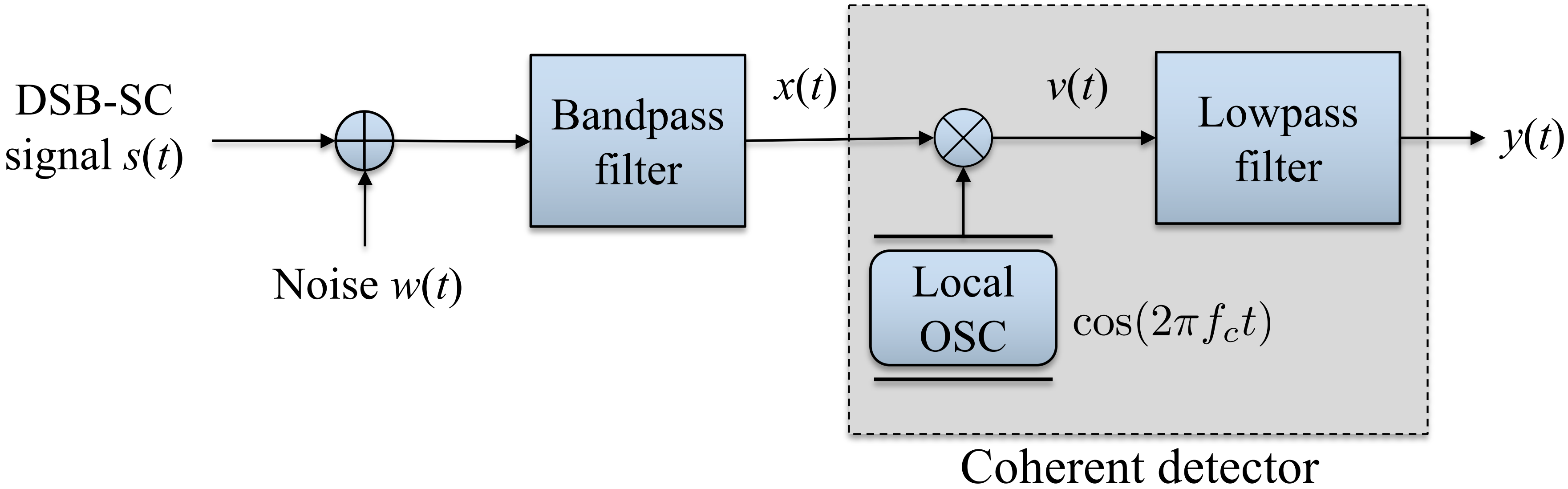
- Notes
  - $SNR_C$ has nothing to do with the receiver structure, but depends on the channel characteristic and modulation approach.
  - $SNR_O$ is receiver-structure dependent.
- Finally, define the *figure of merit* for the receiver as:

$$\text{figure of merit} = \frac{SNR_O}{SNR_C}$$

# Noise in Linear Coherent Receivers

- Recall that for demodulation of AM signal
  - when the carrier is suppressed, **linear** coherent detection is used.
  - when the carrier is additionally transmitted, **nonlinear** envelope detection is used.
- The noise analysis of the above two cases are respectively addressed in the sequel.

# Noise in Linear Coherent Receivers




$m(t)$ : stationary with zero meam and PSD $S_M(f)$ bandlimited to $W$

$$s(t) = A_c m(t) \cos(2\pi f_c t)$$

# Noise in Linear Coherent Receivers

- Average signal power

Here, we assume $m(t)$ stationary.

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[s^2(t)]\mathrm{d}t &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[A_c^2\cos^2(2\pi f_c t)m^2(t)]\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A_c^2\cos^2(2\pi f_c t)E[m^2(t)]\mathrm{d}t \\
&= A_c^2 P\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2(2\pi f_c t)\mathrm{d}t \\
&= \frac{1}{2}A_c^2 P
\end{aligned}$$

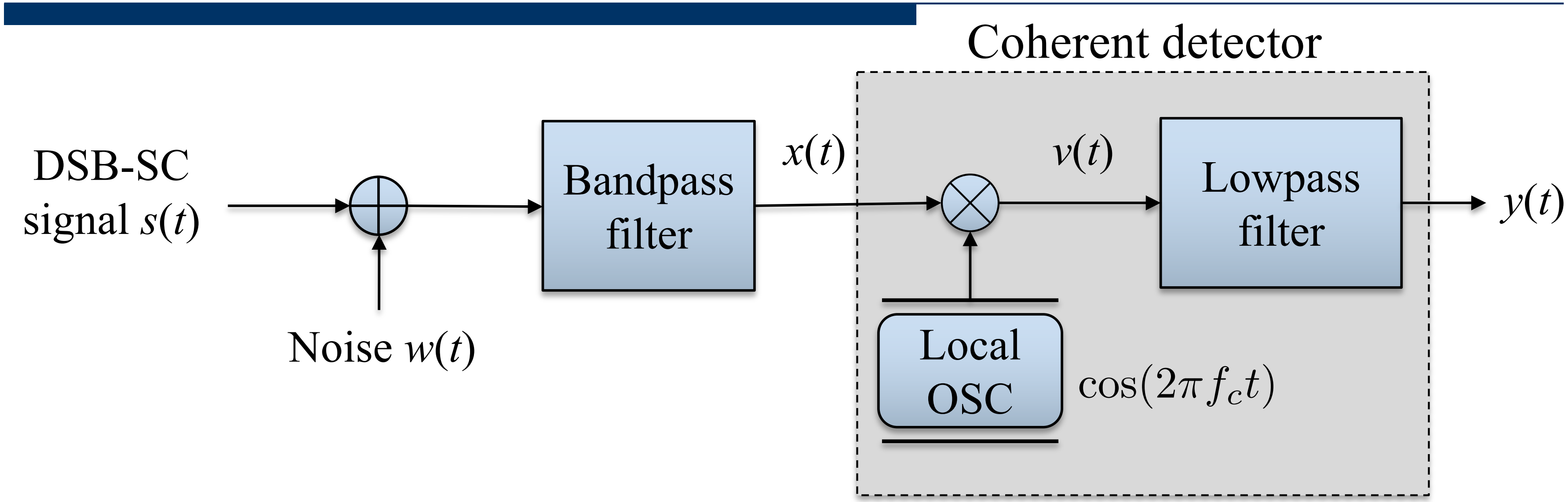
where $P = E[m^2(t)] = \int_{-W}^{W} S_M(f)df$ is the message power.

# Noise in Linear Coherent Receivers

- Noise power in the message bandwidth

$$\int_{-W}^{W} S_w(f)df = \int_{-W}^{W} \frac{N_0}{2} df = WN_0$$

# Noise in Linear Coherent Receivers




- Channel SNR for DSB-SC

$$\text{SNR}_{\text{C,DSB-SC}} = \frac{A_c^2 P/2}{W N_0} = \frac{A_c^2 P}{2 W N_0}$$

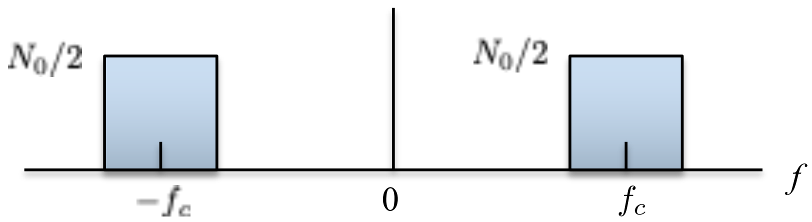
- Next, we calculate the output SNR (observed at $y(t)$) under the condition that the transmitter and the receiver are perfectly synchronized.

$$
\begin{aligned}
x(t) &= s(t) + n(t) \\
&= A_c m(t)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow v(t) &= x(t)\cos(2\pi f_c t) \\
&= [A_c m(t)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)]\cos(2\pi f_c t) \\
&= A_c m(t)\cos^2(2\pi f_c t) + n_I(t)\cos^2(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\
&\stackrel{\text{LowPass}}{\longrightarrow} \frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t)
\end{aligned}
$$

$$
\Rightarrow y(t) = \frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t)
$$

$$
\Rightarrow \text{SNR}_{\text{O,DSB-SC}} = \frac{E[A_c^2 m^2(t)/4]}{E[n_I^2(t)/4]} = \frac{A_c^2 P}{E[n^2(t)]} = \frac{A_c^2 P}{2WN_0}
$$



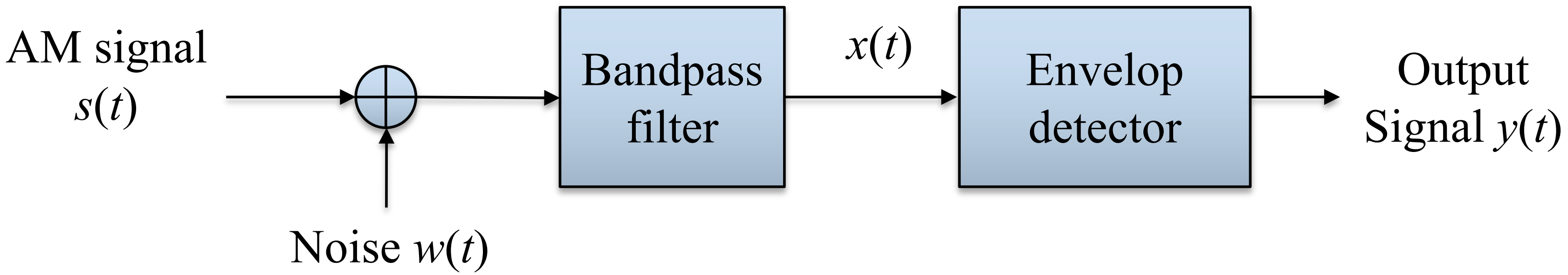

Recall $E[n^2(t)] = E[n_I^2(t)] = E[n_Q^2(t)]$.

$\Rightarrow$ Figure of merit for DSB-SC and coherent detection $= 1$.

Similar derivation on SSB and coherent detection yields the same *figure of merit*.

- Conclusions
  - Coherent detection for SSB performs the same as coherent detection for DSB-SC.
  - There is no SNR degradation for SSB and DSB-SC coherent receivers. The only effect of these modulation and demodulation processes is to translate the message signal to a different frequency band to facilitate its transmission over a band-pass channel.
  - No trade-off between noise performance and bandwidth. This may become a problem when high quality transceiving is required.

# Noise in Envelope Detector for AM




$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[s^2(t)]\,dt = A_c^2 E\left[(1+k_a m(t))^2\right]\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2(2\pi f_c t)\,dt$$

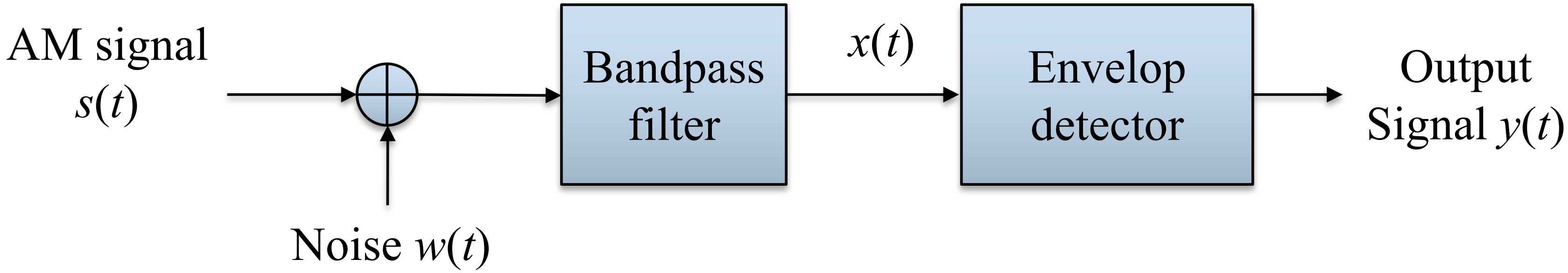
$$= \frac{A_c^2}{2}(1 + k_a^2 P) \qquad \text{(Assume } m(t) \text{ is zero mean.)}$$

Also, $\int_{-W}^{W} S_w(f)df = \int_{-W}^{W} \frac{N_0}{2} df = WN_0$

- Hence, channel SNR for DSB-C is equal to:

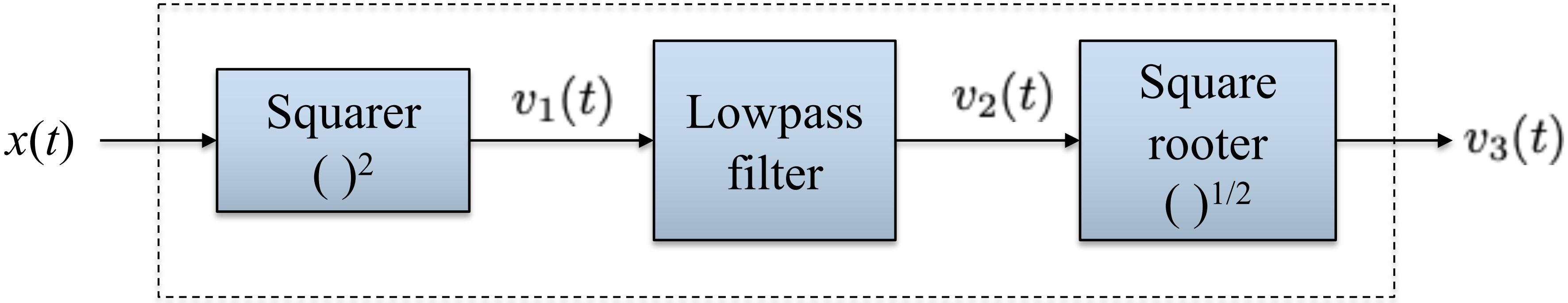
$$\Rightarrow SNR_{C,\text{AM}} = \frac{A_c^2(1+k_a^2 P)}{2WN_o}$$

- Next, we calculate the output SNR (observed at $y(t)$).




