Part 3 Hilbert Transform and Canonical Representation of Signals

Hilbert Transform

- How to obtain $g_+(t)$?
- Answer: *Hilbert Transformer*.

Proof: Observe that

$$2u(f) = 1 + \text{sgn}(f), \text{ where } \text{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

Then by the next slide, we learn that

$$2u(f) \xrightarrow{\text{Inverse Fourier}} \delta(t) + j\frac{1}{\pi t} \cdot \mathbf{1}\{t \neq 0\}$$

By extended Fourier transform,

$$\begin{aligned}
\int_{-\infty}^{\infty} \text{sgn}(f) e^{-a|f|+j2\pi f t} df &= \int_{0}^{\infty} e^{-a|f|+j2\pi f t} df - \int_{-\infty}^{0} e^{-a|f|+j2\pi f t} df \\
&= \int_{0}^{\infty} e^{-(a-j2\pi t)f} df - \int_{-\infty}^{0} e^{(a+j2\pi t)f} df \\
&= \frac{1}{a-j2\pi t} - \frac{1}{a+j2\pi t} \\
&= \frac{j4\pi t}{a^2+4\pi^2 t^2}
\end{aligned}$$

$$\text{sgn}(f) \overset{\text{InverseFourier}}{\longrightarrow} \lim_{a\downarrow 0} j\frac{4\pi t}{a^2+4\pi^2 t^2} = \begin{cases} \dfrac{j}{\pi t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

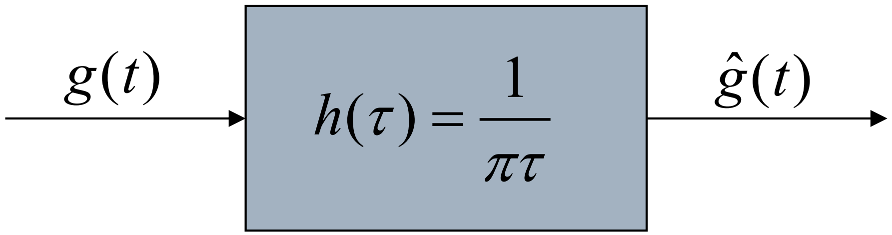
$$2u(f) = 1 + \text{sgn}(f) \overset{\text{InverseFourier}}{\longrightarrow} \delta(t) + \frac{j}{\pi t} \cdot \mathbf{1}\{t \neq 0\}$$

Hilbert Transform

$$
\begin{aligned}
g_+(t) &= Fourier^{-1}\{2u(f)G(f)\} \\
&= Fourier^{-1}\{2u(f)\} * Fourier^{-1}\{G(f)\} \\
&= \left(\delta(t) + j\frac{1}{\pi t}\cdot \mathbf{1}\{t \neq 0\}\right) * g(t) \\
&= g(t) + j\frac{1}{\pi t}\cdot \mathbf{1}\{t \neq 0\} * g(t) \\
&= g(t) + j\hat{g}(t),
\end{aligned}
$$

where $\hat{g}(t) = \int_{-\infty}^{\infty} \frac{g(\tau)}{\pi(t-\tau)} d\tau$ is named the Hilbert Transform of $g(t)$.

Hilbert Transform

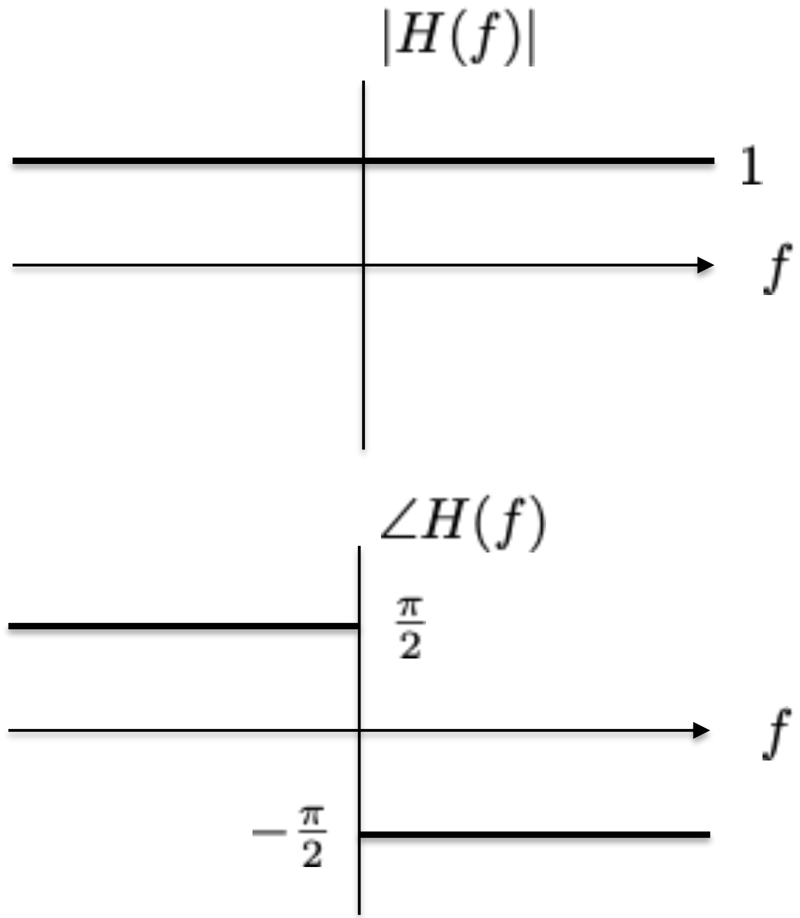


$$h(\tau) = \frac{1}{\pi\tau} \xrightarrow{Fourier} H(f) = -j\,\mathrm{sgn}(f), \text{ where } \mathrm{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

$$\Rightarrow \hat{G}(f) = -j\,\mathrm{sgn}(f) \cdot G(f) = \begin{cases} |G(f)| \exp\{j[\angle G(f) - \pi/2]\}, & f > 0 \\ 0, & f = 0 \\ |G(f)| \exp\{j[\angle G(f) + \pi/2]\}, & f < 0 \end{cases}$$

Hilbert Transform

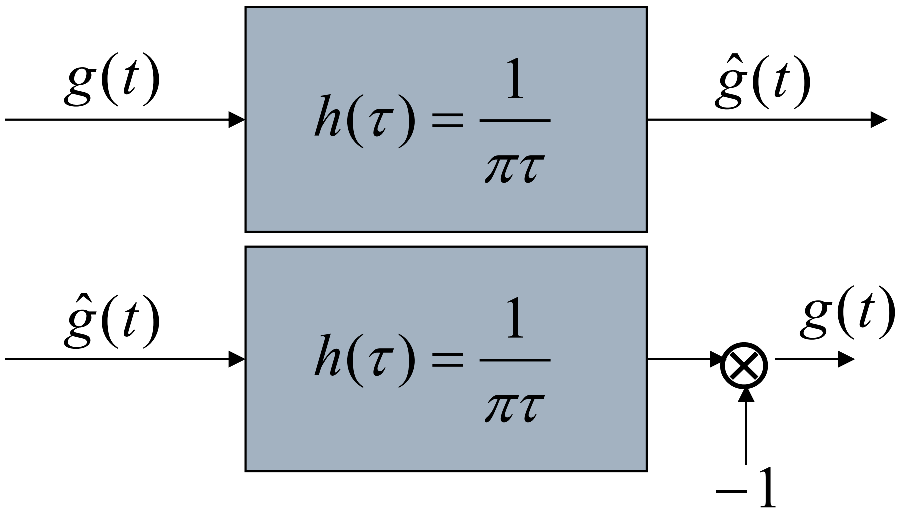
- Hence, Hilbert Transform is basically a *90 degree phase shifter*.



Hilbert Transform

Hilbert Transform Pair

$$\begin{cases} \hat{g}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{g(\tau)}{t-\tau}\, d\tau \\ g(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\hat{g}(\tau)}{t-\tau}\, d\tau \end{cases}$$



Hilbert Transform

- An important property of Hilbert Transform is that:

$g(t)$ and $\hat{g}(t)$ are orthogonal in the sense of Integration.
In other words, $\int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = 0$.
(See the proof in the next slide.)

The real and imaginary parts of $g_+(t) = g(t) + j\hat{g}(t)$ are orthogonal to each other.

(*Examples of Hilbert Transform Pairs can be found in Table A6.4.*)

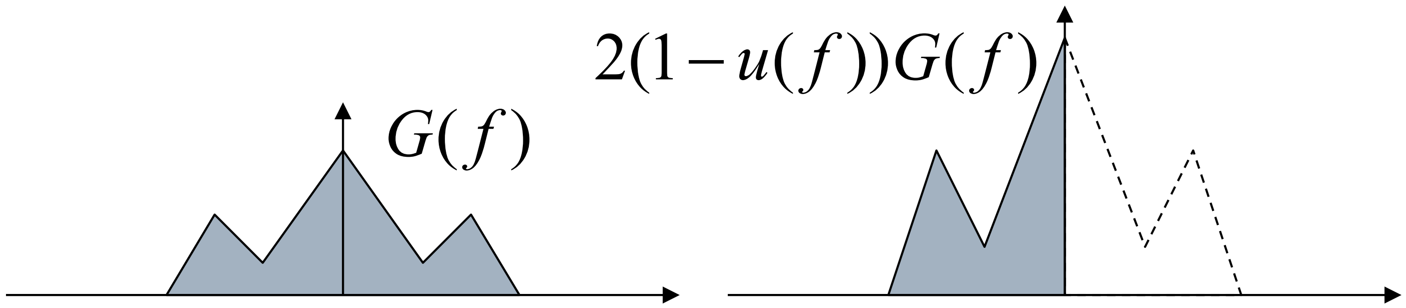
$$
\begin{aligned}
\int_{-\infty}^{\infty} g(t)\hat{g}(t)dt &= \int_{-\infty}^{\infty} g(t)\left(\int_{-\infty}^{\infty} \hat{G}(f)e^{j2\pi ft}df\right)dt \\
&= \int_{-\infty}^{\infty} \hat{G}(f)\left(\int_{-\infty}^{\infty} g(t)e^{j2\pi ft}dt\right)df \\
&= \int_{-\infty}^{\infty} \hat{G}(f)G(-f)df \\
&= -j\int_{-\infty}^{\infty} \operatorname{sgn}(f)G(f)G(-f)df \\
&= -j\left(\int_{0}^{\infty} G(f)G(-f)df - \int_{-\infty}^{0} G(f)G(-f)df\right) \\
&= -j\left(\int_{0}^{\infty} G(f)G(-f)df - \int_{0}^{\infty} G(-f)G(f)df\right) \\
&= 0, \text{ if } \int_{0}^{\infty} G(f)G(-f)df < \infty.
\end{aligned}
$$

Note that $G(f)G(-f) = G(f)G^*(f) = |G(f)|^2$.