$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c[1 + k_a m(t)]\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= [A_c(1 + k_a m(t)) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

## Envelop detector




$$
\begin{aligned}
y(t) &= \sqrt{(x^2(t))_{\text{LowPass}}} \\
&= \frac{1}{\sqrt{2}}\sqrt{[A_c(1+k_a m(t)) + n_I(t)]^2 + \cancel{n_Q^2(t)}} \\
&\approx \frac{1}{\sqrt{2}}[A_c(1+k_a m(t)) + n_I(t)] \quad \text{if } A_c[1+k_a m(t)] \gg |\tilde{n}(t)|
\end{aligned}
$$

(Refer to Slides 5-20 and 5-22.)

$$
\stackrel{\text{block DC}}{=} \frac{1}{\sqrt{2}}[A_c k_a m(t) + n_I(t)]
$$

$$\Rightarrow \mathrm{SNR}_{\mathrm{O,AM}} \approx \frac{E[A_c^2 k_a^2 m^2(t)/2]}{E[n_I^2(t)/2]} = \frac{A_c^2 k_a^2 P}{E[n^2(t)]} = \frac{A_c^2 k_a^2 P}{2WN_0}$$

$$\Rightarrow \frac{SNR_{O,\mathrm{AM}}}{SNR_{C,\mathrm{AM}}} \approx \frac{A_c^2 k_a^2 P/(2WN_0)}{A_c^2(1+k_a^2 P)/(2WN_o)} = \frac{k_a^2 P}{1+k_a^2 P} < 1$$

- Conclusion
  - Even if the noise power is small when it is compared to the average carrier power at the envelope detector output, the noise performance of a full AM (DSC-C) receiver is inferior to that of a DSB-SC receiver due to the wastage of transmitter power.

# Noise in Envelope Detector for AM

- Assume $m(t) = A_m \cos(2\pi f_m t)$

  $$\Rightarrow s(t) = A_c[1 + k_a A_m \cos(2\pi f_m t)]\cos(2\pi f_c t)$$

  Hence,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[s^2(t)]dt = A_c^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[1 + k_a A_m \cos(2\pi f_m t)]^2 \cos^2(2\pi f_c t)dt$$

$$= A_c^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\cos^2(2\pi f_c t) + 2k_a A_m \cos(2\pi f_m t)\cos^2(2\pi f_c t) + k_a^2 A_m^2 \cos^2(2\pi f_m t)\cos^2(2\pi f_c t)\Big)dt$$

$$= A_c^2\left(\frac{1}{2} + 0 + \frac{k_a^2 A_m^2}{4}\right) = \frac{A_c^2}{2}(1 + k_a^2 P)$$

where $P = \lim_{T\to\infty} \frac{1}{T}\int_0^T A_m^2 \cos^2(2\pi f_m t)dt = \frac{A_m^2}{2}$.

$\Rightarrow$ Following similar procedure as previous discussion,

$$\frac{SNR_{O,\mathrm{AM}}}{SNR_{C,\mathrm{AM}}} \approx \frac{k_a^2 P}{1+k_a^2 P} = \frac{k_a^2 A_m^2/2}{1+k_a^2 A_m^2/2}.$$

So even if for 100% percent modulation ($k_a A_m = 1$), the figure of merit = 1/3. This means that *an AM system with envelope detection* must transmit **three** times as much average power as *DSB-SC with coherent detector* to achieve the same quality of noise performance.

# Threshold Effect

- What if $A_c[1+k_a m(t)] \gg |\tilde{n}(t)|$ is violated in *AM modulation with envelope detection*?

$$
\begin{aligned}
x(t) &= s(t) + n(t) \\
&= A_c[1+k_a m(t)]\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\
&= [A_c(1+k_a m(t)) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)
\end{aligned}
$$

$$B \gg |\tilde{n}| \Rightarrow \sqrt{(B+n_I)^2 + n_Q^2} = \sqrt{B^2 + 2n_I B + n_I^2 + n_Q^2} \approx \sqrt{B^2 + 2n_I B + n_I^2} = B + n_I$$

$$B \ll |\tilde{n}| \Rightarrow \sqrt{(B+n_I)^2 + n_Q^2} = \sqrt{B^2 + 2n_I B + |\tilde{n}|^2} \approx \sqrt{B^2 + 2|\tilde{n}|B + |\tilde{n}|^2} = B + |\tilde{n}|$$

Assume $A_c[1+k_a m(t)] \ll |\tilde{n}(t)|$

$$
\begin{aligned}
y(t) &= \sqrt{(x^2(t))_{\text{LowPass}}} \\
&= \frac{1}{\sqrt{2}}\sqrt{[A_c(1+k_a m(t))+n_I(t)]^2+n_Q^2(t)} \\
&= \frac{1}{\sqrt{2}}\sqrt{A_c^2(1+k_a m(t))^2+2n_I(t)A_c(1+k_a m(t))+|\tilde{n}(t)|^2} \\
&\approx \frac{1}{\sqrt{2}}\sqrt{A_c^2(1+k_a m(t))^2+2|\tilde{n}(t)|A_c(1+k_a m(t))+|\tilde{n}(t)|^2} \\
&\approx \frac{1}{\sqrt{2}}\sqrt{[A_c(1+k_a m(t))+|\tilde{n}(t)|]^2} \\
&= \frac{1}{\sqrt{2}}\left(A_c(1+k_a m(t))+|\tilde{n}(t)|\right) \\
&\overset{\text{Block DC}}{=} \frac{1}{\sqrt{2}}\left(A_c k_a m(t)+|\tilde{n}(t)|\right)
\end{aligned}
$$

$$\sqrt{2}y(t) = \begin{cases} A_c k_a m(t) + n_I(t), & A_c[1 + k_a m(t)] \gg |\tilde{n}(t)| \\ A_c k_a m(t) + |\tilde{n}(t)|, & A_c[1 + k_a m(t)] \ll |\tilde{n}(t)| \end{cases}$$

$$\Rightarrow \text{SNR}_{\text{O,AM}} = \frac{E[A_c^2 k_a^2 m^2(t)]}{E[|\tilde{n}(t)|^2]} = \frac{A_c^2 k_a^2 P}{E[n_I^2(t)] + E[n_Q^2(t)]} = \frac{A_c^2 k_a^2 P}{4WN_0}$$

$$\Rightarrow \frac{\text{SNR}_{O,\text{AM}}}{\text{SNR}_{C,\text{AM}}} = \frac{A_c^2 k_a^2 P/(4WN_0)}{A_c^2(1 + k_a^2 P)/(2WN_0)} = \frac{k_a^2 P}{2(1 + k_a^2 P)} < \frac{1}{6}$$

(See Slide 5-19.)

$$\begin{cases} \lim_{B\to 0}\left[\sqrt{(B+n_I)^2 + n_Q^2} - (B+n_I)\right] = |\tilde{n}| - n_I \\ \lim_{B\to\infty}\left[\sqrt{(B+n_I)^2 + n_Q^2} - (B+n_I)\right] = \lim_{B\to\infty} \dfrac{n_Q^2}{\sqrt{(B+n_I)^2 + n_Q^2} + (B+n_I)} = 0 \end{cases}$$

$$\Rightarrow \sqrt{(B+n_I)^2 + n_Q^2} \approx \begin{cases} (B+n_I) + |\tilde{n}| - n_I = B + |\tilde{n}|, & \text{if } B \to 0 \\ B + n_I, & \text{if } B \to \infty \end{cases}$$

Also, $\dfrac{d}{dB}\left[\sqrt{(B+n_I)^2 + n_Q^2} - (B+n_I)\right] = \dfrac{(B+n_I) - \sqrt{(B+n_I)^2 + n_Q^2}}{\sqrt{(B+n_I)^2 + n_Q^2}} < 0.$

# Threshold Effect

$$\rho = \frac{A_c^2/2}{2WN_0} = \frac{A_c^2}{4WN_0}$$

- Threshold effect
  - For AM with envelope detection, there exists a *carrier-to-noise ratio* $\rho$ (namely, the power ratio between unmodulated carrier $A_c\cos(2\pi f_c t)$ and the passband noise $n(t)$) below which the noise performance of a detector deteriorates rapidly.

$$\text{SNR}_{O,\text{AM}} = \begin{cases} \dfrac{A_c^2 k_a^2 P}{2WN_0} = 2k_a^2 P\rho, & \text{if } A_c[1+k_a m(t)] \gg |\tilde{n}(t)| \text{ i.e. } \rho \gg 1 \\ \dfrac{A_c^2 k_a^2 P}{4WN_0} = k_a^2 P\rho, & \text{if } A_c[1+k_a m(t)] \ll |\tilde{n}(t)| \text{ i.e. } \rho \ll 1 \end{cases}$$

# General $SNR_O$ in Envelope Detection

- For envelope detector, the noise is no longer additive; thus, the original definition of $SNR_O$ (which is based on additive noise) may not be applied.
- A new definition should be given:
  - **Definition**. The (general) *output signal-to-noise ratio* for an output $y(t)$ due to a carrier input is defined as

$$SNR_O = \frac{s_o^2}{\text{Var}[y(t)]}$$

where $s_o = E[y(t)] - E[y_o(t)]$, and $y_o(t)$ is equal to $y(t)$ in the presense of noise alone.

Conceptually, $y(t) = s_o + y_o(t)$.

# General $SNR_O$ in Envelope Detection

- $s_o$ is named the *mean output signal*. $s_o = E[y(t)] - E[y_o(t)]$
- Var[$y(t)$] is named the *mean output noise power*.
- **Example**. $y(t) = A + n_I(t)$, where $n_I(t)$ is zero mean.

$$\begin{cases} s_o = E[A + n_I(t)] - E[n_I(t)] = A \\ \mathrm{Var}[y(t)] = \mathrm{Var}[n_I(t)] = E[n_I^2(t)] \end{cases}$$

$$\Rightarrow SNR_O = \frac{A^2}{E[n_I^2(t)]}$$

*This shows the backward compatibility of the new definition.*

# General $SNR_O$ in Envelope Detection

- Now, for an envelope detector, the output due to a carrier input and additive Gaussian noise channel is given by:

$$y(t) = \sqrt{(A + n_I(t))^2 + n_Q^2(t)}$$

$I_0(\ )$ = modified Bessel function of the first kind of zero order.

$\Rightarrow y(t)$ is Rician distributed with pdf

$$f_{y(t)}(y) = \frac{y}{\sigma_N^2} \exp\left(-\frac{y^2 + A^2}{2\sigma_N^2}\right) I_0\left(\frac{Ay}{\sigma_N^2}\right) \text{ for } y \geq 0, \text{ where } \sigma_N^2 = E[n^2(t)] = 2WN_0$$

$\Rightarrow y_o(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$ is Rayleigh distributed with pdf

$$f_{y_0(t)}(y) = \frac{y}{\sigma_N^2} \exp\left(-\frac{y^2}{2\sigma_N^2}\right) \text{ for } y \geq 0, \text{ where } \sigma_N^2 = E[n^2(t)] = 2WN_0$$

$$
\begin{aligned}
E[y(t)] &= \int_0^\infty y f_{y(t)}(y)dy \\
&= \int_0^\infty \frac{y^2}{\sigma_N^2} \exp\left(-\frac{y^2+A^2}{2\sigma_N^2}\right) I_0\left(\frac{Ay}{\sigma_N^2}\right) dy \\
&= \frac{\sigma_N}{(2\rho)^{3/2}} \exp(-\rho) \int_0^\infty u^2 \exp\left(-\frac{u^2}{4\rho}\right) I_0(u)du, \\
&\quad \text{by taking } u = Ay/\sigma_N^2 \text{ and } \rho = A^2/(2\sigma_N^2) = A^2/(4WN_0).
\end{aligned}
$$

$$
\begin{aligned}
E[y_o(t)] &= \int_0^\infty y f_{y_o(t)}(y)dy = \int_0^\infty \frac{y^2}{\sigma_N^2} \exp\left(-\frac{y^2}{2\sigma_N^2}\right) dy \\
&= \sigma_N \int_0^\infty z^2 e^{-z^2/2} dz = \sigma_N \int_0^\infty z \cdot \left(z e^{-z^2/2}\right) dz \\
&= \sigma_N \left( z \cdot \left(-e^{-z^2/2}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-z^2/2}\right) dz \right) \\
&= \sigma_N \int_0^\infty e^{-z^2/2} dz = \sigma_N \sqrt{2\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \sigma_N \sqrt{\frac{\pi}{2}}
\end{aligned}
$$

# Confluent Hypergeometric Functions

- The *Kummer confluent hypergeometric function* is a solution of *Kummer's equation*

$$x\frac{d^2y}{dx^2}+(b-x)\frac{dy}{dx}-ay=0 \text{ for } a,b \text{ complex}$$

with boundary conditions $y(0)=1$ and $y'(0)=a/b$.

- For $b \neq 0, -1, -2, \ldots$, the *Kummer confluent hypergeometric function* is equal to ${}_1F_1(a;b;x)$.

Generalized hypergeometric function

$${}_pF_q(\vec{a};\vec{b};x)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\cdot\frac{x^k}{k!}, \text{ where } \begin{cases}(a)_k=a(a+1)\cdots(a+k-1)\\(a)_0=1\end{cases}.$$

# Properties of Confluent Hypergeometric Functions

1. ${}_1F_1(a;b;x) \approx 1 + \frac{a}{b}x \quad \text{as} \quad x \to 0.$

2. ${}_1F_1(-1;1;x) = 1 - x.$

3. ${}_1F_1(-1/2;1;-x) = \exp\left(-\frac{x}{2}\right) \times \left((1+x)I_0\left(\frac{x}{2}\right) + xI_2\left(\frac{x}{2}\right)\right)$

$$\approx 2\sqrt{\frac{x}{\pi}} \text{ as } x \to \infty.$$

4. $\int_0^\infty u^{m-1}\exp(-b^2u^2)I_0(u)du = \frac{\Gamma(m/2)}{2b^m}\left[{}_1F_1\left(\frac{m}{2};1;\frac{1}{4b^2}\right)\right]$

5. $\exp(-u)\cdot {}_1F_1(\alpha;\beta;u) = {}_1F_1(\beta-\alpha;\beta;-u)$

# General $SNR_O$ in Envelope Detection

- Hence,

$$E[y(t)] = \frac{\sigma_N}{(2\rho)^{3/2}} \exp(-\rho) \int_0^\infty u^2 \exp\left(-\frac{u^2}{4\rho}\right) I_0(u)\,du$$

$$= \frac{\sigma_N}{(2\rho)^{3/2}} \exp(-\rho) \frac{\Gamma(3/2)}{2(4\rho)^{-3/2}} \left[ {}_1F_1\left(\frac{3}{2};1;\rho\right) \right] \quad \textit{By Property 4}$$

$$= \sqrt{\frac{\pi}{2}} \sigma_N \exp(-\rho) \left[ {}_1F_1\left(\frac{3}{2};1;\rho\right) \right]$$

$$= \sqrt{\frac{\pi}{2}} \sigma_N \left[ {}_1F_1\left(-\frac{1}{2};1;-\rho\right) \right] \quad \textit{By Property 5}$$

# General $SNR_O$ in Envelope Detection

- As a result,

$$s_o = E[y(t)] - E[y_o(t)] = \sqrt{\frac{\pi}{2}}\sigma_N\left[{}_1F_1\left(-\frac{1}{2};1;-\frac{A^2}{2\sigma_N^2}\right) - 1\right].$$

- Similarly, we can obtain:

$$\text{Var}[y(t)] = 2\sigma_N^2\left[{}_1F_1\left(-1;1;-\frac{A^2}{2\sigma_N^2}\right) - \frac{\pi}{4}\left[{}_1F_1\left(-\frac{1}{2};1;-\frac{A^2}{2\sigma_N^2}\right)\right]^2\right]$$

$$= 2\sigma_N^2\left[1 + \frac{A^2}{2\sigma_N^2} - \frac{\pi}{4}\left[{}_1F_1\left(-\frac{1}{2};1;-\frac{A^2}{2\sigma_N^2}\right)\right]^2\right] \quad \textit{By Property 2}$$

# General $SNR_O$ in Envelope Detection

- This concludes to:

$$\mathrm{SNR}_O = \frac{\left[{}_1F_1(-1/2;1;-\rho)-1\right]^2}{\frac{4}{\pi}(1+\rho)-\left[{}_1F_1(-1/2;1;-\rho)\right]^2}, \text{ where } \rho=\frac{A^2}{2\sigma_N^2}$$

$$\approx \begin{cases} \dfrac{[2\sqrt{\rho/\pi}-1]^2}{\frac{4}{\pi}(1+\rho)-[2\sqrt{\rho/\pi}]^2}, & \text{as } \rho\to\infty \quad \text{(Property 3)} \\ \dfrac{[(1+\rho/2)-1]^2}{\frac{4}{\pi}(1+\rho)-(1+\rho/2)^2}, & \text{as } \rho\to 0 \quad \text{(Property 1)} \end{cases}$$

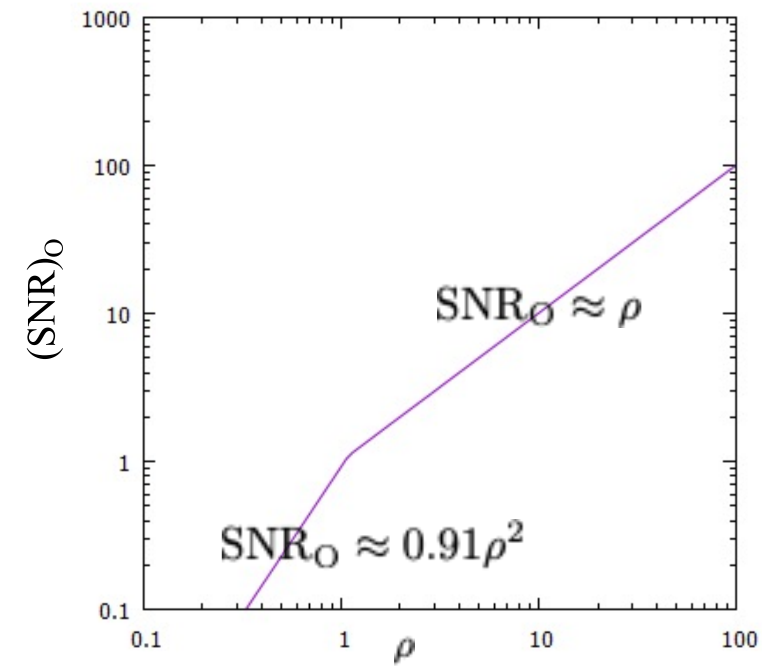
(Continue from the previous slide.)

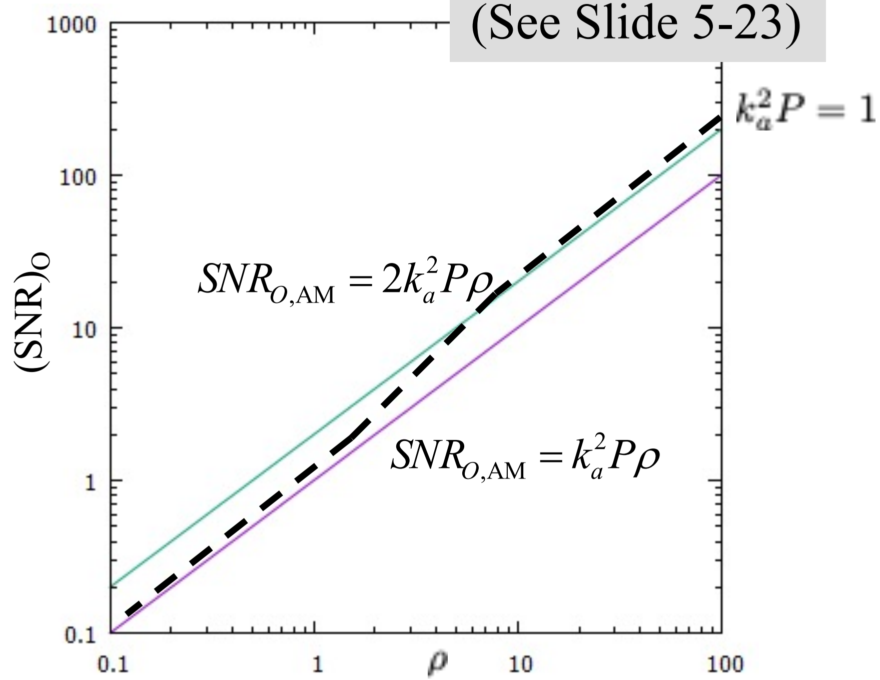
$$
\begin{aligned}
&= \begin{cases} \rho + \frac{\pi}{4} - \sqrt{\pi\rho}, & \text{as } \rho \to \infty \\ \dfrac{\pi\rho^2}{16(1+\rho) - \pi(2+\rho)^2}, & \text{as } \rho \to 0 \end{cases} \\
&\approx \begin{cases} \rho, & \text{as } \rho \to \infty \\ \frac{\pi\rho^2}{16-4\pi}, & \text{as } \rho \to 0 \end{cases} \\
&= \begin{cases} \rho, & \text{as } \rho \to \infty \\ 0.91\rho^2, & \text{as } \rho \to 0 \end{cases}
\end{aligned}
$$

# General $SNR_O$ in Envelope Detection

$$\text{Curve of } SNR_O = \frac{\left[{}_1F_1(-1/2;1;-\rho)-1\right]^2}{\frac{4}{\pi}(1+\rho)-\left[{}_1F_1(-1/2;1;-\rho)\right]^2}$$

and the two limiting approximates.



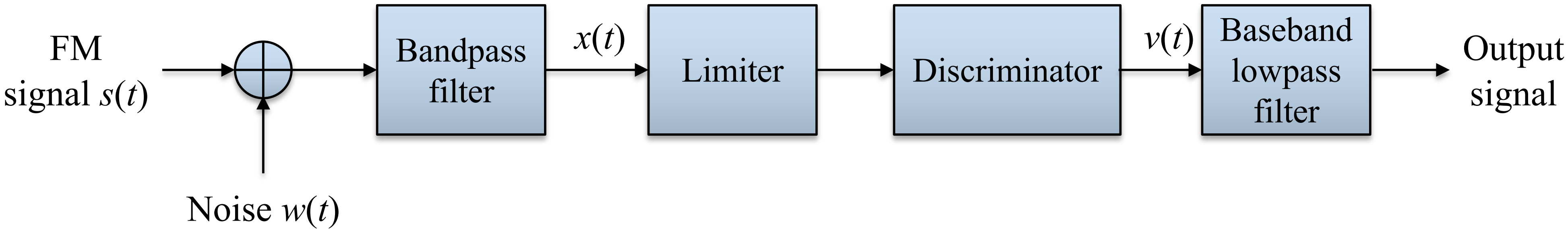

(See Slide 5-23)

# General $SNR_O$ in Envelope Detection

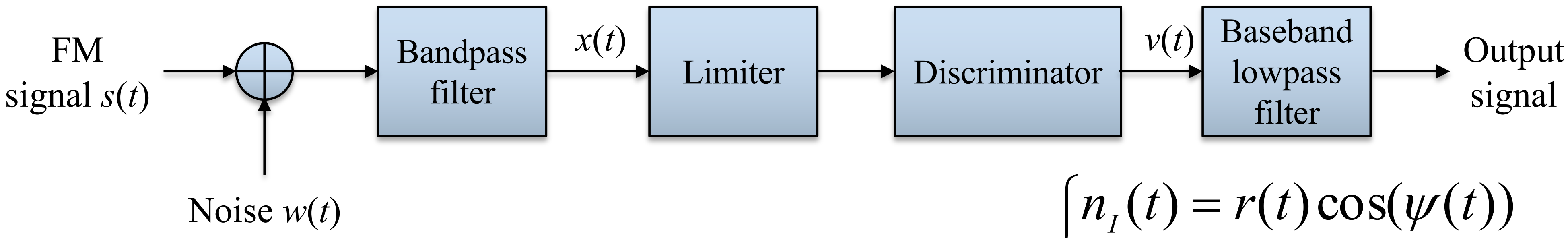
Slide 5-12: $\text{SNR}_{\text{O,DSB-SC}} = \frac{A_c^2 P}{2WN_0} = \frac{A_c^2 P}{\sigma_N^2} = 2P\rho$

- Remarks
  - For large *carrier-to-noise ratio* $\rho$, the envelope detector behaves like a coherent detector in the sense that the output SNR is proportional to $\rho$.
  - For small *carrier-to-noise ratio* $\rho$, the (newly defined) output signal-to-noise ratio of the envelope detector degrades faster than a linear function of $\rho$ (decrease at a rate of $\rho^2$).
  - From "threshold effect" and "general $\text{SNR}_O$," we can see that the envelope detector favors a strong signal. This is sometimes called "*weak signal suppression*."