Complex Representation of Signals and Systems

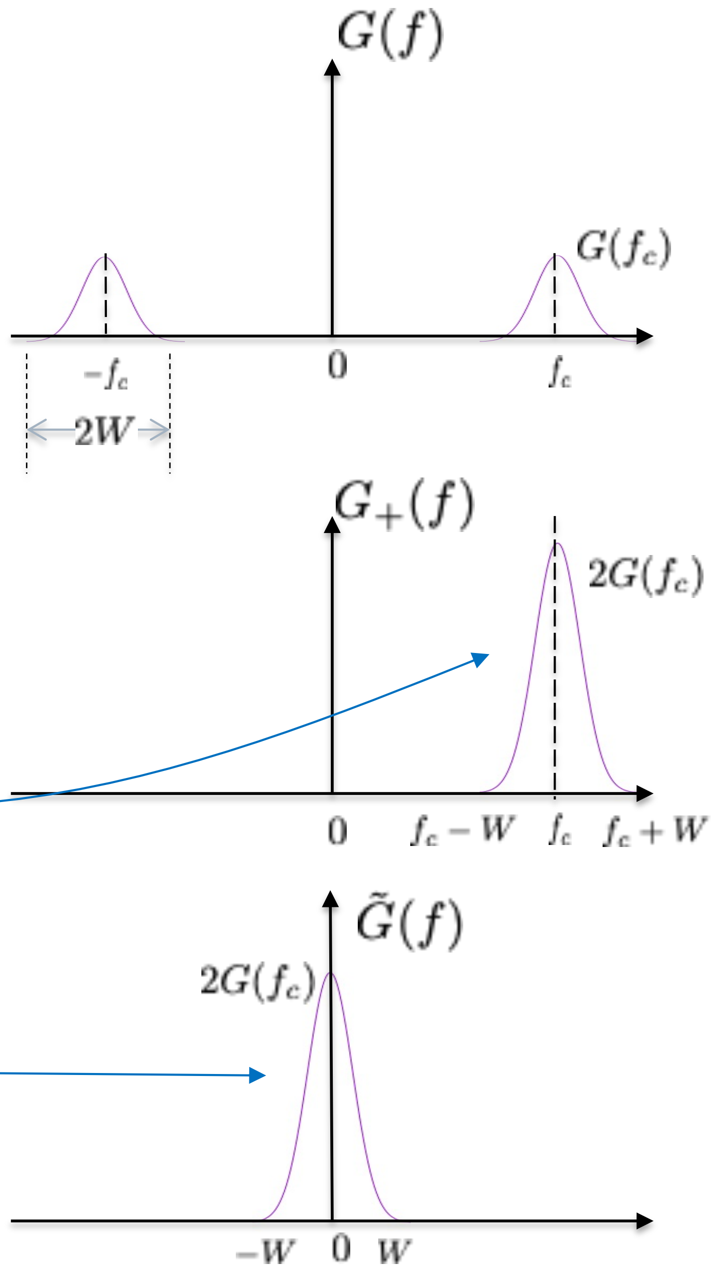
- $g_+(t)$ is named the *pre-envelope,* or *analytical signal,* of $g(t)$.
- We can similarly define

$$g_-(t) = g(t) - j\hat{g}(t)$$



Canonical Representation of Bandpass Signals

- Now let $G(f)$ be a *narrow-band signal* for which $2W << f_c$.
  - Then we can obtain its pre-envelope $G_+(f)$.
  - Afterwards, we can shift the pre-envelope to its low-pass signal $\tilde{G}(f) = G_+(f + f_c)$



Canonical Representation of Bandpass Signal

- These steps give the relation between the complex *lowpass signal* (baseband signal) and the real *bandpass signal* (passband signal).

$$g(t) = \mathrm{Re}\{g_+(t)\} = \mathrm{Re}\{\tilde{g}(t)\exp(j2\pi f_c t)\}$$

  - Quite often, the real and imaginary parts of complex lowpass signal are respectively denoted by $g_\mathrm{I}(t)$ and $g_\mathrm{Q}(t)$.

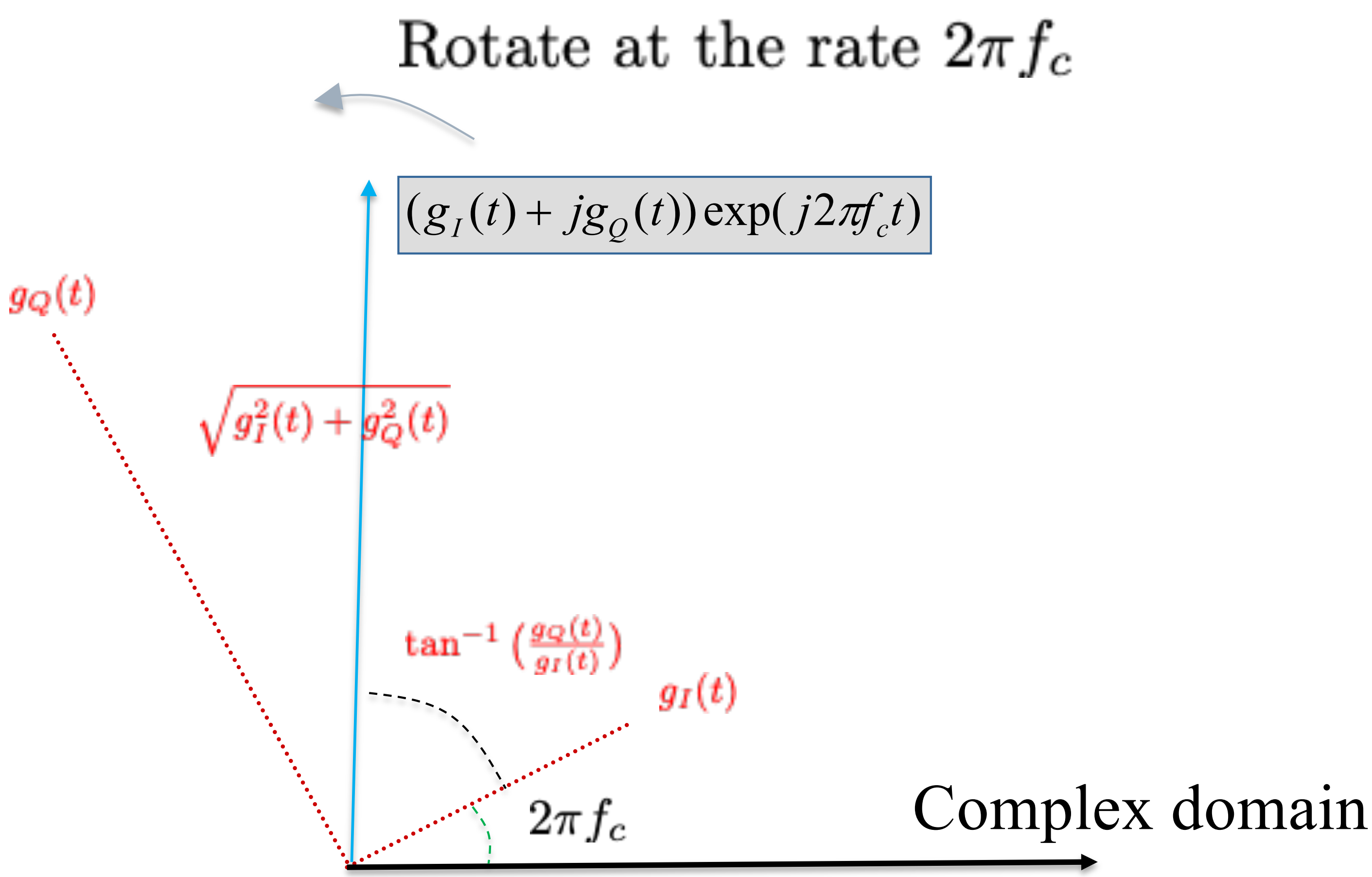
Canonical Representation of Bandpass Signal

- In terminology,

$$\begin{cases} g_+(t) & \text{pre-envelope} \\ \tilde{g}(t) & \text{complex envelope} \\ g_I(t) & \text{in-phase component of the band-pass signal } g(t) \\ g_Q(t) & \text{quadrature component of the band-pass signal } g(t) \end{cases}$$

This leads to a *canonical*, or *standard*, expression for $g(t)$.

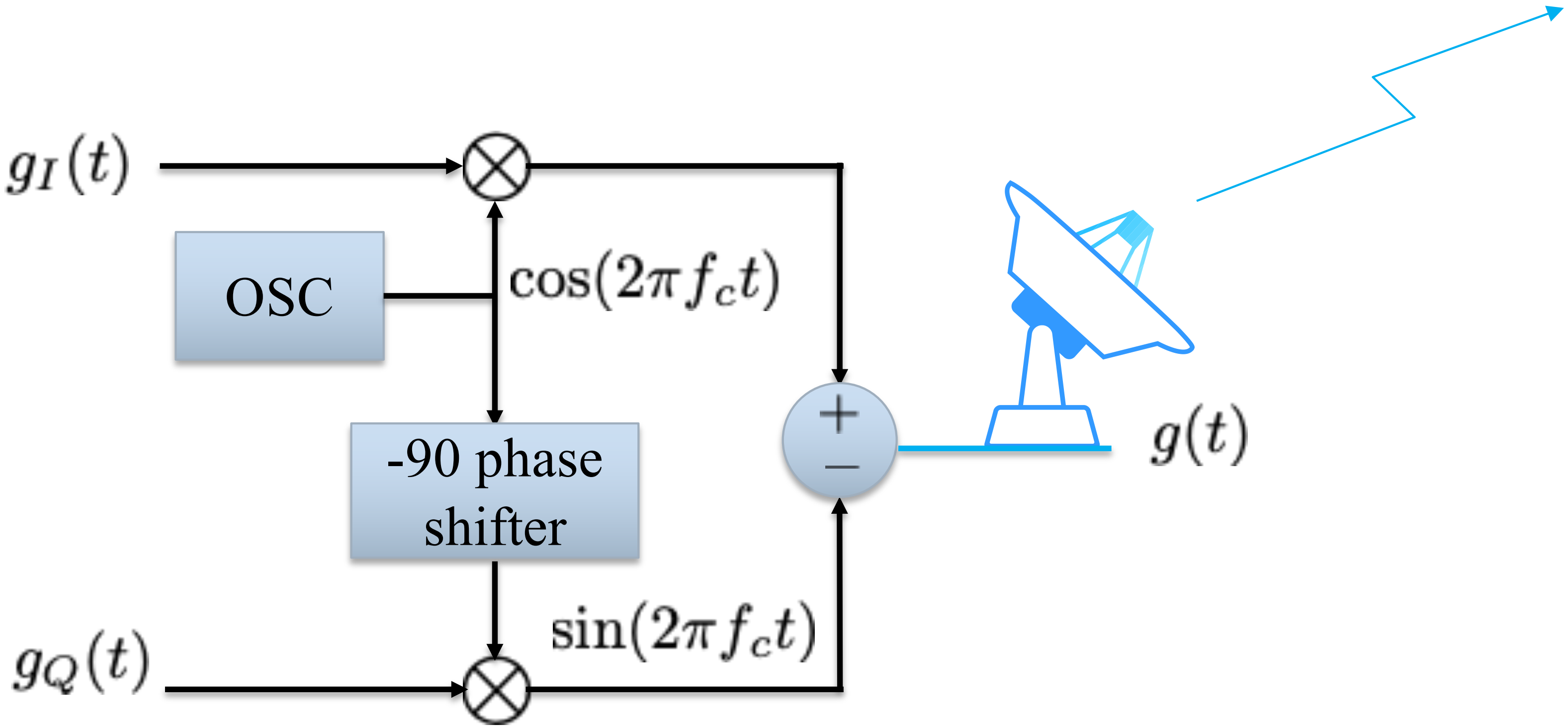
$$\begin{aligned} g(t) &= \text{Re}\left\{(g_I(t) + j g_Q(t))\exp(j2\pi f_c t)\right\} \\ &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \end{aligned}$$