# Impact of Noise in FM Receivers




- To simplify the system analysis, we assume:
  - ideal band-pass filter that is just wide enough to pass the modulated signal $s(t)$ without distortion,
  - ideal demodulator,
  - Gaussian distributed white noise process.
- So, the only source of imperfection is from the noise.

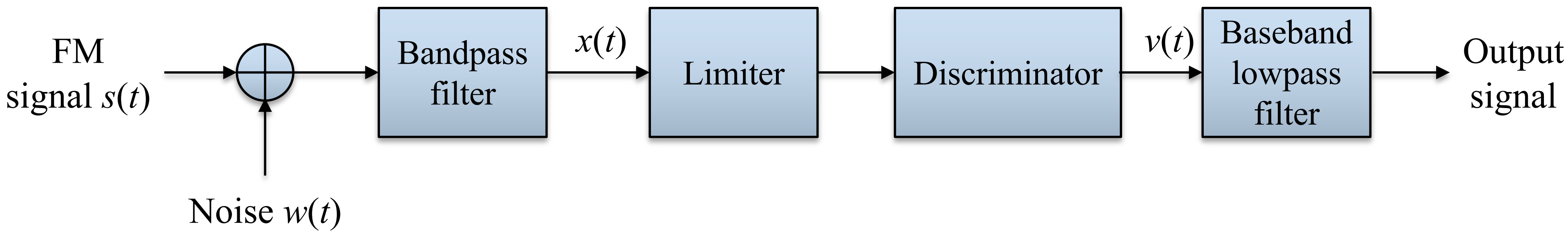



$$\begin{cases} n_I(t) = r(t)\cos(\psi(t)) \\ n_Q(t) = r(t)\sin(\psi(t)) \end{cases}$$

$$s(t) = A_c \cos[2\pi f_c t + \phi(t)], \text{ where } \phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau.$$

$$\begin{aligned} x(t) &= A_c \cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \psi(t)] \\ &= A_c \cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \phi(t) + \psi(t) - \phi(t)] \\ &= \left(A_c + r(t)\cos[\psi(t) - \phi(t)]\right)\cos[2\pi f_c t + \phi(t)] \\ &\quad - r(t)\sin[\psi(t) - \phi(t)]\sin[2\pi f_c t + \phi(t)] \\ &= \sqrt{\left(A_c + r(t)\cos[\psi(t) - \phi(t)]\right)^2 + r^2(t)\sin^2[\psi(t) - \phi(t)]}\cos[2\pi f_c t + \theta(t)] \end{aligned}$$

$$\text{where } \theta(t) = \phi(t) + \tan^{-1}\left\{\frac{r(t)\sin[\psi(t) - \phi(t)]}{A_c + r(t)\cos[\psi(t) - \phi(t)]}\right\}$$




$$x(t)=\sqrt{\left(A_c+r(t)\cos[\psi(t)-\phi(t)]\right)^2+r^2(t)\sin^2[\psi(t)-\phi(t)]}\cos[2\pi f_c t+\theta(t)]$$

$$\xrightarrow{\text{Limiter}} A\cdot\cos[2\pi f_c t+\theta(t)]$$

---

**Next, the signal will be passed through a Discriminator.**

Recall on Slides 4-87 ~ 4-94, we have talked about the *Balanced Frequency Discriminator*, whose input and output satisfy:

$$\text{Input } s(t)=A\cos(2\pi f_c t+\theta(t)) \quad \text{Output } \tilde{s}_o(t)=2aA\theta'(t)$$

$$\text{Recall } s(t)=A_c\cos\left[2\pi f_c t+2\pi k_f\int_0^t m(\tau)d\tau\right]\Rightarrow \tilde{s}_o(t)=4\pi k_f a A_c m(t)$$

Specifically, with $\tilde{s}(t) = A\exp(j\theta(t))$, we have:

$$\tilde{s}_1(t) = a\left[\frac{d\tilde{s}(t)}{dt} + j\pi B_T \tilde{s}(t)\right] = aAj\left[\theta'(t) + \pi B_T\right]\exp\left[j\theta(t)\right]$$

$$\tilde{s}_2(t) = -a\left[\frac{d\tilde{s}(t)}{dt} - j\pi B_T \tilde{s}(t)\right] = -aAj\left[\theta'(t) - \pi B_T\right]\exp\left[j\theta(t)\right]$$

$$\Rightarrow \tilde{s}_o(t) = |\tilde{s}_1(t)| - |\tilde{s}_2(t)| = 2aA\theta'(t)$$

Thus, after passing through the discriminator

$$v(t) = 2aA\theta'(t) = 2aA\frac{d\left(\phi(t) + \tan^{-1}\left\{\frac{r(t)\sin[\psi(t)-\phi(t)]}{A_c + r(t)\cos[\psi(t)-\phi(t)]}\right\}\right)}{dt}$$

Let $\alpha(t) = \psi(t) - \phi(t)$.

$$v(t) = 2aA\theta'(t) = 2aA\frac{d\left(\phi(t) + \tan^{-1}\left\{\frac{r(t)\sin(\alpha(t))}{A_c + r(t)\cos(\alpha(t))}\right\}\right)}{dt}$$

**Claim 1**: $\alpha(t) = \psi(t) - \phi(t)$ is uniformly distributed over $[0, 2\pi)$, and is independent of $m(t)$ and $r(t)$.

Thus, $r(t)\cos(\alpha(t))$ and $r(t)\sin(\alpha(t))$ have the same distributions as $n_I(t) = r(t)\cos(\psi(t))$ and $n_Q(t) = r(t)\sin(\psi(t))$.

1. $\alpha(t)$ being independent of $r(t)$ should be self-justified since both $\psi(t)$ and $m(t)$ are independent of $r(t)$.
2. $\psi(t)$ is independent of $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$ and is uniformly distributed over $[0, 2\pi)$ for given $m(t)$. Thus, $\psi(t) - \phi(t)$ is uniformly distributed over $[0, 2\pi)$ and is independent of $m(t)$.

$$
\begin{aligned}
v(t) &= 2aA\theta'(t) = 2aA\frac{d}{dt}\left(\phi(t) + \tan^{-1}\left\{\frac{n_Q(t)}{A_c + n_I(t)}\right\}\right) \\
&= 2aA\left(\phi'(t) + \frac{1}{1 + \left(\frac{n_Q(t)}{A_c+n_I(t)}\right)^2} \times \left(\frac{n'_Q(t)}{A_c + n_I(t)}\right.\right. \\
&\qquad \left.\left. - \frac{n_Q(t)n'_I(t)}{(A_c + n_I(t))^2}\right)\right)
\end{aligned}
$$

**Assumption 2**: $A_c \gg r(t)$ with high probability.

So that $A_c \gg |n_I(t)|$ and $A_c \gg |n_Q(t)|$ imply $A_c + n_I(t) \approx A_c$.

$$
v(t) \approx 2aA\left(\phi'(t) + \frac{1}{1 + \left(\frac{n_Q(t)}{A_c}\right)^2} \times \left(\frac{n'_Q(t)}{A_c} - \frac{n_Q(t)}{A_c}\cdot\frac{n'_I(t)}{A_c}\right)\right)
$$

**Assumption 2** implies $\frac{n_Q(t)}{A_c} \ll 1$ and $\frac{n_Q(t)}{A_c} \cdot \frac{n_I'(t)}{A_c} \ll \frac{n_Q'(t)}{A_c}$.

$$v(t) \approx 2aA\left(\phi'(t) + \frac{n_Q'(t)}{A_c}\right)$$

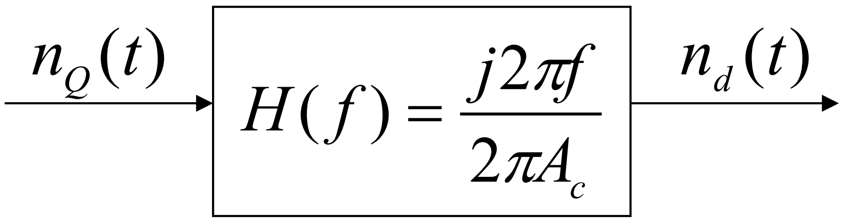
**Assumption 3**: $2aA = \frac{1}{2\pi}$.

$$2\pi v(t) \approx \phi'(t) + \frac{n_Q'(t)}{A_c} = 2\pi k_f m(t) + 2\pi n_d(t),$$

where $n_d(t) = \frac{n_Q'(t)}{2\pi A_c}$.

We then obtain the desired "additive" form.

Table 6.2 : 8. $\frac{d}{dt} g(t) \leftrightarrow j2\pi f G(f)$

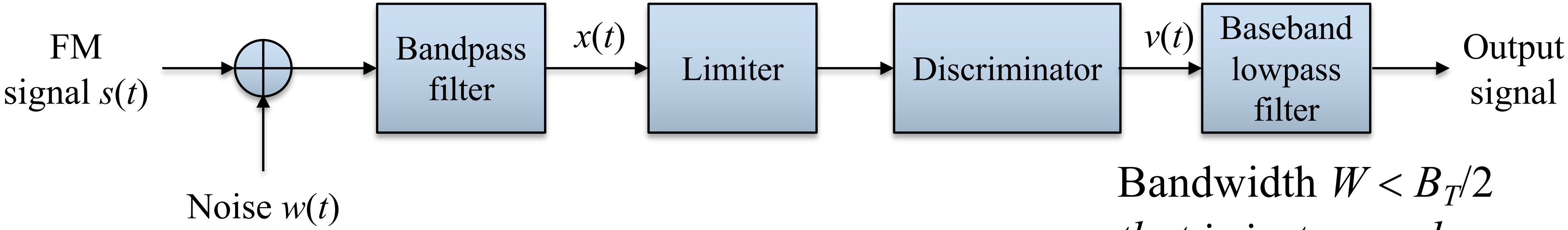


$$\Rightarrow S_{n_d}(f) = |H(f)|^2 S_{n_Q}(f) = \frac{f^2}{A_c^2} S_{n_Q}(f)$$

$$\Rightarrow S_{n_d}(f) = \begin{cases} \frac{f^2}{A_c^2} N_0, & |f| \le \frac{B_T}{2}; \\ 0, & \text{otherwise} \end{cases}$$

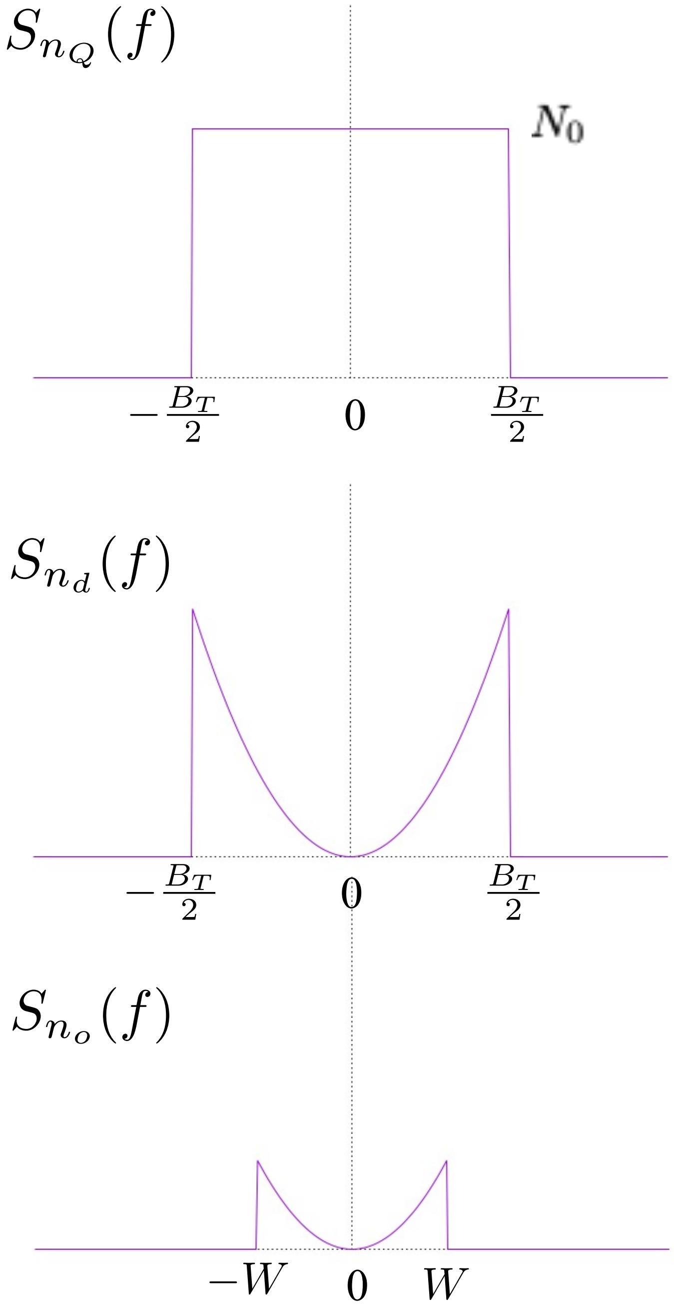
$$S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & \text{for } |f| < B_T/2 \\ 0, & \text{otherwise} \end{cases}$$

$v_o(t) \approx k_f m(t) + n_o(t)$




Bandwidth $W < B_T/2$
*that is just enough*
*to pass* $m(t)$.

$$\Rightarrow S_{n_o}(f) = \begin{cases} \frac{f^2}{A_c^2} N_0, & |f| \le W; \\ 0, & \text{otherwise} \end{cases}$$




$$\Rightarrow E[n_o^2(t)] = \frac{N_0}{A_c^2}\int_{-W}^{W} f^2 df = \frac{2N_0W^3}{3A_c^2}$$

**Observation from the above formula:**

In an FM system, increasing carrier power $A_c^2$ = Decreasing noise power.

This is named the **noise quieting effect.**

As $v_o(t) \approx k_f m(t) + n_o(t)$,

$$\Rightarrow SNR_{O,\mathrm{FM}} = \frac{k_f^2 E[m^2(t)]}{\frac{2N_0W^3}{3A_c^2}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}, \text{provided } A_c >> r(t).$$

---

We next turn to $SNR_{C,\mathrm{FM}}$.

$s(t) = A_c \cos[2\pi f_c t + \phi(t)]$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$.

$\Rightarrow$ average power in the modulated signal $s(t)$ is $A_c^2/2$.

Average noise power in the message bandwidth is $\int_{-W}^{W} \frac{N_0}{2} df = WN_0$.

$$\Rightarrow SNR_{C,\mathrm{FM}} = \frac{A_c^2/2}{N_0 W} = \frac{A_c^2}{2N_0 W}.$$

$$\Rightarrow \frac{SNR_{O,\mathrm{FM}}}{SNR_{C,\mathrm{FM}}} = \frac{\dfrac{3A_c^2 k_f^2 P}{2N_0 W^3}}{\dfrac{A_c^2}{2WN_0}} = \frac{3k_f^2 P}{W^2}.$$

**Remarks**: For fixed $W$, increasing $B_T \Leftrightarrow$ increasing $\dfrac{SNR_{O,\mathrm{FM}}}{SNR_{C,\mathrm{FM}}}$.

1. Deviation ratio $D = \dfrac{\Delta f}{W} \propto \dfrac{k_f P^{1/2}}{W}$.

$$\Delta f = k_f \max |m(t)|$$

Hence, $\dfrac{SNR_{O,\mathrm{FM}}}{SNR_{C,\mathrm{FM}}} \propto D^2$.

$$\frac{\mathrm{SNR}_{O,\mathrm{FM}}}{\mathrm{SNR}_{C,\mathrm{FM}}} \propto B_T^2$$

2. $B_{T,\mathrm{Carson}} = 2\Delta f\left(1+\dfrac{1}{D}\right) = 2DW\left(1+\dfrac{1}{D}\right) = 2W(D+1)$

(From Slide 5-19, $2P = A_m^2$)

(From Slide 4-60, $D = \beta = \frac{k_f A_m}{f_m} = \frac{k_f\sqrt{2P}}{W}$ )

# Summary

$$\frac{\text{SNR}_\text{O}}{\text{SNR}_\text{C}} = \begin{cases} \frac{3k_f^2 P}{W^2} = \frac{3}{2}\left(\frac{B_{T,\text{Carson}}}{2W} - 1\right)^2, & \text{FM } (D^2 = \frac{2k_f^2 P}{W^2}) \\ 1, & \text{DSB-SC} \\ \frac{k_a^2 P}{1+k_a^2 P}, & \text{AM } (\rho \text{ large}, k_a^2 P < \frac{1}{2}) \\ \frac{1}{2}\left(\frac{k_a^2 P}{1+k_a^2 P}\right), & \text{AM } (\rho \text{ small}, k_a^2 P < \frac{1}{2}) \end{cases}$$

- Specifically,
  - for high *carrier-to-noise ratio* $\rho$ (equivalent to the assumption made in Assumption 2), an increase in transmission bandwidth $B_T$ provides a corresponding quadratic increase in figure of merit of a FM system.
    - So, there is a tradeoff between $B_T$ and figure of merit.
- Notably, figure of merit for an AM system has nothing to do with $B_T$.

# Single-Tone FM Signal with Noise

- $m(t) = A_m \cos(2\pi f_m t)$
- Then we can represent the figure of merit in terms of modulation index (or deviation ratio) $\beta$ as:

$$\Rightarrow \frac{\text{SNR}_{\text{O,FM}}}{\text{SNR}_{\text{C,FM}}} = \frac{3k_f^2 P}{W^2} = \frac{3k_f^2 (A_m^2/2)}{f_m^2} = \frac{3}{2}\frac{(\Delta f)^2}{f_m^2} = \frac{3}{2}\beta^2$$

- In order to make the figure of metric for an FM system to be superior to that for an AM system with 100% modulation, it requires:

$$\frac{3}{2}\beta^2 \geq \frac{1}{3} \Rightarrow \beta > \frac{\sqrt{2}}{3} \approx 0.471$$

$$B_{\text{T,Carson}} = 2\Delta f\left(1 + \frac{1}{\beta}\right) = 2\beta f_m \left(1 + \frac{1}{\beta}\right) = 2(\beta + 1) f_m$$

# Capture Effect

- Recall that in **Assumption 2**, we assume $A_c >> r(t)$.
- This somehow hints that the *noise suppression* of an FM modulation works well when the noise (*or other unwanted modulated signal that cannot be filtered out by either bandpass or lowpass filters*) is weaker than the desired FM signal.
- What if the unwanted FM signal is stronger than the desired FM signal.
  - The FM receiver will *capture* the unwanted FM signal!
- What if the unwanted FM signal has nearly equal strength as the desired FM signal.
  - The FM receiver will fluctuate back and forth between them!

# FM Threshold Effect

- Recall that in **Assumption 2**, we assume $A_c >> r(t)$ (equivalently, a high *carrier-to-noise ratio*) to simplify $\theta(t)$ so that the next formula holds.

$$SNR_{O,\mathrm{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}$$

- However, a further decrease of carrier-to-noise ratio will break the FM receiver (from a clicking sound down to a crackling sound).
- As the same as the AM modulation, this is also named the *threshold effect*.

# FM Threshold Effect

- Consider a simplified case with $m(t) = 0$ (no message signal). From Slide 5-41, we have

$$2\pi v(t) = \underbrace{\phi'(t)}_{=0} + \frac{n'_Q(t)[A_c + n_I(t)] - n_Q(t)n'_I(t)}{[A_c + n_I(t)]^2 + n_Q^2(t)}$$

To facilitate the understanding of "clicking" sound effect, we let $r(t) = \lambda A_c$, a constant ratio of $A_c$.

$\Rightarrow n_I(t) = \lambda A_c \cos[\psi(t)]$ and $n_Q(t) = \lambda A_c \sin[\psi(t)]$ imply

$$\begin{aligned} 2\pi v(t) &= \frac{\lambda A_c \psi'(t)\cos[\psi(t)](A_c + \lambda A_c \cos[\psi(t)]) + \lambda^2 A_c^2 \psi'(t)\sin^2[\psi(t)]}{A_c^2 + 2\lambda A_c^2 \cos[\psi(t)] + \lambda^2 A_c^2} \\ &= \frac{\lambda\psi'(t)\cos[\psi(t)] + \lambda^2\psi'(t)}{1 + 2\lambda\cos[\psi(t)] + \lambda^2} \end{aligned}$$

# FM Threshold Effect

$$\Rightarrow 2\pi v(t) = \theta'(t) = \lambda\psi'(t)\frac{\cos[\psi(t)] + \lambda}{1 + 2\lambda\cos[\psi(t)] + \lambda^2}$$

Then at the time, say, $\psi(t) \approx \pi$, and $\lambda > 1$ but $\lambda \approx 1$

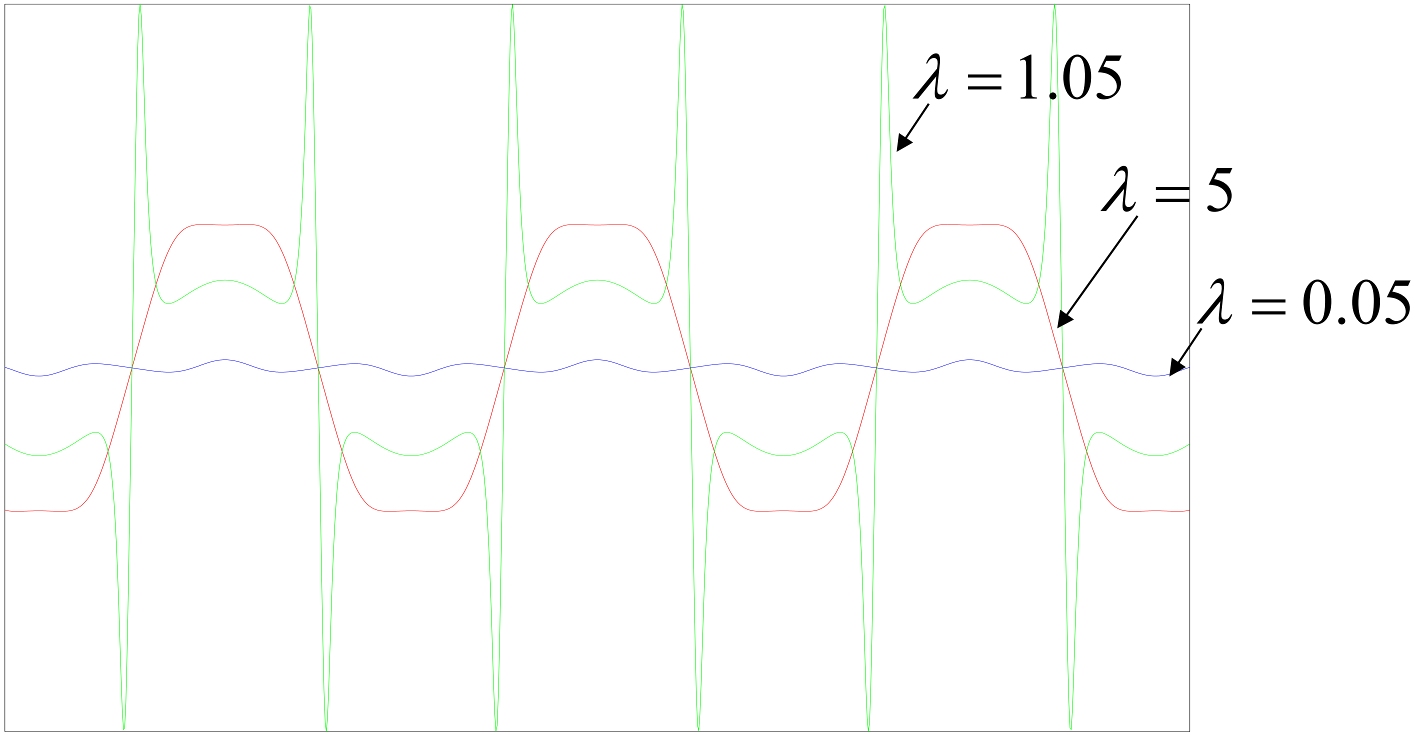
$$\Rightarrow 2\pi v(t) = \theta'(t) = \lambda\psi'(t)\frac{\cos[\psi(t)] + \lambda}{1 + 2\lambda\cos[\psi(t)] + \lambda^2} \approx \frac{\lambda}{\lambda - 1}\psi'(t)$$