Rotate at the rate $2\pi f_c$
$(g_I(t) + jg_Q(t))\exp(j2\pi f_c t)$
$g_Q(t)$
$\sqrt{g_I^2(t) + g_Q^2(t)}$
$\tan^{-1}\left(\frac{g_Q(t)}{g_I(t)}\right)$
$g_I(t)$
$2\pi f_c$
Complex domain

Canonical Representation of Bandpass Signal

- Canonical transmitter

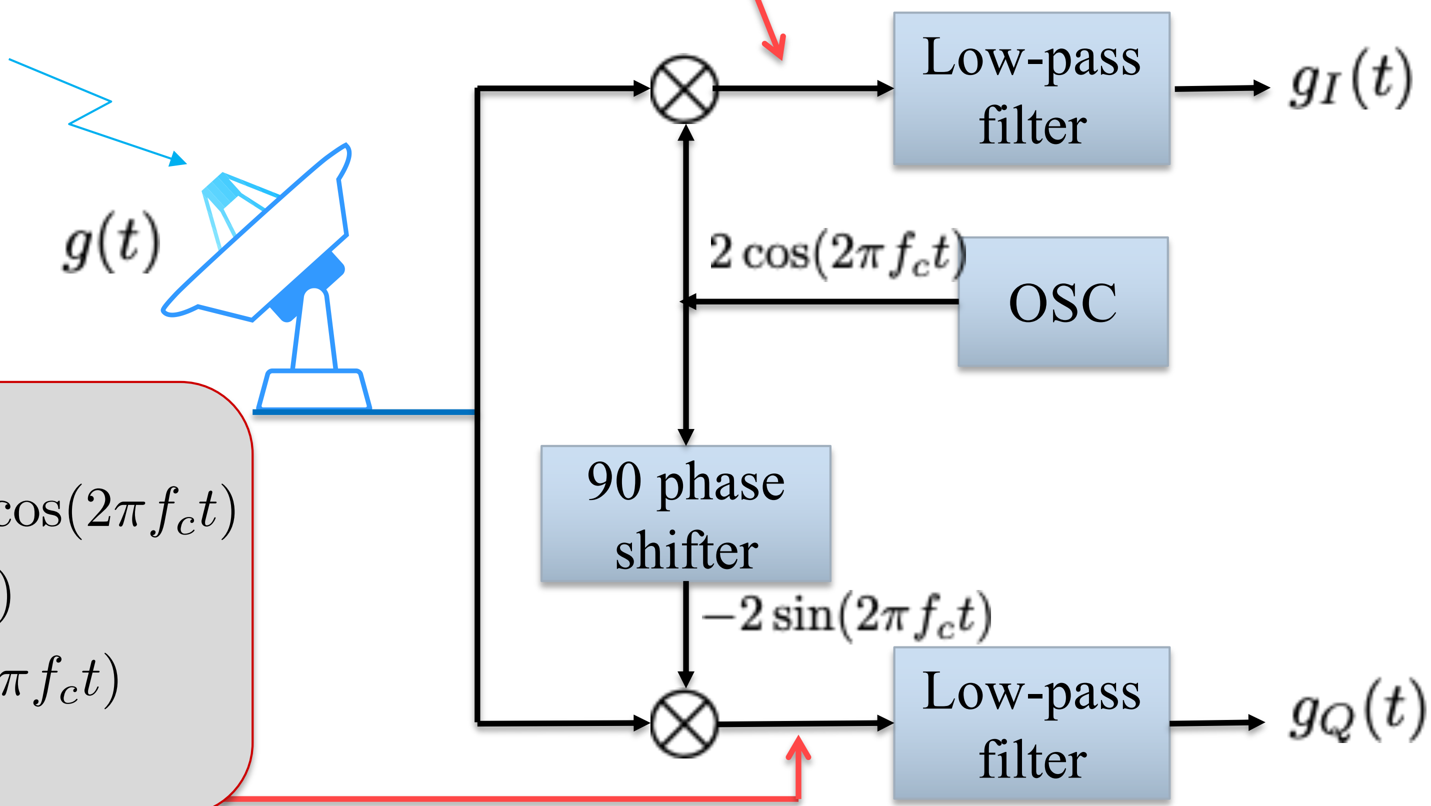
$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$$



Canonical Representation of Bandpass Signal

- Canonical receiver

$$
\begin{aligned}
& 2g(t)\cos(2\pi f_c t) \\
= \; & 2g_I(t)\cos^2(2\pi f_c t) - 2g_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\
= \; & g_I(t) + g_I(t)\cos(4\pi f_c t) - g_Q(t)\sin(4\pi f_c t)
\end{aligned}
$$



$$
\begin{aligned}
& -2g(t)\sin(2\pi f_c t) \\
= \; & -2g_I(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\
& +2g_Q(t)\sin^2(2\pi f_c t) \\
= \; & g_Q(t) - g_Q(t)\cos(4\pi f_c t) \\
& -g_I(t)\sin(4\pi f_c t)
\end{aligned}
$$

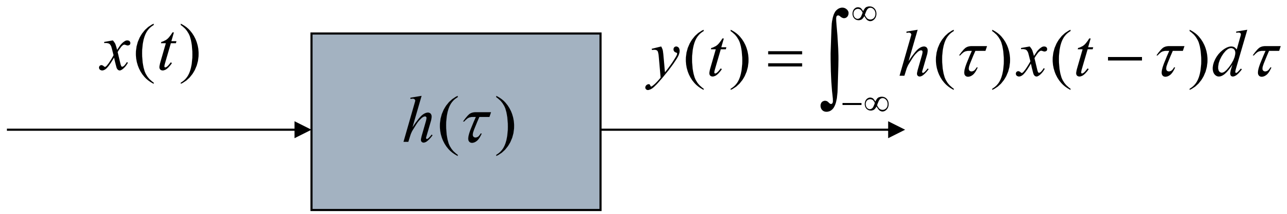
More Terminology

$$\begin{cases} g_+(t) & \text{pre - envelope} \\ \tilde{g}(t) & \text{complex envelope} \\ g_I(t) & \text{in - phase component of the band - pass signal } g(t) \\ g_Q(t) & \text{quadrature component of the band - pass signal } g(t) \end{cases}$$

$$\begin{cases} a(t) = |g_+(t)| = |\tilde{g}(t)| = \sqrt{g_I^2(t) + g_Q^2(t)} \\ \qquad\qquad \text{natural envelope or envelope of } g(t) \\ \phi(t) = \tan^{-1}\left(\dfrac{g_Q(t)}{g_I(t)}\right) \text{ phase of } g(t) \end{cases}$$

Bandpass System

- Consider the case of passing a bandpass signal $x(t)$ through a real LTI filter $h(\tau)$ to yield an output $y(t)$.



- Can we have a *lowpass* ***equivalent*** *system* for the bandpass system?

- Similar to the previous analysis, we have:

Assumption : The spectrum of $x(t)$ is limited to within $\pm W$ Hz of the carrier frequency f_c, and $W < f_c$.

$$\begin{cases} x(t) = \mathrm{Re}\left\{ \tilde{x}(t) e^{j2\pi f_c t} \right\} = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \\ \tilde{x}(t) = x_I(t) + j x_Q(t) \end{cases}$$

Assumption : The spectrum of $h(\tau)$ is limited to within $\pm B$ Hz of the carrier frequency f_c, and $B < f_c$.

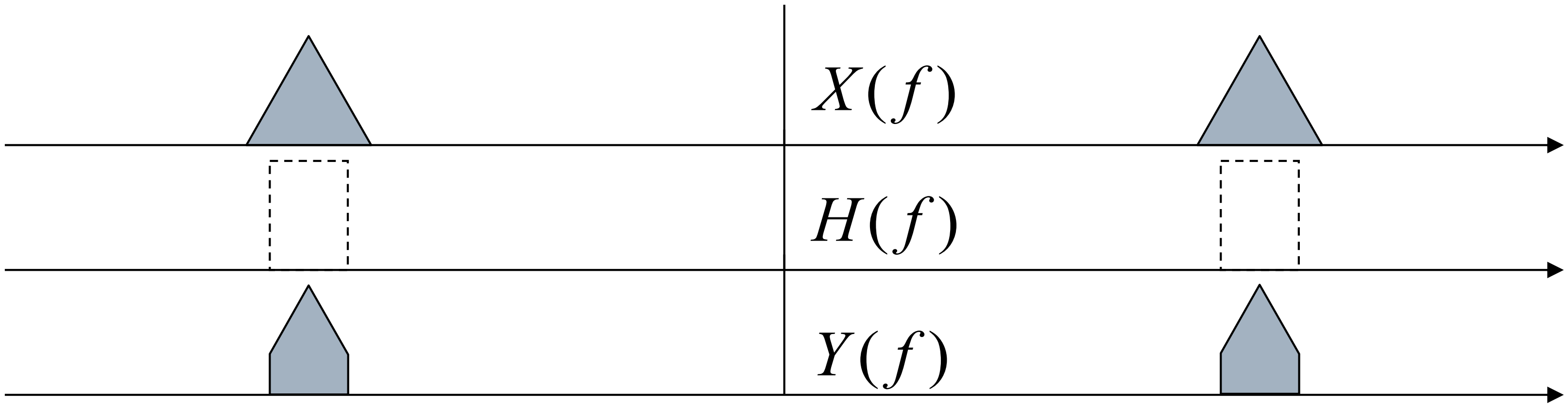
$$\begin{cases} h(\tau) = \mathrm{Re}\left\{ \tilde{h}(\tau) e^{j2\pi f_c \tau} \right\} = h_I(\tau)\cos(2\pi f_c \tau) - h_Q(\tau)\sin(2\pi f_c \tau) \\ \tilde{h}(\tau) = h_I(\tau) + j h_Q(\tau) \quad \text{complex impulse response} \end{cases}$$

- Now, is the filter output $y(t)$ also a bandpass signal?

$$Y(f) = X(f)H(f)$$

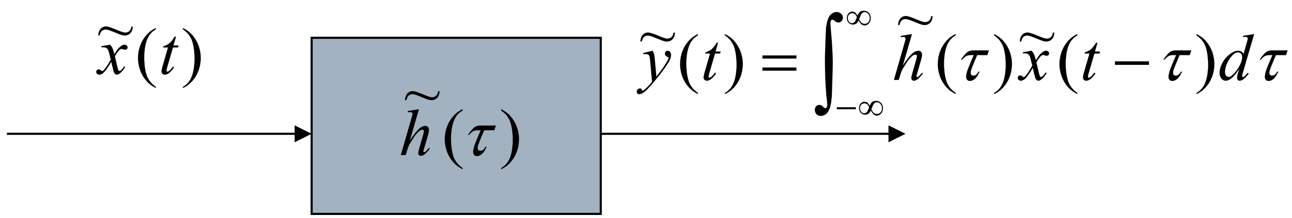
$\Rightarrow$ The spectrum of $y(t)$ is limited to within $\pm\min\{W, B\}$ Hz of the carrier frequency f_c, provided that $\max\{W, B\} < f_c$

$$\begin{cases} y(t) = \mathrm{Re}\left\{\tilde{y}(t)e^{j2\pi f_c t}\right\} = y_I(t)\cos(2\pi f_c t) - y_Q(t)\sin(2\pi f_c t) \\ \tilde{y}(t) = y_I(t) + j y_Q(t) \end{cases}$$



Bandpass System

- Question: Is the following system valid?



  - The advantage of the above equivalent system is that there is no need to deal with the *carrier frequency* in the system analysis.

- The answer to the question is YES (with some modification)! It will be substantiated in the sequel.

It suffices to show that $\tilde{Y}(f) = \tilde{X}(f)\tilde{H}(f)$.

Observe that $\begin{cases} Y_+(f) = 2u(f)Y(f) \\ X_+(f) = 2u(f)X(f) \\ H_+(f) = 2u(f)H(f) \end{cases}$ and $\begin{cases} \tilde{Y}(f) = Y_+(f+f_c) \\ \tilde{X}(f) = X_+(f+f_c). \\ \tilde{H}(f) = H_+(f+f_c) \end{cases}$

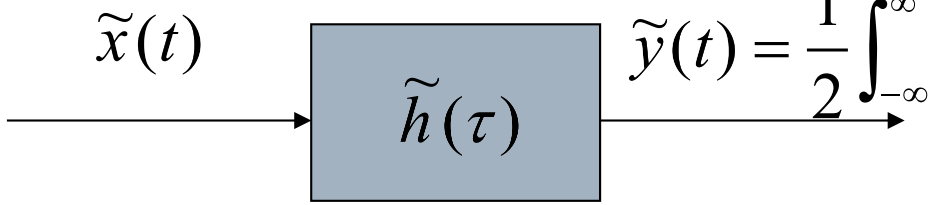
Consequently,

$$\begin{aligned} \tilde{X}(f)\tilde{H}(f) &= X_+(f+f_c)H_+(f+f_c) \\ &= \big(2u(f+f_c)X(f+f_c)\big)\big(2u(f+f_c)H(f+f_c)\big) \\ &= 4u(f+f_c)X(f+f_c)H(f+f_c) \\ &= 4u(f+f_c)Y(f+f_c) \\ &= 2Y_+(f+f_c) \\ &= 2\tilde{Y}(f) \end{aligned}$$

There is an additional multiplicative constant **2** at the output!

Bandpass System

- Conclusion:



$$\tilde{y}(t)=\frac{1}{2}\int_{-\infty}^{\infty}\tilde{h}(\tau)\tilde{x}(t-\tau)d\tau$$

Bandpass System

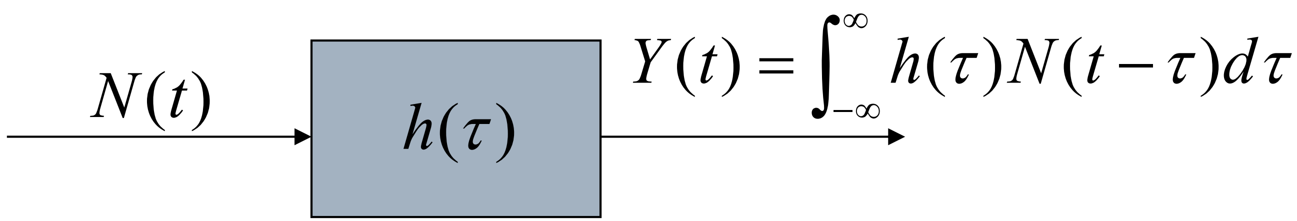
- Final note on bandpass system
  - Some books define $H_+(f) = u(f)H(f)$ for a **filter**, instead of $H_+(f) = 2u(f)H(f)$ (as for a **signal**).
  - As a result of this definition (i.e., $H_+(f) = u(f)H(f)$),

$$h(\tau) = 2\,\mathrm{Re}\left\{\tilde{h}(\tau)e^{j2\pi f_c\tau}\right\} \text{ and } \tilde{y}(t) = \int_{-\infty}^{\infty}\tilde{h}(\tau)\tilde{x}(t-\tau)d\tau$$

  - It is justifiable to remove 2 in $H_+(f) = u(f)H(f)$, because a filter is used to filter out the signal; hence, it is not necessary to *make the total area constant*.

Representation of Narrowband Noise using In-Phase and Quadrature Components

- The bandpass system representation discussed previously is based on *deterministic signals*.
- How about a random process? Can we have a low-pass isomorphism system to a bandpass random process.
  - Take the noise process $N(t)$ as an example.



Representation of Narrowband Noise using In-Phase and Quadrature Components

- A WSS *real-valued* *zero-mean* noise process $N(t)$ is a bandpass process if its PSD $S_N(f) \neq 0$ only for $|f - f_c| < B$ and $|f + f_c| < B$, and also $B < f_c$.
  - Similar to the analysis for deterministic signals, let

$$\begin{cases} N(t) = N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t) \\ \tilde{N}(t) = N_I(t) + jN_Q(t) \end{cases}$$

    for some joint zero-mean WSS of $N_I(t)$ and $N_Q(t)$.
  - Notably, the joint WSS of $N_I(t)$ and $N_Q(t)$ immediately imply WSS of $\tilde{N}(t)$.

PSD of $N_I(t)$ and $N_Q(t)$

- First, we note that joint WSS of $N_I(t)$ and $N_Q(t)$ and the WSS of $N(t)$ imply:

$$\begin{cases} R_{N_I}(\tau) = R_{N_Q}(\tau) \\ R_{N_I,N_Q}(\tau) = -R_{N_Q,N_I}(\tau) \end{cases}$$

(See the proof in the sequel.)

$$
\begin{aligned}
R_N(\tau) &= E[N(t+\tau)N(t)] \\
&= E[(N_I(t+\tau)\cos(2\pi f_c(t+\tau)) - N_Q(t+\tau)\sin(2\pi f_c(t+\tau))) \\
&\quad \times(N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t))] \\
&= R_{N_I}(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&\quad + R_{N_Q}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\
&\quad - R_{N_I,N_Q}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\
&\quad - R_{N_Q,N_I}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&= R_{N_I}(\tau)\frac{\cos(2\pi f_c\tau) + \cos(2\pi f_c(2t+\tau))}{2} \\
&\quad + R_{N_Q}(\tau)\frac{\cos(2\pi f_c\tau) - \cos(2\pi f_c(2t+\tau))}{2} \\
&\quad - R_{N_I,N_Q}(\tau)\frac{\sin(2\pi f_c(2t+\tau)) - \sin(2\pi f_c\tau)}{2} \\
&\quad - R_{N_Q,N_I}(\tau)\frac{\sin(2\pi f_c(2t+\tau)) + \sin(2\pi f_c\tau)}{2}
\end{aligned}
$$

(Continue from the previous slide.)

$$
\begin{aligned}
R_N(\tau) &= \frac{1}{2}[R_{N_I}(\tau) + R_{N_Q}(\tau)]\cos(2\pi f_c\tau) \\
&+ \frac{1}{2}[R_{N_I,N_Q}(\tau) - R_{N_Q,N_I}(\tau)]\sin(2\pi f_c\tau) \\
&+ \frac{1}{2}[R_{N_I}(\tau) - R_{N_Q}(\tau)]\cos(2\pi f_c(2t+\tau)) \\
&- \frac{1}{2}[R_{N_I,N_Q}(\tau) + R_{N_Q,N_I}(\tau)]\sin(2\pi f_c(2t+\tau))
\end{aligned}
$$

These two terms must equal zero, since $R_N(\tau)$ is not a function of t.

$$
\Rightarrow \begin{cases} R_{N_I}(\tau) = R_{N_Q}(\tau) \\ R_{N_I,N_Q}(\tau) = -R_{N_Q,N_I}(\tau) \end{cases}
$$

(Property 1)

$$
\Rightarrow R_N(\tau) = R_{N_I}(\tau)\cos(2\pi f_c\tau) - R_{N_Q,N_I}(\tau)\sin(2\pi f_c\tau)
$$

(Property 2)

PSD of $N_I(t)$ and $N_Q(t)$

- Some other properties

$$
\begin{aligned}
R_{\tilde{N}}(\tau) &= E[\tilde{N}(t+\tau)\tilde{N}^*(t)] \\
&= E[(N_I(t+\tau)+jN_Q(t+\tau)) \times (N_I(t)-jN_Q(t))] \\
&= R_{N_I}(\tau)+R_{N_Q}(\tau)+j[R_{N_Q,N_I}(\tau)-R_{N_I,N_Q}(\tau)] \\
&= 2R_{N_I}(\tau)+j2R_{N_Q,N_I}(\tau) \quad \textbf{(Property 3)}
\end{aligned}
$$

PSD of $N_I(t)$ and $N_Q(t)$

Properties 2 and 3 jointly imply

$$R_N(\tau) = \frac{1}{2}\text{Re}\{R_{\tilde{N}}(\tau)\exp(j2\pi f_c\tau)\}$$ **(Property 4)**