**Thus, a sign change in $\psi'(t)$ will cause a spike!**

*Notably, when $\lambda$ = 0 (no noise), the output equals $m(t) = 0$ as desired.*

# FM Threshold Effect

$$\psi(t) = \pi \sin(t) \Rightarrow 2\pi\nu(t) = \theta'(t) = \lambda\pi\cos(t)\frac{\cos[\pi\sin(t)] + \lambda}{1 + 2\lambda\cos[\pi\sin(t)] + \lambda^2}$$



# How to Avoid “Clicking” Sound?

Fix modulation index (or deviation ratio) $\beta$ and message signal bandwidth $W$:

1. Determine $B_T$ by either Carson’s rule or Universal curve.
2. For a specified noise level $N_0$, select $A_c$ to satisfy:

$$10\log_{10}\left(\frac{A_c^2}{2B_T N_0}\right) \geq 13\,\text{dB} \quad \text{or equvalently, } \frac{A_c^2}{2B_T N_0} \geq 20.$$

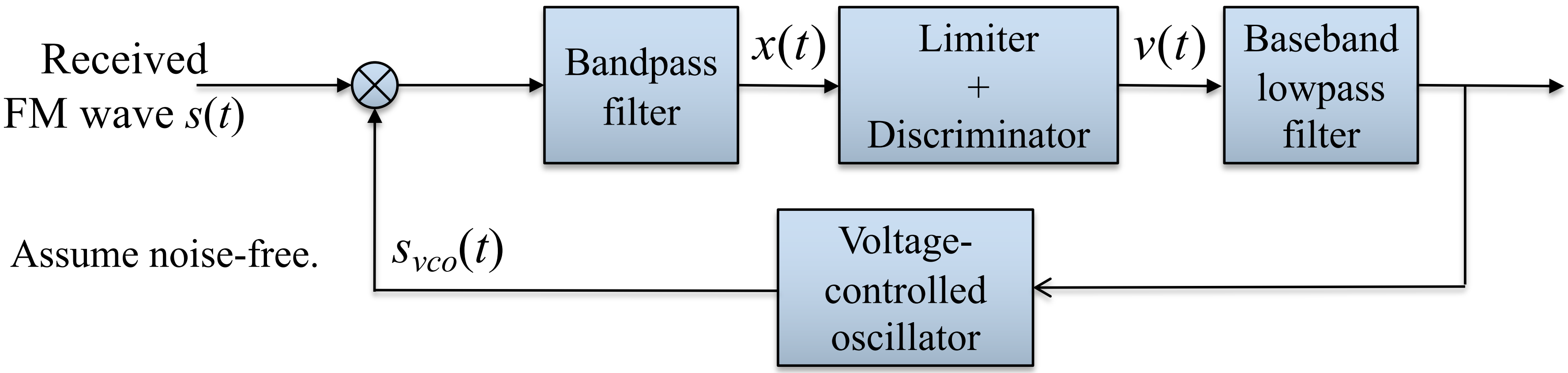
Experiments found that occasional clicks are heard at $\rho$ around 13 dB.

# Threshold Reduction

- After our learning that FM modulation has threshold effect, the next question is naturally on "*how to reduce the threshold*?"

# Threshold Reduction

- Threshold reduction in FM receivers may be achieved by
  1. negative feedback (commonly referred to as an *FMFB demodulator*), or
  2. *phase-locked loop demodulator*.
     - Why PLL can reduce threshold effect is not covered in this course.




$$s(t) = A_c \cos[2\pi f_c t + \phi(t)], \text{ where } \phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau.$$

$$s_{vco}(t) = 2\cos[2\pi f_{vco} t + \phi_{vco}(t)], \text{ where } \phi_{vco}(t) = 2\pi\alpha k_f \int_0^t m(\tau)d\tau.$$

$$s(t)s_{vco}(t) = 2A_c \cos[2\pi f_c t + \phi(t)]\cos[2\pi f_{vco} t + \phi_{vco}(t)]$$

$$\xrightarrow{\text{Bandpass}} A_c \cos[2\pi(f_c - f_{vco})t + (1-\alpha)\phi(t)]$$

The new frequency deviation $\Delta f_{new} = (1-\alpha)\Delta f_{original}$.

Thus, the bandpass filter can conceptually have a smaller passband as wide as $(1-\alpha)B_T$, centered at $(f_c - f_{vco})$.

$$n(t)s_{vco}(t) = 2n(t)\cos[2\pi f_{vco}t + \phi_{vco}(t)]$$

$\Rightarrow$ The noise at the Mixer output can be treated white with the same noise level as the input white noise.

$$\begin{aligned} x(t) &= A_c \cos[2\pi f_c' t + (1-\alpha)\phi(t)] + r(t)\cos[2\pi f_c' t + \psi(t)] \\ &\stackrel{\text{Limiter}}{\longrightarrow} \cos[2\pi f_c' t + \theta(t)] \end{aligned}$$

$f_c' = f_c - f_{\text{vco}}$

where $E[n_I^2(t)] = E[n_Q^2(t)] = E[n^2(t)] = (1-\alpha)B_T N_0$,

and $\theta(t) = (1-\alpha)\phi(t) + \tan^{-1}\left\{\dfrac{r(t)\sin[\psi(t) - (1-\alpha)\phi(t)]}{A_c + r(t)\cos[\psi(t) - (1-\alpha)\phi(t)]}\right\}$.

Since $E[n_I^2(t)] = E[n_Q^2(t)]$ is smaller, and $A_c$ remains the same, the condition $A_c \gg r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$ holds with higher probability.

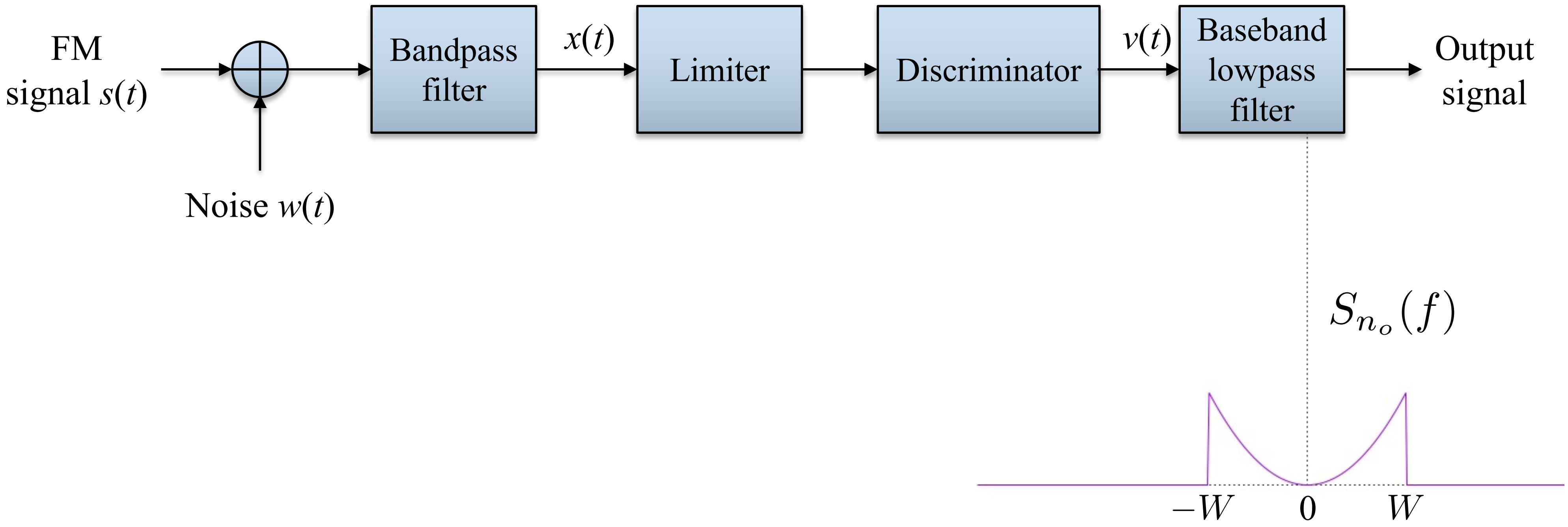
Experiments show that an FMFB receiver is capable of realizing a threshold extension on the order of 5~7 dB.

# Threshold Reduction of an FMFB Receiver

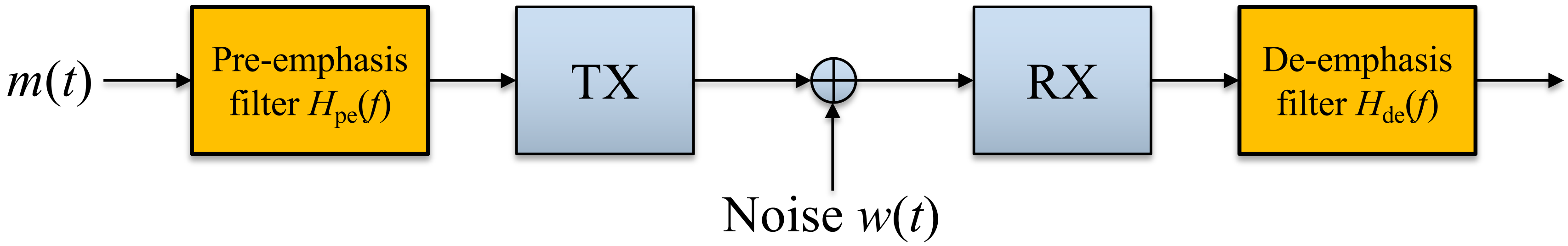
- To sum up:
  - An FMFB demodulator is essentially a *tracking filter* that can track only the *slowly varying frequency of a FM signal*.

# Pre-Emphasis and De-emphasis for FM

- Recall that the noise PSD at the output shapes like a bowel.
- If we can "equalize" the signal-to-noise power ratios over the entire message band, a better noise performance should result.



# Pre-Emphasis and De-emphasis for FM

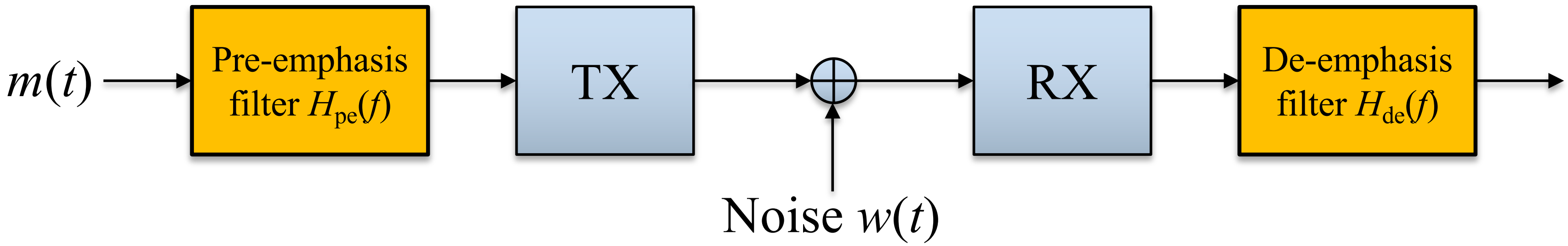



- Now instead of change/equalize the signal PSD, we produce an *undistorted* version of the original message at the receiver output with

$$H_{pe}(f)H_{de}(f) = 1 \text{ for } -W \le f \le W.$$

- This relation guarantees the intactness of the message power.
- Next, we need to find $H_{de}(f)$ such that the noise power is optimally suppressed.