- As the inconsistency encountered in Slide 3-24,
  - some books define

$$\begin{cases} R_{\tilde{X}}(\tau) = \frac{1}{2}E[\tilde{X}(t+\tau)\tilde{X}^*(t)] & \text{for complex } \tilde{X}(t); \\ R_X(\tau) = E[X(t+\tau)X(t)] & \text{for real } X(t) \end{cases}$$

  - As a result of the two "inconsistent" definitions, a "simpler" relation is obtained: $R_N(\tau) = \text{Re}\{R_{\tilde{N}}(\tau)\exp(j2\pi f_c\tau)\}$
  - For consistence, we let $R_{\tilde{X}}(\tau) = E[\tilde{X}(t+\tau)\tilde{X}^*(t)]$ for complex $\tilde{X}(t)$

Summary of Spectrum Properties

1. $R_{N_I}(\tau) = R_{N_Q}(\tau)$ and $R_{N_I,N_Q}(\tau) = -R_{N_Q,N_I}(\tau)$
2. $R_N(\tau) = R_{N_I}(\tau)\cos(2\pi f_c\tau) - R_{N_Q,N_I}(\tau)\sin(2\pi f_c\tau)$

   $\Rightarrow R_N(0) = R_{N_I}(0) = R_{N_Q}(0)$
3. $R_{\tilde{N}}(\tau) = 2 \cdot R_{N_I}(\tau) + j2 \cdot R_{N_Q,N_I}(\tau)$
4. $R_N(\tau) = \frac{1}{2} \cdot \mathrm{Re}\{R_{\tilde{N}}(\tau)\exp(j2\pi f_c\tau)\}$
5. $S_{\tilde{N}}(f)$ is real-valued. $\longrightarrow$ By $R_{\tilde{N}}(\tau) = R^*_{\tilde{N}}(-\tau)$.
6. $S_{N_I}(f) = S_{N_Q}(f)$ and $S_{N_I,N_Q}(f) = -S_{N_Q,N_I}(f)$ [From 1.]
7. $S_N(f) = \frac{1}{4}\left(S_{\tilde{N}}(f - f_c) + S_{\tilde{N}}(-f - f_c)\right)$ [From 4. See the next slide.]

$$
\begin{aligned}
S_N(f) &= \int_{-\infty}^{\infty} R_N(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\text{Re}\{R_{\tilde{N}}(\tau)e^{j2\pi f_c\tau}\} e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \frac{1}{4}\left(R_{\tilde{N}}(\tau)e^{j2\pi f_c\tau} + (R_{\tilde{N}}(\tau)e^{j2\pi f_c\tau})^*\right) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \frac{1}{4}\left(R_{\tilde{N}}(\tau)e^{j2\pi f_c\tau} + R^*_{\tilde{N}}(\tau)e^{-j2\pi f_c\tau}\right) e^{-j2\pi f\tau} d\tau \\
&= \frac{1}{4}\int_{-\infty}^{\infty} R_{\tilde{N}}(\tau)e^{-j2\pi(f-f_c)\tau} d\tau + \frac{1}{4}\left(\int_{-\infty}^{\infty} R_{\tilde{N}}(\tau)e^{-j2\pi(-f-f_c)\tau} d\tau\right)^* \\
&= \frac{1}{4}\left(S_{\tilde{N}}(f-f_c) + S^*_{\tilde{N}}(-f-f_c)\right) \\
&= \frac{1}{4}\left(S_{\tilde{N}}(f-f_c) + S_{\tilde{N}}(-f-f_c)\right)
\end{aligned}
$$

8. $N_I(t)$ and $N_Q(t)$ are orthogonal.

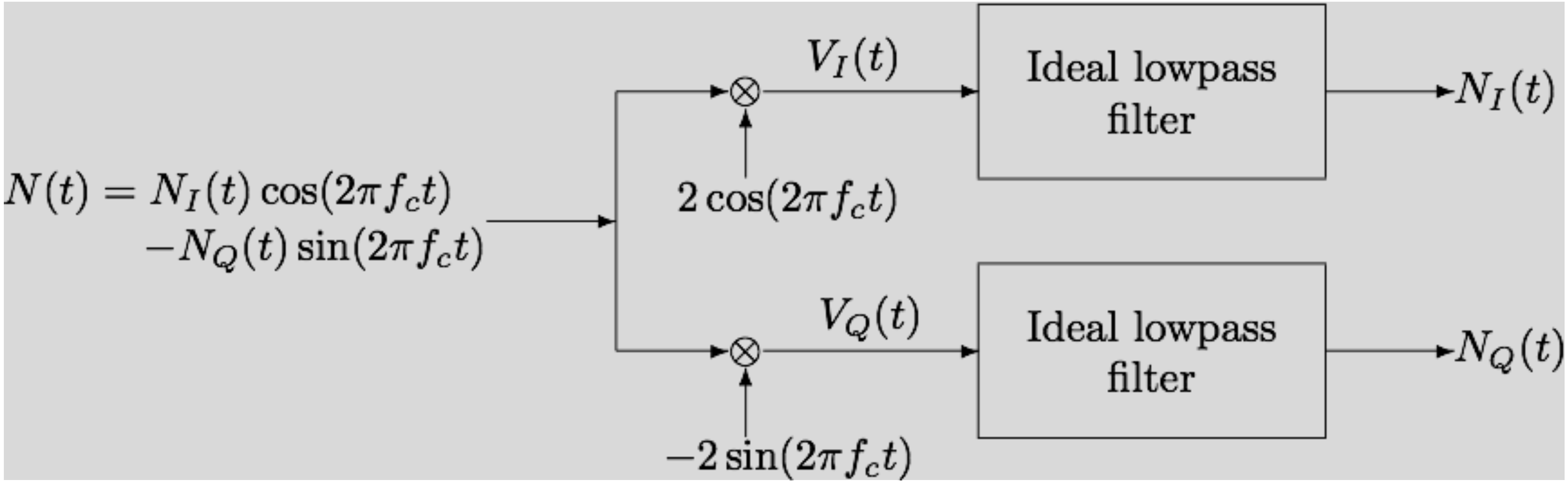
$$\begin{cases} R_{N_I,N_Q}(\tau) = -R_{N_Q,N_I}(\tau) \\ R_{N_I,N_Q}(-\tau) = E[N_I(t)N_Q(t+\tau)] = R_{N_Q,N_I}(\tau) \end{cases}$$

$$\Rightarrow R_{N_I,N_Q}(\tau) = -R_{N_I,N_Q}(-\tau)$$

$$\Rightarrow R_{N_I,N_Q}(0) = -R_{N_I,N_Q}(0)$$

$$\Rightarrow R_{N_I,N_Q}(0) = E[N_I(t)N_Q(t)] = 0.$$

9. $S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & |f| < B \\ 0, & \text{otherwise} \end{cases}$



$$\begin{aligned} R_{V_I}(t,u) &= E[V_I(t)V_I(u)] \\ &= 4E[N(t)\cos(2\pi f_c t)N(u)\cos(2\pi f_c u)] \\ &= 4R_N(t,u)\cos(2\pi f_c t)\cos(2\pi f_c u) \end{aligned}$$

Notably, $V_I(t)$ is not WSS.

$$
\begin{aligned}
\bar{R}_{V_I}(\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{V_I}(t+\tau,\tau)\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} 4R_N(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t)\mathrm{d}t \\
&= R_N(\tau)\lim_{T\to\infty}\frac{2}{T}\int_{-T}^{T}[\cos(2\pi f_c(2t+\tau))+\cos(2\pi f_c\tau)]\mathrm{d}t \\
&= R_N(\tau)\lim_{T\to\infty}\frac{2}{T}\int_{-T}^{T}\cos(2\pi f_c(2t+\tau)\mathrm{d}t + 2R_N(\tau)\cos(2\pi f_c\tau) \\
&= 2R_N(\tau)\cos(2\pi f_c\tau)
\end{aligned}
$$

$$
\boxed{\bar{S}_{V_I}(f) = S_N(f-f_c) + S_N(f+f_c)}
$$

$$
\bar{S}_{N_I}(f) = |H(f)|^2\ \bar{S}_{V_I}(f), \text{ where } |H(f)|^2 = \begin{cases} 1, & |f| \le B \\ 0, & f > B \end{cases}
$$

$$\Rightarrow \; S_{N_I}(f) = \bar{S}_{N_I}(f) = [S_N(f - f_c) + S_N(f + f_c)]\,|H(f)|^2$$

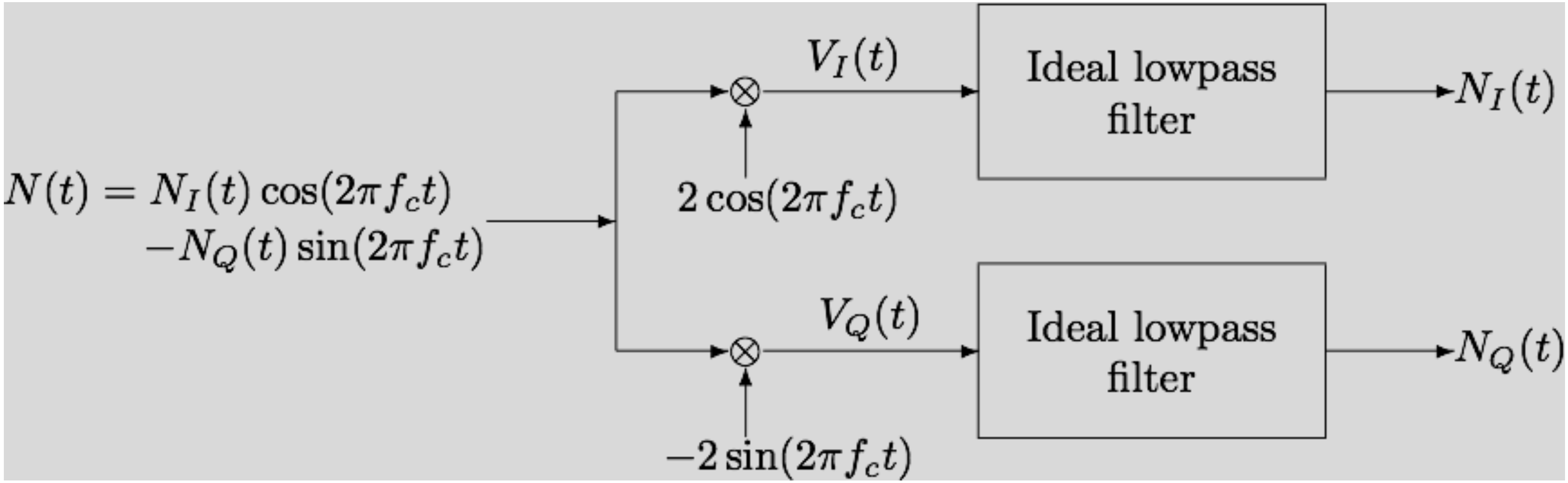
$$= \begin{cases} S_N(f - f_c) + S_N(f + f_c), & |f| < B \\ 0, & \text{otherwise} \end{cases}$$

Similarly,

$$S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & |f| < B \\ 0, & \text{otherwise} \end{cases}$$

This result (namely, $S_{N_I}(f) = S_{N_Q}(f)$) coincides with Property 1, for which $R_{N_I}(\tau) = R_{N_Q}(\tau)$.

10. $S_{N_I,N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & |f| < B \\ 0, & \text{otherwise} \end{cases}$



$$\begin{aligned} R_{V_I,V_Q}(t,u) &= E[V_I(t)V_Q(u)] \\ &= -4E\left[N(t)\cos(2\pi f_c t)N(u)\sin(2\pi f_c u)\right] \\ &= -4R_N(t,u)\cos(2\pi f_c t)\sin(2\pi f_c u) \end{aligned}$$

$$R_{V_I,V_Q}(t+\tau,t) = -4R_N(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)$$

$$\begin{aligned}\bar{R}_{V_I,V_Q}(\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{V_I,V_Q}(t+\tau,t)\mathrm{d}t \\ &= -R_N(\tau)\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}[\sin(2\pi f_c(2t+\tau)) - \sin(2\pi f_c\tau]\mathrm{d}t \\ &= 2R_N(\tau)\sin(2\pi f_c\tau)\end{aligned}$$

$$\bar{S}_{V_I,V_Q}(f) = j[S_N(f+f_c) - S_N(f-f_c)]$$

$$\begin{aligned}R_{N_I,N_Q}(t,u) &= E\left[N_I(t)N_Q(u)\right] \\ &= E\left[\int_{-\infty}^{\infty} h_I(\tau_1)V_I(t-\tau_1)d\tau_1 \cdot \int_{-\infty}^{\infty} h_Q(\tau_2)V_Q(u-\tau_2)d\tau_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)E[V_I(t-\tau_1)V_Q(u-\tau_2)]d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)R_{V_I,V_Q}(t-\tau_1,u-\tau_2)d\tau_1 d\tau_2\end{aligned}$$

$$
\begin{aligned}
&\bar{R}_{N_I,N_Q}(\tau) \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{N_I,N_Q}(t+\tau,t)dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)R_{V_I,V_Q}(t+\tau-\tau_1,t-\tau_2)d\tau_1 d\tau_2 dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{V_I,V_Q}(t+\tau-\tau_1,t-\tau_2)dt d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)\bar{R}_{V_I,V_Q}(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2
\end{aligned}
$$

$$\bar{S}_{N_I,N_Q}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)\bar{R}_{V_I,V_Q}(\tau_2+\tau-\tau_1)e^{-j2\pi f\tau}d\tau d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)\bar{R}_{V_I,V_Q}(u)e^{-j2\pi f(u-\tau_2+\tau_1)}du d\tau_1 d\tau_2,$$

(Let $u = \tau_2 + \tau - \tau_1$.)

$$= H_I(f)H_Q(-f)\bar{S}_{V_I,V_Q}(f)$$

Here, $H_Q(-f) = H_Q^*(f)$.

Take $H_I(f) = H_Q(f) = H(f)$.
Then $H_I(f)H_Q(-f) = H(f)H(-f) = H(f)H^*(f) = |H(f)|^2$.

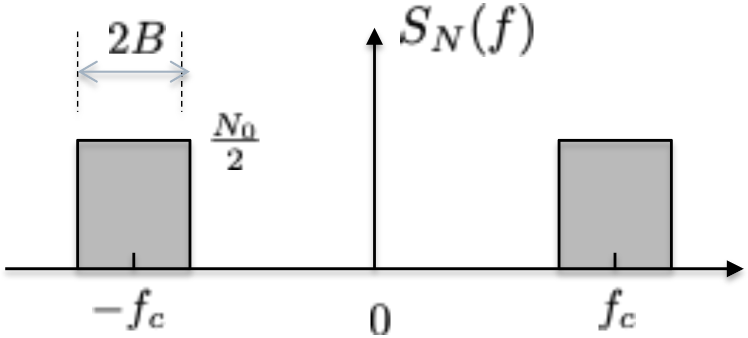
$$S_{N_I,N_Q}(f) = \bar{S}_{N_I,N_Q}(f)$$

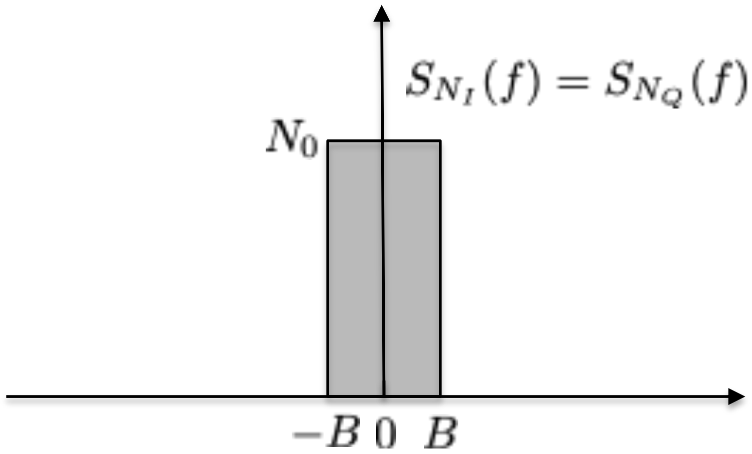
$$= j[S_N(f+f_c) - S_N(f-f_c)]\,|H(f)|^2$$

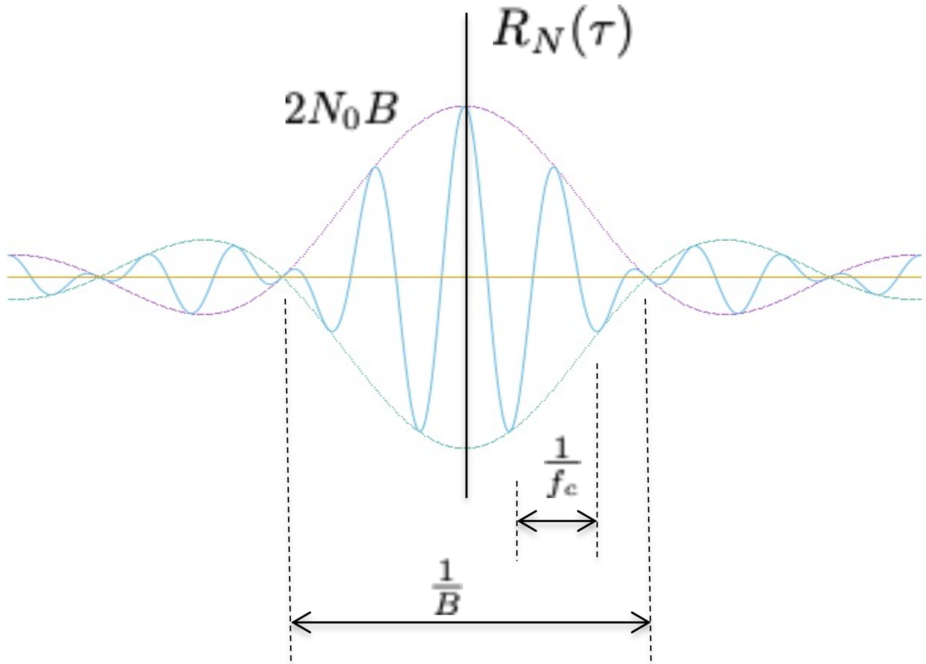
$$= \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & |f| < B \\ 0, & \text{otherwise} \end{cases}$$

Ideal Bandpass Filtered White Noise Revisited

$$
\begin{aligned}
R_N(\tau) &= \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} e^{j2\pi f\tau} df \\
&\quad + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} e^{j2\pi f\tau} df \\
&= 2BN_0 \text{sinc}(2B\tau)\cos(2\pi f_c\tau) \\
R_{N_I}(\tau) &= 2BN_0 \text{sinc}(2B\tau)
\end{aligned}
$$







Representation of Narrowband Noise using Envelope and Phase Components

- Now we turn to *envelope $R(t)$* and *phase $\Psi(t)$* components of a random process of the form

$$N(t) = N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t)$$
$$= R(t)\cos[2\pi f_c t + \Psi(t)]$$

where $R(t) = \sqrt{N_I^2(t) + N_Q^2(t)}$ and $\Psi(t) = \tan^{-1}[N_Q(t)/N_I(t)]$.

Pdf of $R(t)$ and $\Psi(t)$

- Assume that $N(t)$ is a white Gaussian process with two-sided PSD $\sigma^2 = N_0/2$.
- For convenience, let N_I and N_Q be snapshot samples of $N_I(t)$ and $N_Q(t)$.
- Then

$$f_{N_I,N_Q}(n_I,n_Q) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{n_I^2+n_Q^2}{2\sigma^2}\right)$$

Pdf of $R(t)$ and $\Psi(t)$

- By $n_I = r\cos(\psi)$ and $n_Q = r\sin(\psi)$,

$$\int_{A(n_I,n_Q)} \frac{1}{2\pi\sigma^2}\exp\left(-\frac{n_I^2+n_Q^2}{2\sigma^2}\right)dn_I dn_Q = \int_{A(r,\psi)} \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)\left|\begin{matrix}\frac{dn_I}{dr} & \frac{dn_Q}{dr}\\ \frac{dn_I}{d\psi} & \frac{dn_Q}{d\psi}\end{matrix}\right| dr d\psi$$

$$= \int_{A(r,\psi)} \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) r dr d\psi$$

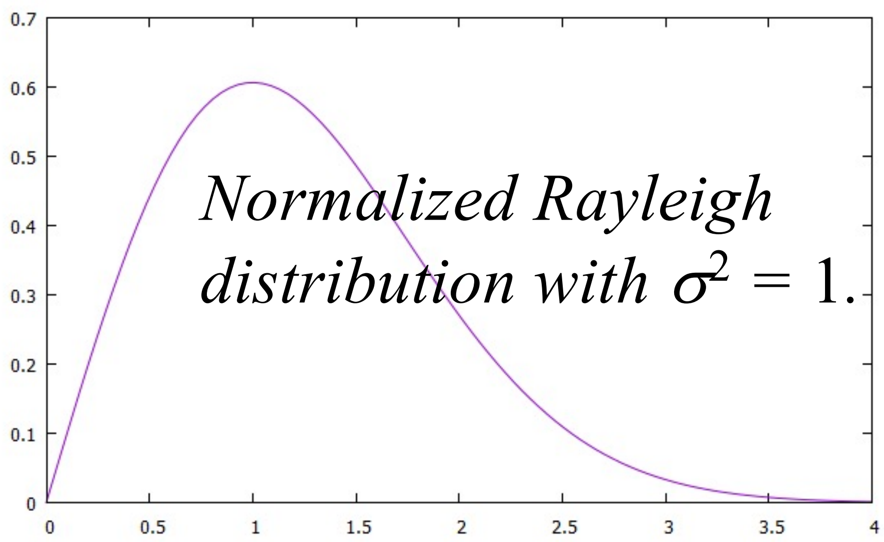
So $f_{R,\Psi}(r,\psi) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) = \frac{1}{2\pi}\times\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right).$

Pdf of $R(t)$ and $\Psi(t)$

- R and Ψ are therefore independent.

$$\begin{cases} f_R(r) = \dfrac{r}{\sigma^2}\exp\left(-\dfrac{r^2}{2\sigma^2}\right) \text{ for } r \geq 0. \\ f_\Psi(\psi) = \dfrac{1}{2\pi} \text{ for } 0 \leq \psi < 2\pi. \end{cases}$$

Rayleigh distribution.