# Pre-Emphasis and De-emphasis for FM




- Under the assumption of high carrier-to-noise ratio, the noise PSD at the de-emphasis filter output is given by:

$$|H_{\rm de}(f)|^2 S_{n_o}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}|H_{\rm de}(f)|^2, & |f| \le W; \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \text{Average noise power} = \int_{-W}^{W} \frac{N_0 f^2}{A_c^2} |H_{de}(f)|^2 \, df$$

# Pre-Emphasis and De-emphasis for FM

- Since the message power remains the same, the improvement factor of the output signal-to-noise ratio after and before pre/de-emphasis is:

$$I = \frac{\int_{-W}^{W} \frac{N_0 f^2}{A_c^2} df}{\int_{-W}^{W} \frac{N_0 f^2}{A_c^2} |H_{de}(f)|^2 \, df} = \frac{\int_{-W}^{W} f^2 df}{\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df} = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df}$$

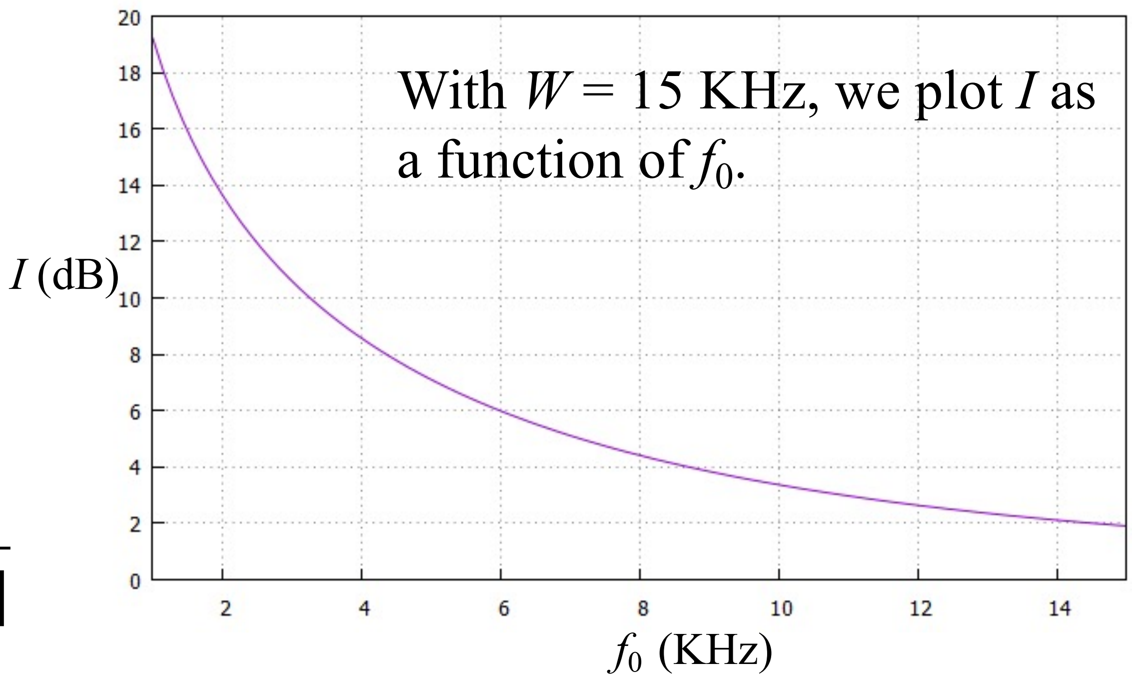
# Pre-Emphasis and De-emphasis for FM

First order filter: $H_{de}(f) = \frac{1}{1+j(f/f_0)}$

$$\Rightarrow I = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df}$$

$$= \frac{2W^3}{3\int_{-W}^{W} \frac{f^2}{1+(f/f_0)^2} \, df}$$

$$= \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]}$$



# Pre-Emphasis and De-emphasis for FM

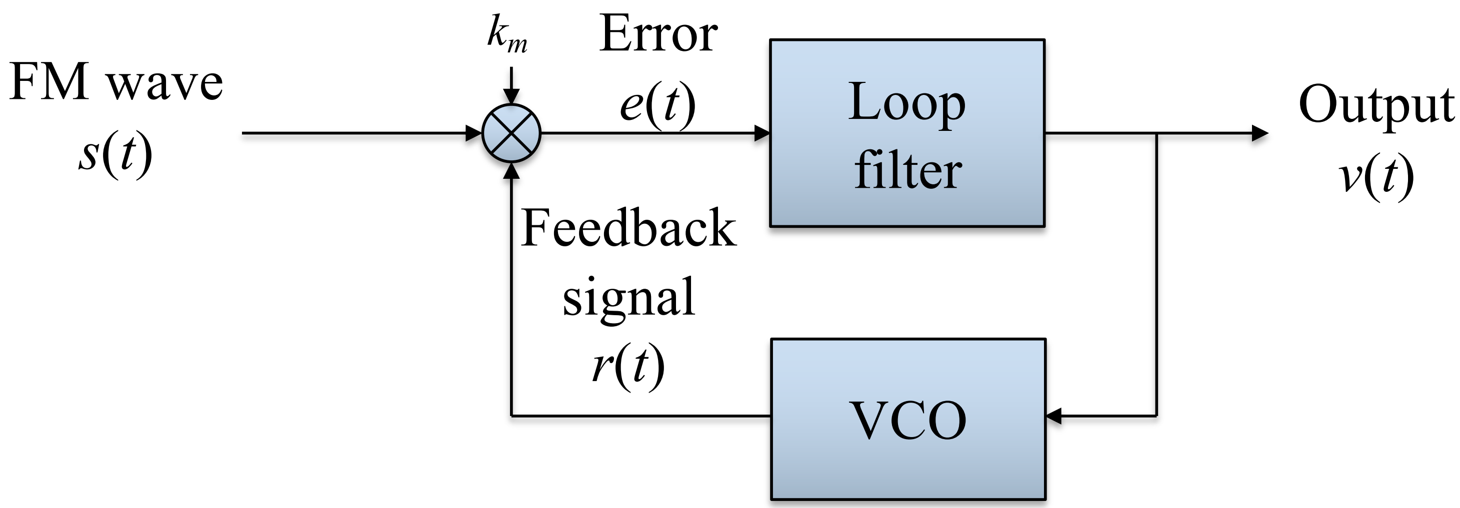
- Final remarks:
  - In the previous trial, we simply use a first order linear filter to improve the system.
  - Nonlinear pre-emphasis and de-emphasis filters have been applied to applications like tape recording. These techniques, known as Dolby-A, Dolby-B, and DBX systems, use a combination of filtering and dynamic range compression to reduce the effects of noise.

# Phase-Locked Loop

- Phase-locked loop

$$s(t) = A_c \sin[2\pi f_c t + \phi_1(t)], \text{ where } \phi_1(t) = 2\pi k_f \int_0^t m(\tau)d\tau.$$

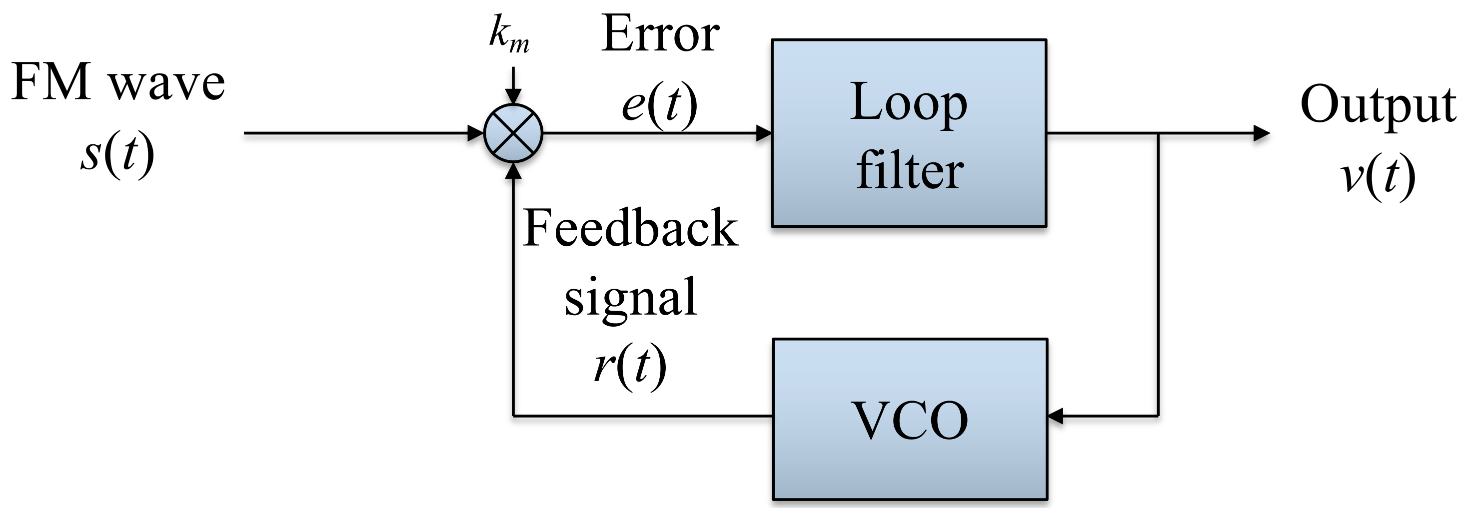
$$r(t) = A_v \cos[2\pi f_c t + \phi_2(t)], \text{ where } \phi_2(t) = 2\pi k_v \int_0^t v(\tau)d\tau.$$



## Loop filter = lowpass filter + filter $h(\tau)$.

$$
\begin{aligned}
e(t) &= k_m s(t) r(t) \\
&= k_m A_c \sin[2\pi f_c t + \phi_1(t)] \cdot A_v \cos[2\pi f_c t + \phi_2(t)] \\
&= \frac{k_m A_c A_v}{2} \left( \sin[4\pi f_c t + \phi_1(t) + \phi_2(t)] + \sin[\phi_1(t) - \phi_2(t)] \right) \\
&\xrightarrow{\text{Low Pass}} \frac{k_m A_c A_v}{2} \sin[\phi_e(t)], \quad \text{where } \phi_e(t) = \phi_1(t) - \phi_2(t).
\end{aligned}
$$

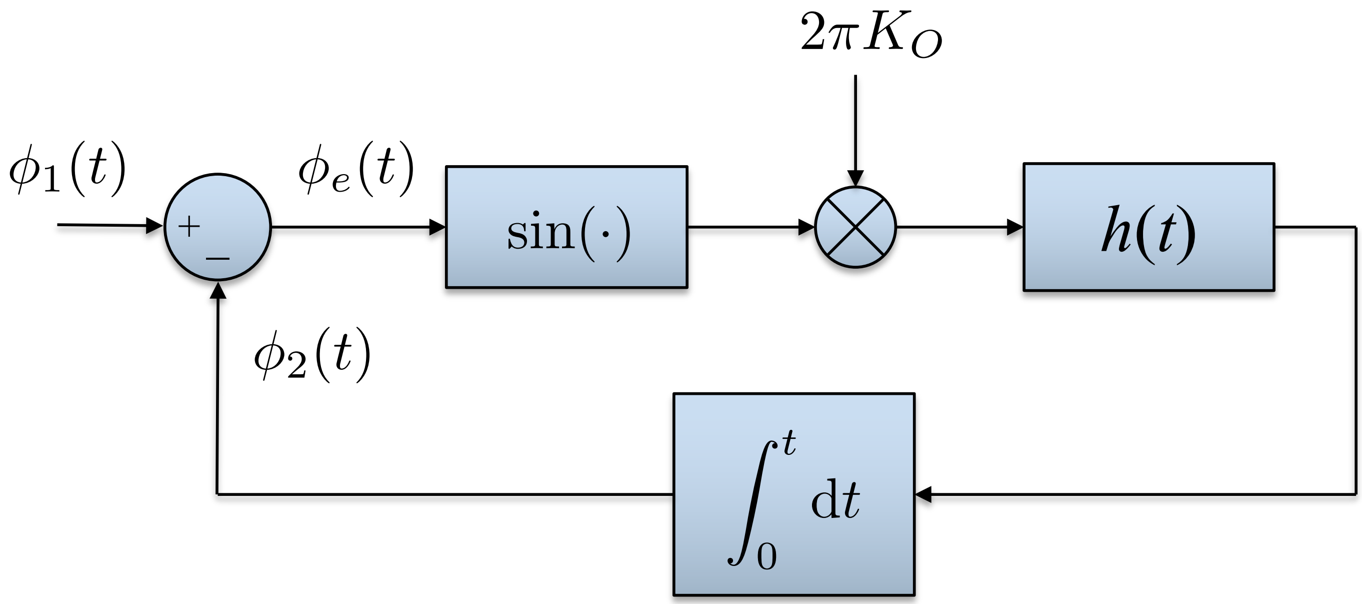
Also, $v(t) = \int_{-\infty}^{\infty} \{e(\tau)\}_{\text{LowPass}} h(t-\tau) d\tau$.