Sine Wave + Gaussian Noise

- Now suppose the Gaussian white noise is added to a sinusoid of amplitude A.
- Then

$$\begin{aligned} x(t) &= A\cos(2\pi f_c t) + n(t) \\ &= A\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \end{aligned}$$

  - Uncorrelation for Gaussian $n_I(t)$ and $n_Q(t)$ implies their independence.

Sine Wave + Gaussian Noise

- This gives the pdf of $x_I(t)$ and $x_Q(t)$ as:

$$f_{X_I,X_Q}(x_I,x_Q)=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x_I-A)^2+x_Q^2}{2\sigma^2}\right]$$

- By $x_I = r\cos(\psi)$ and $x_Q = r\sin(\psi)$,

$$f_{R,\Psi}(r,\psi)=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(r\cos(\psi)-A)^2+r^2\sin^2(\psi)}{2\sigma^2}\right]\left|\begin{matrix}\frac{dx_I}{dr} & \frac{dx_Q}{dr}\\ \frac{dx_I}{d\psi} & \frac{dx_Q}{d\psi}\end{matrix}\right|$$

$$=\frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+A^2-2Ar\cos(\psi)}{2\sigma^2}\right)$$

Sine Wave + Gaussian Noise

- Notably, in this case, R and Ψ are no longer independent.
- We are more interested in the marginal distribution of R.

$$
\begin{aligned}
f_R(r) &= \int_0^{2\pi} f_{R,\Psi}(r,\psi)\,d\psi \\
&= \int_0^{2\pi} \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2 - 2Ar\cos(\psi)}{2\sigma^2}\right) d\psi
\end{aligned}
$$

Sine Wave + Gaussian Noise

$$f_R(r) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right)\int_0^{2\pi}\exp\left(\frac{2Ar\cos(\psi)}{2\sigma^2}\right)d\psi$$

$$= \frac{r}{\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right)I_0\left(\frac{Ar}{\sigma^2}\right),$$

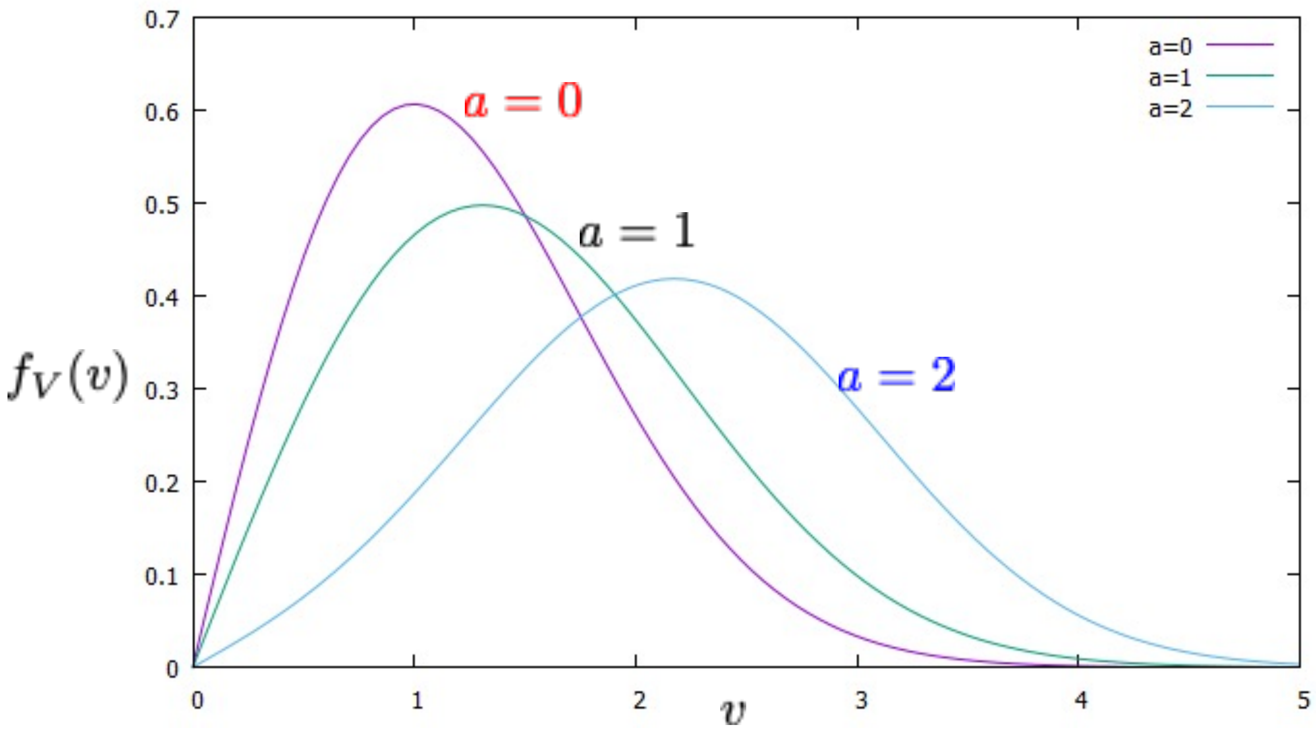
where $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\exp(x\cos(\psi))d\psi$ is the modified Bessel function of the first kind of zero order.

This distribution is named the ***Rician distribution***.

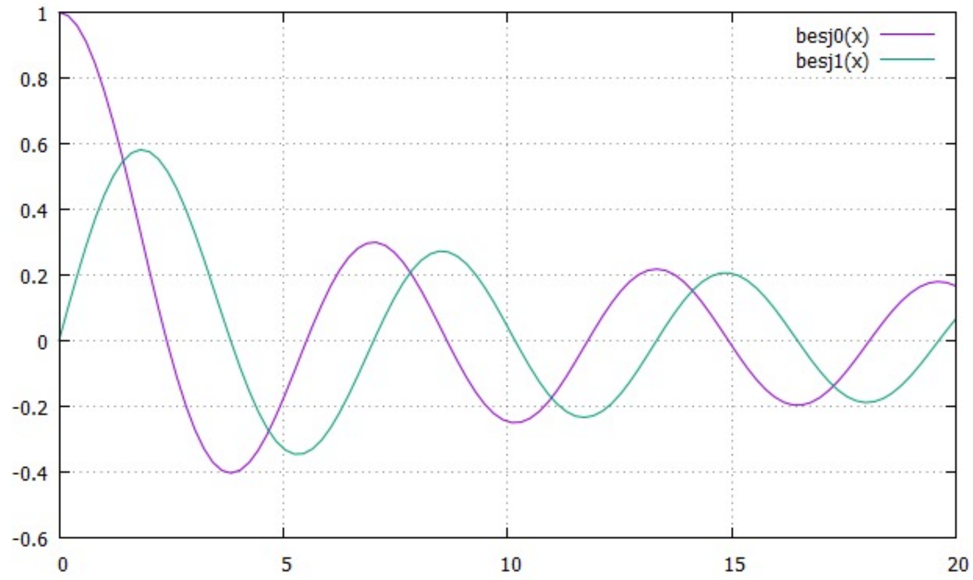
Normalized Rician distribution

Let $v = \frac{r}{\sigma}$ and $a = \frac{A}{\sigma}$.

$$f_V(v) = v \cdot \exp\left(-\frac{v^2 + a^2}{2}\right) I_0(av),$$



Bessel Functions



- Bessel's equation of order n

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

- Its solution $J_n(x)$ is the *Bessel function of the first kind of order* n.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin(\theta) - n\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(jx\sin\theta - jn\theta\right)d\theta = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!}$$

Properties of the Bessel Function

1. $J_n(x) = (-1)^n J_{-n}(x) = (-1)^n J_n(-x)$

2. $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ 3. When x small, $J_n(x) \approx \frac{x^n}{2^n n!}$.

4. When x large, $J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2} \right)$

5. $\lim_{n\to\infty} J_n(x) = 0$. 6. $\sum_{n=-\infty}^{\infty} J_n(x) \exp(jn\phi) = \exp(jx\sin(\phi))$

7. $\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$.

Modified Bessel Function

- Modified Bessel's equation of order n

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(j^2x^2-n^2)y=0 \qquad (\text{i.e., } x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y=0)$$

- Its solution $I_n(x)$ is the *Modified Bessel function of the first kind of order n*.

$$I_n(x)=j^{-n}J_n(jx)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(x\cos(\theta))\cos(n\theta)\,d\theta$$

$$=\sum_{m=0}^{\infty}\frac{(x/2)^{n+2m}}{m!(n+m)!}$$

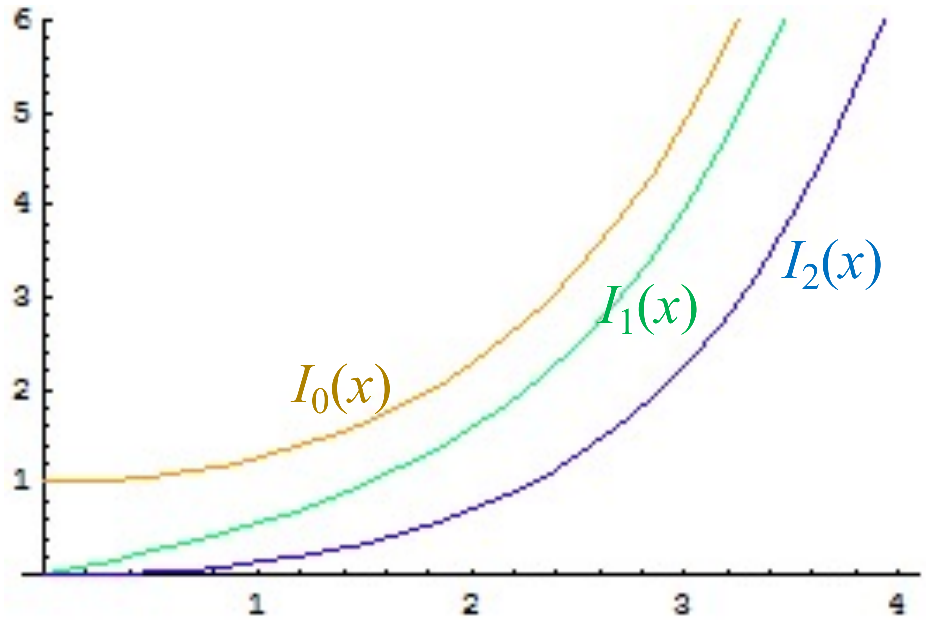
  - The modified Bessel function is monotonically increasing in x for all n.

Properties of Modified Bessel Function

$$3'.\ \lim_{x\downarrow 0} I_n(x) = \begin{cases} 1, & n=0 \\ 0, & n\geq 1 \end{cases}.$$

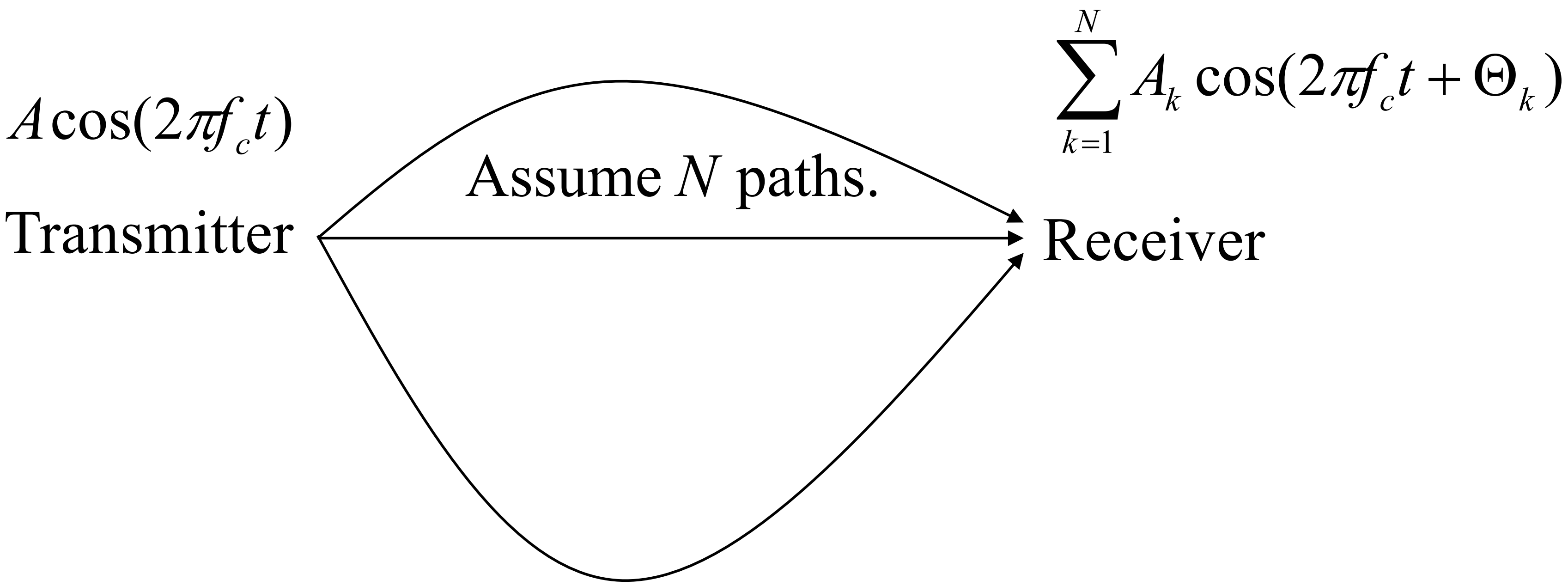
$$4'.\ \text{When } x \text{ large, } I_n(x) \approx \frac{\exp(x)}{\sqrt{2\pi x}}.$$

$$6'.\ \sum_{n=-\infty}^{\infty} I_n(x)\exp(jn\phi) = \exp(x\cos(\phi))$$



Flat-Fading Channel

- Model of a multi-path channel



Flat-Fading Channel

$$Y(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_c t + \Theta_k) = Y_I \cos(2\pi f_c t) - Y_Q \sin(2\pi f_c t)$$

where $Y_I = \sum_{k=1}^{N} A_k \cos(\Theta_k)$ and $Y_Q = \sum_{k=1}^{N} A_k \sin(\Theta_k)$.

$\Rightarrow$ Input $\widetilde{X}(t) = A$ induces output $\widetilde{Y}(t) = Y_I + jY_Q$,

which is independent of t.

Assume $\{(A_k, \Theta_k)\}$ i.i.d., and A_k uniform over $[-1,+1)$ and Θ_k uniform over $[0,2\pi)$.

Flat-Fading Channel

- By Central Limit Theorem,

$$\frac{Y_I}{\sqrt{N/6}} = \frac{A_1\cos(\Theta_1) + \cdots + A_N\cos(\Theta_N)}{\sqrt{N/6}} \to Normal(0,1)$$

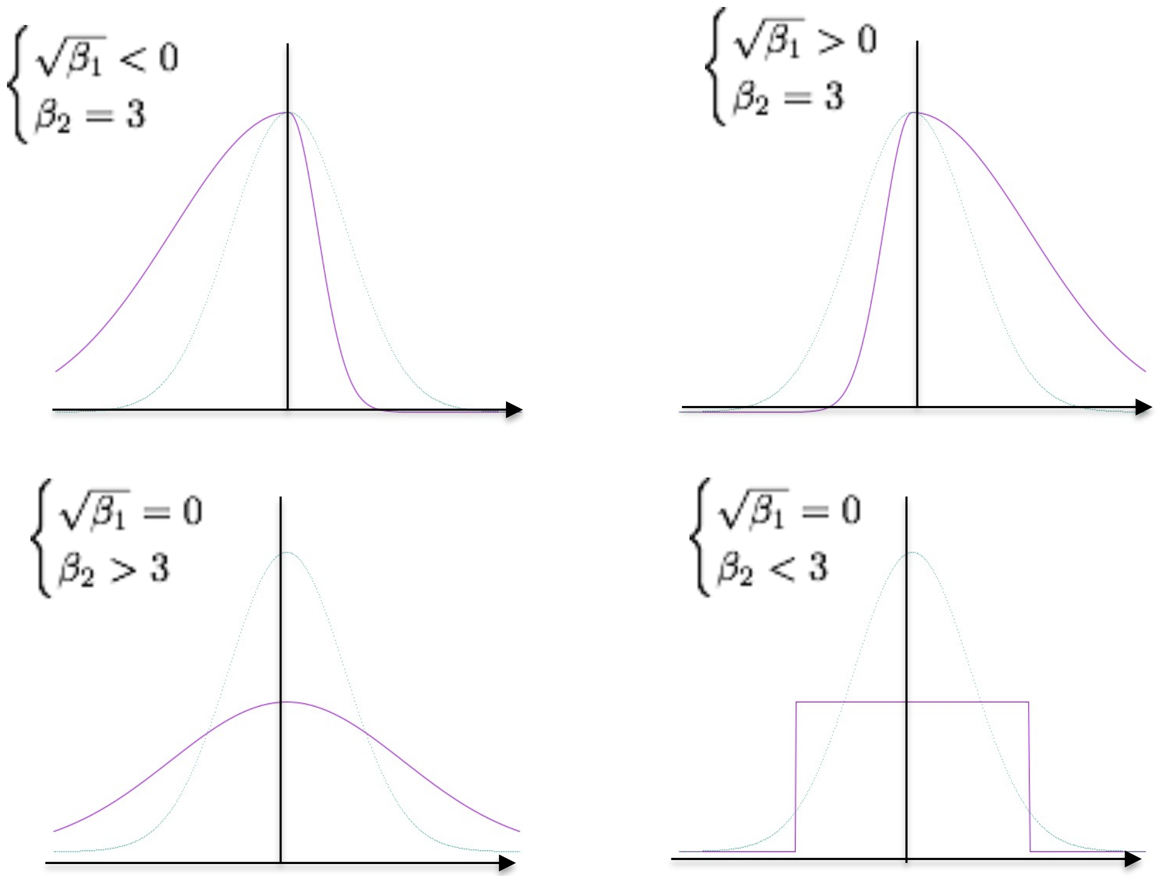
So Y_I is approximately Gaussian distributed with mean 0 and variance $(N/6)$.

For any Gaussian random variable G, we have

$$\sqrt{\beta_1} = \frac{E[(G-\mu)^3]}{E^{3/2}[(G-\mu)^2]} = 0 \text{ and } \beta_2 = \frac{E[(G-\mu)^4]}{E^2[(G-\mu)^2]} = 3.$$

Flat-Fading Channel

$\sqrt{\beta_1}$ skewness
β_2 kurtosis



Flat-Fading Channel

- Normality Test: We can therefore use β_1 and β_2 to examine the degree of resemblance to Gaussian for a random variable.

$$S_N = (U_1 + \cdots + U_N)/a_N \quad \text{for zero-mean i.i.d. } \{U_k\}$$

$$\Rightarrow \tilde{\beta}_2 = \frac{E[S_N^4]}{E^2[S_N^2]} = \frac{E[(U_1 + \cdots + U_N)^4]}{E^2[(U_1 + \cdots + U_N)^2]} = \frac{NE[U^4] + 3N(N-1)E^2[U^2]}{N^2E^2[U^2]} = 3 + \frac{E[U^4]/E^2[U^2] - 3}{N}$$

$$\Rightarrow \begin{cases} E[A\cos(\Theta)] = E[A\sin(\Theta)] = 0 \\ E[A^2\cos^2(\Theta)] = E[A^2\sin^2(\Theta)] = 1/6 \\ E[A^4\cos^4(\Theta)] = E[A^4\sin^4(\Theta)] = 3/40 \end{cases}$$

$$\begin{cases} N = 10 & 100 & 1000 & 10000 \\ 2.97 & 2.997 & 2.9997 & 2.99997 \end{cases}$$

Summary

- Narrowband process
  - Hilbert transform
  - Bandpass system
- Gaussian, Rayleigh and Rician
  - Central Limit Theorem