$$
\begin{aligned}
\phi_e(t) &= \phi_1(t) - \phi_2(t) \\
&= \phi_1(t) - 2\pi k_v \int_0^t v(\tau)\mathrm{d}\tau \\
&= \phi_1(t) - 2\pi k_v \int_0^t \left( \int_{-\infty}^{\infty} \frac{k_m A_c A_v}{2} \sin[\phi_e(u)] h(\tau - u)\mathrm{d}u \right) \mathrm{d}\tau \\
&= \phi_1(t) - \int_0^t \left( \int_{-\infty}^{\infty} 2\pi k_0 \sin[\phi_e(u)] h(\tau - u)\mathrm{d}u \right) \mathrm{d}\tau
\end{aligned}
$$

where $k_0 = k_m k_v A_c A_v / 2.$

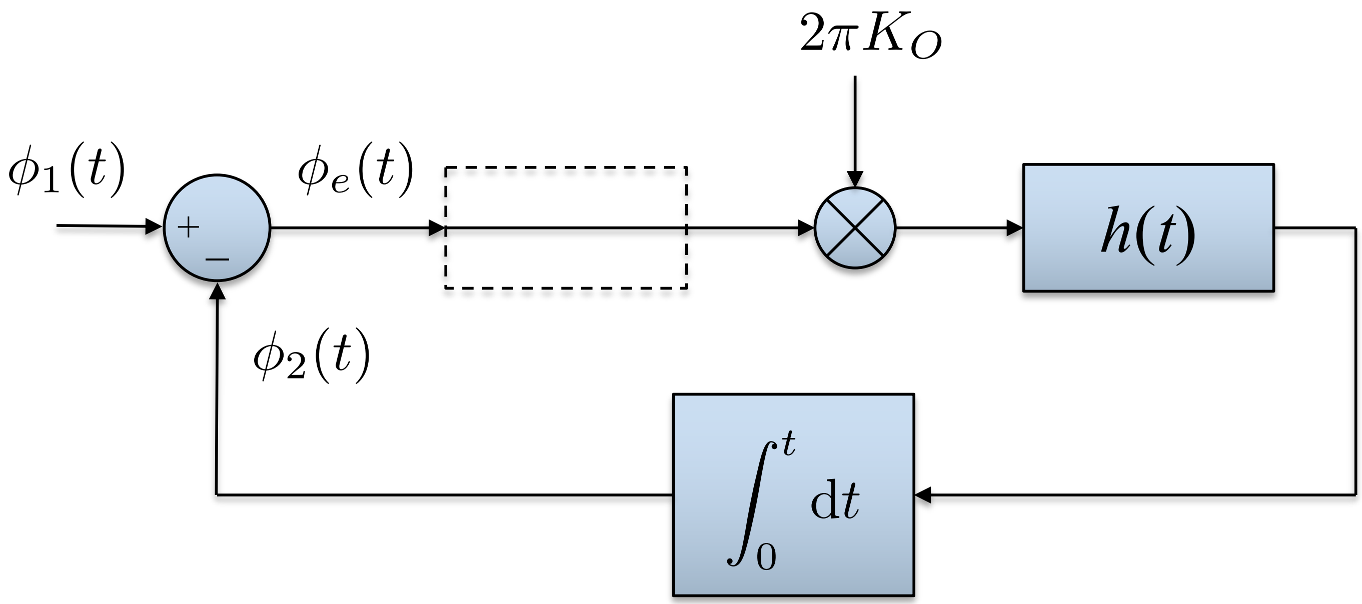
The previous formula suggests an equivalent analytical model for PLL.




When $\phi_e(t) = 0$, the system is said to be in *phase-lock*.
In this case, $\phi_1(t) = \phi_2(t)$ or equivalently, $k_v v(t) = k_f m(t)$.

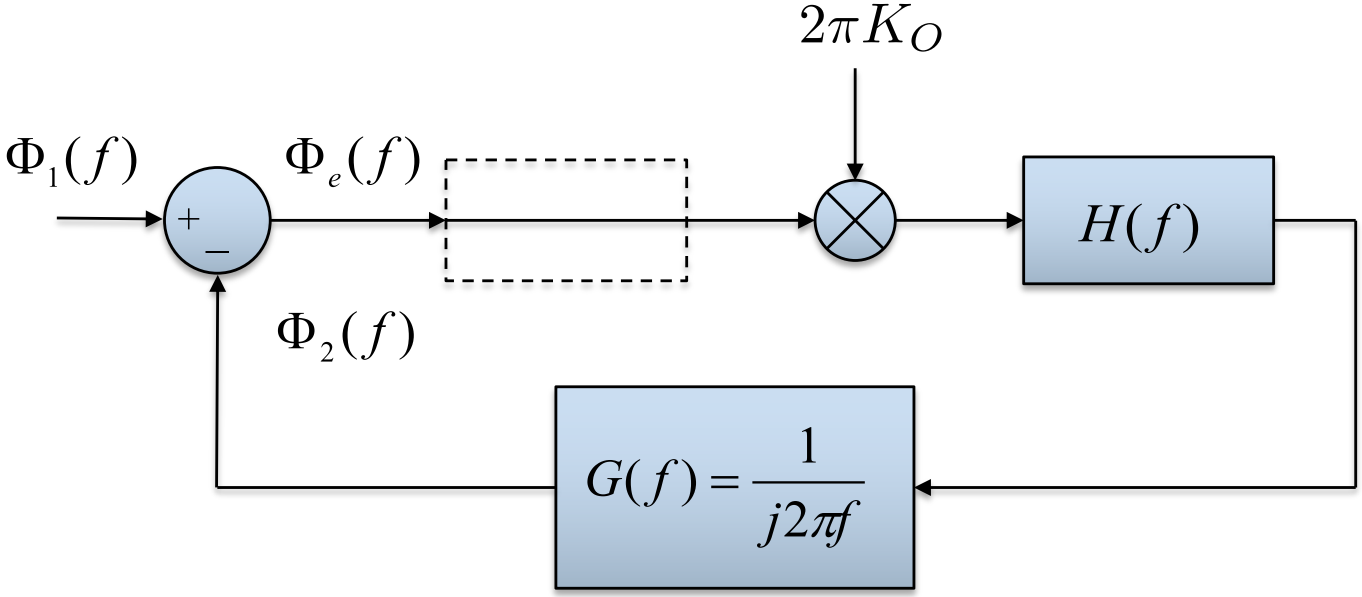
When $\phi_e(t)$ is small (< 0.5 radians), the system is said to be *nearly phase-locked*.
In this case, we can approximate $\sin[\phi_e(t)]$ by $\phi_e(t)$; hence, a linear approximate model is resulted.

Linearization approximation model for PLL.




We can transform the above time-domain system to its equivalent frequency domain system to facilitate its analysis.

# Linearization approximation model for PLL.




$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{\Phi_e(f)}{[\Phi_1(f) - \Phi_2(f)] + \Phi_2(f)}$$

$$= \frac{\Phi_e(f)}{\Phi_e(f) + 2\pi k_0 \Phi_e(f) H(f) G(f)}$$

$$= \frac{1}{1 + 2\pi k_0 H(f) G(f)} = \frac{jf}{jf + k_0 H(f)}$$

# First-Order PLL

$H(f) = 1.$

$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{j(f/k_0)}{1 + j(f/k_0)}$$

A parameter $k_0$ controls both the loop gain and bandwidth of the filter. In other words, it is impossible to adjust the loop gain without changing the filter bandwidth.

# Second-Order PLL

$H(f) = 1 + a/(jf)$ and using linear PLL model.

$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{jf}{jf + k_0 H(f)} = \frac{jf}{jf + k_0(1 + a/(jf))} = \frac{(jf)^2}{(jf)^2 + k_0(jf) + k_0 a}$$

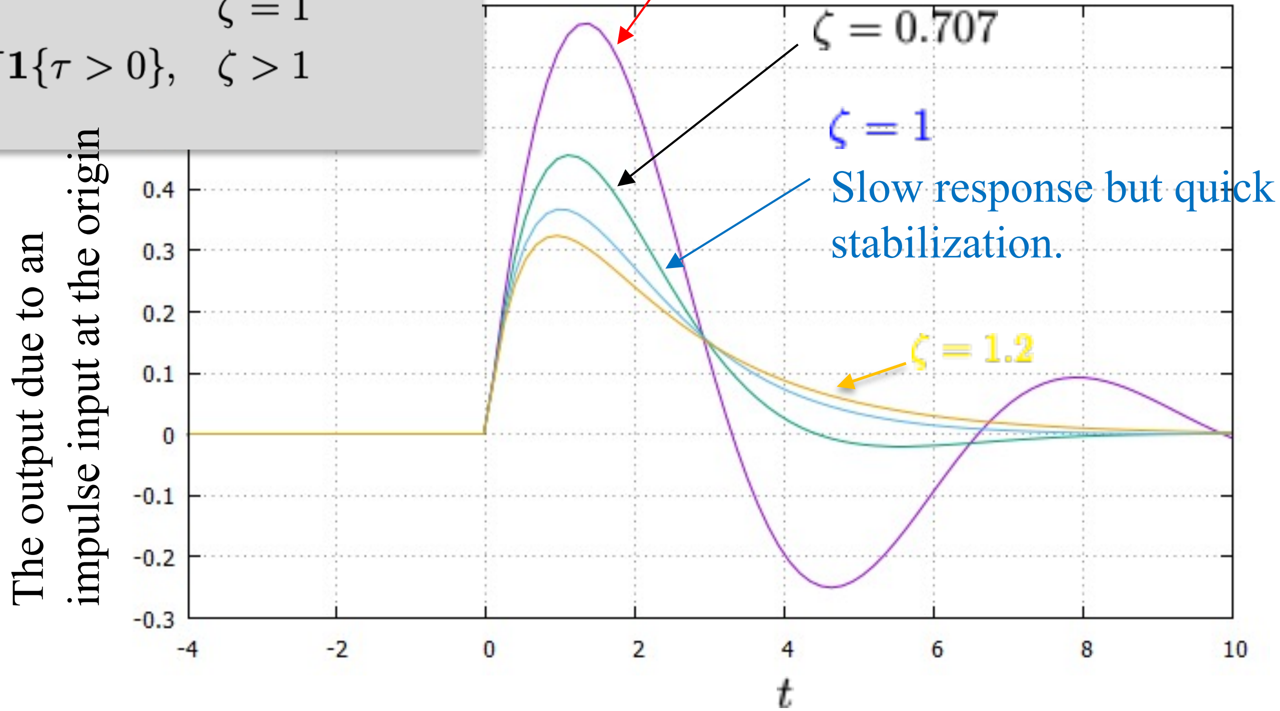
$$= \frac{(jf/f_n)^2}{1 + 2\zeta(jf/f_n) + (jf/f_n)^2}$$

where natural frequency $f_n = \sqrt{ak_0}$ and damping factor $\zeta = \sqrt{k_0/(4a)}$.

# Second-Order PLL

$$\mathcal{F}^{-1}\left\{\frac{1}{1+2\zeta(j2\pi f)+(j2\pi f)^2}\right\} \quad \text{with } f_n=\frac{1}{2\pi}$$

$$= \begin{cases} \frac{1}{\sqrt{1-\zeta^2}}\sin(\tau\sqrt{1-\zeta^2})e^{-\zeta\tau}\mathbf{1}\{\tau>0\}, & 0<\zeta<1 \\ \tau e^{-\tau}, & \zeta=1 \\ \frac{1}{\sqrt{\zeta^2-1}}\sinh(\tau\sqrt{\zeta^2-1})e^{-\zeta\tau}\mathbf{1}\{\tau>0\}, & \zeta>1 \end{cases}$$



# Summary

- Notes
  1. SSB modulation is optimum in noise performance and bandwidth conservation in AM family.
  2. FM improves the noise performance of AM family at the expense of an excessive transmission bandwidth.
  3. FM offers the tradeoff between transmission bandwidth and noise performance.